Constant Detours do not Matter and so $\mathcal{P}^-_* = \mathcal{E}^0_*$

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1 Introduction

We have modified in November 2007 the original paper from November 2006 in order to simplify and standardize the notation. This was done by using the definitions and notation of [1]. In this paper we prove $\mathcal{P}^-_* = \mathcal{E}^0_*$. This means that the both author's class $\mathcal{P}^-$ and the Gregorczyk's $\mathcal{E}^0$ contain the same characteristic functions of predicates. This is so in spite the fact that basically the only difference between the two classes is that the former class does not contain the successor function which is in the latter class.

Actually, we will in the Main Theorem 2.2 prove a stronger result that for any k and any detour function d we have

$$\mathcal{P}^-(d)_\star = \mathcal{P}^-(\lambda x.d(x) + k)_\star$$

where the starred subscripts mean that the nongrowing function of both classes $\mathcal{P}^-(d)$ and $\mathcal{P}^-(\lambda x.d(x) + k)$ are identical.

2 Reduction of Constant Detours

2.1 Classes of nongrowing functions. A function f over $\mathbb{N}$ is *nongrowing* if $f(\vec{x}) \leq \max(\vec{x}, 1)$ for all $\vec{x}$.

For a class of functions $\mathcal{F}$ we designate by $\mathcal{F}_\star$ the subset of $\mathcal{F}$ consisting of the nongrowing functions. Note the use of $\star$ instead of $*$.

We have $\mathcal{P}^-_\star = \mathcal{P}^-$ because the functions of of $\mathcal{P}^-$ are nongrowing.

2.2 Main theorem.

1. *For any detour function d and any k we have*

$$\mathcal{P}^-(d)_\star = \mathcal{P}^-(\lambda x.d(x) + k)_\star \ ,$$

2. $\mathcal{P}^- = \mathcal{E}^0_\star$,
3. $\mathcal{P}^-_* = \mathcal{E}^0_*$.

Proof. Part I. We first note that the characteristic functions of predicates are nongrowing and so the point 3 follows from the point 2.

In order to show that the point 2 follows from the point 1 we note that trivially $\mathcal{P}^- \subseteq \mathcal{E}^0$. On the other hand, if a nongrowing f is in $\mathcal{E}^0$ then there is a constant k such that all functions in the derivations of f are bounded by $\max(\vec{x}) + k$. Thus $f \in \mathcal{P}^-(\lambda x.x + k)_\star$. Incidentally, this is the import of the part *(i)* of the Theorem 7 of [1]. It now follows from the point 1 that $f \in \mathcal{P}^-(\lambda x.x)_* = \mathcal{P}^-_* = \mathcal{P}^-$.

It remains to prove 1. We first note that trivially $\mathcal{P}^-(d) \subseteq \mathcal{P}^-(\lambda x.d(x) + k)$. The converse starred inclusion is $\mathcal{P}^-(\lambda x.d(x)+k)_\star \subseteq \mathcal{P}^-(d)_*$. This can be proved by a straightforward induction on k from the assumption:

$$\mathcal{P}^-(\lambda x.d(x) + 1)_\star \subseteq \mathcal{P}^-(d)_\star \ . \tag{1}$$

We have to delay the proof of the assumption until Par. 2.6 after we have proved the Lemmas 2.4 and 2.5 dealing with the simultaneous definitions of functions. We will, namely, need to encode the detour value $d(\max(\vec{x})) + 1$ by a sequence of values $1, 0$ and we will encode every $y < d(\max(\vec{x})) + 1$ by the sequence $0, y$. We will thereby decrease the detour to $d(\max(\vec{x}))$.

2.3 Sequence notation and simultaneous definitions. For a sequence of functions $\vec{g}$ and $\vec{h}$ we say that the sequence $\vec{f}$ is defined by *simultaneous primitive recursion* if

$$\begin{aligned} \vec{f}(0, \vec{y}) &= \vec{g}(\vec{y}) \\ \vec{f}(x + 1, \vec{y}) &= \vec{h}(x, \vec{y}, \vec{f}(x, \vec{y})) \ . \end{aligned}$$

Here we use the sequence abbreviations:

$$(f_1, \ldots, f_n)(\vec{x}) \equiv f_1(\vec{x}), \ldots, f_n(\vec{x})$$

and

$$(f_1, \ldots, f_n)(\vec{x}) = (g_1, \ldots, g_n)(\vec{x}) \equiv f_1(\vec{x}) = g_1(\vec{x}) \wedge \cdots \wedge f_n(\vec{x}) = g_n(\vec{x}) \ .$$

We will need to show in Lemma 2.4 that $\mathcal{P}^-$ is closed under *special limited recursion* which is a twofold simultaneous recursion with $\vec{g} \equiv g_1, g_2$, $\vec{h} \equiv h_1, h_2$, and $\vec{f} \equiv f_1, f_2$ such that for all $x, \vec{y}$ we have

$$\vec{f}(x, \vec{y}) \preceq 1, 0 \ .$$

Here $x, y \preceq v, w$ lexicographically orders pairs, i.e. either $x < v$ or else $x = v$ and $y \leq w$ holds.

In order to reduce the detour by one we need a pair of encoding functions $\vec{m} \equiv m_1, m_2$ explicitly defined to satisfy:

$$\vec{m}(d, x) = \begin{cases} 0, x & \text{if } x < d \\ 1, 0 & \text{otherwise} \end{cases}$$

Finally, we will abbreviate:

$$\vec{m}(d, x_1, \ldots, x_n) \equiv \vec{m}(d, x_1), \ldots, \vec{m}(d, x_n)$$

Note that this consists of a sequence of $2n$ values.

2.4 Main Lemma. *$\mathcal{P}^-$ is closed under special limited recursion.*

Proof. We need to reduce the special limited recursion to the ordinary primitive recursion by 'guessing' two values c, and e such that for a certain x_0 we will have

$$\forall x, \vec{y}\,(x \leq x_0 \rightarrow \vec{f}(x, \vec{y}) \neq 0, c \wedge \vec{f}(x, \vec{y}) \neq 0, e)\ . \tag{1}$$

If this is the case then we can map during the iteration the extra pair $1, 0$ possibly yielded by $\vec{f}$ to c. Toward that end we explicitly define in $\mathcal{P}^-$ the *encoding* function E, the *decoding* pair of functions $\vec{D}$, and a function F by primitive recursion:

$$\begin{aligned}
E(c, e, v, w) &= \begin{cases} c & \text{if } v, w = 1, 0 \\ e & \text{if } v, w = 0, c \\ w & \text{otherwise} \end{cases} \\
\vec{D}(c, e, w) &= \begin{cases} 1, 0 & \text{if } w = c \\ 0, w & \text{otherwise} \end{cases} \\
F(0, c, e, \vec{y}) &= E(c, e, \vec{g}(\vec{y})) \\
F(x+1, c, e, \vec{y}) &= \begin{cases} e & \text{if } F(x, c, e, \vec{y}) = e \\ E(c, e, \vec{h}(c, e, \vec{y}, \vec{D}(c, e, F(x, c, e, \vec{y})))) & \text{otherwise} \end{cases}
\end{aligned}$$

What is the purpose of the second guessed value e? It is an *error* value which will be yielded and propagated by the iteration of F in case we guessed wrongly and the assumption (1) does not hold. We will be able to recognize this only during the iteration of F.

First of all we note that

$$v, w \preceq 1, 0 \wedge v, w \neq 0, c \rightarrow \vec{D}(c, e, E(c, e, v, w)) = v, w\ .$$

By induction on x we then show:

1. If $F(x, c, e, \vec{y}) \neq e$ then $\vec{f}(x, \vec{y}) = \vec{D}(c, e, F(x, c, e, \vec{y}))$,
2. if (1) holds and $x \leq x_0$ then $F(x, c, e, \vec{y}) \neq e$.

We will now find the values c, e, and x_0 satisfying (1). We set $d := \max(x, \vec{y}, 1)$ and note that if $x_0 + 2 \leq d$ then there are at most $x_0 + 1$ different values

$$\vec{f}(0, \vec{y}), \vec{f}(1, \vec{y}), \ldots, \vec{f}(x_0, \vec{y})$$

computed during the iteration of $\vec{f}(x, \vec{y})$ with $x \leq x_0$. On the other hand, we have at our disposal $x_0 + 3$ different values no greater than d. Thus by a twofold search with C we can find a $c \leq d$ such that $0, c$ is avoided by the iteration for a suitable $e \leq d$. By another search with R we can find such an e:

$$\begin{aligned}
&C(x, d, \vec{y}) = (\mu\, c \leq d)[\exists e{\leq}d\, F(x, c, e, \vec{y}) \neq e] \\
&R(x, d, \vec{y}) = (\mu\, e \leq d)[F(x, C(x, d, \vec{y}), e, \vec{y}) \neq e] \\
&\vec{G}(x, d, \vec{y}) = \vec{D}(c, e, F(x, c, e, \vec{y})) \quad \text{where } e = R(x, d, \vec{y}) \text{ and } c = C(x, d, \vec{y}) \\
&\qquad \vec{f}(x, \vec{y}) = \begin{cases} \vec{g}(\vec{y}) & \text{if } x = 0 \\ \vec{h}(0, \vec{y}, \vec{g}(\vec{y})) & \text{if } x = 1 \\ \vec{h}(x \mathbin{\dot{-}} 1, \vec{h}(x \mathbin{\dot{-}} 2, \vec{y}, \vec{G}(x \mathbin{\dot{-}} 2, \max(x, \vec{y}, 1), \vec{y})) & \text{otherwise} \end{cases}
\end{aligned}$$

□

2.5 Lemma *For every* $f \in \mathcal{P}^-$ *there is a pair* $\vec{f} = f_1, f_2 \in \mathcal{P}^-$ *such that*

$$d > 0 \wedge \max(\vec{x}) \leq d \to \vec{m}(d, f(\vec{x})) = \vec{f}(d - 1, \vec{m}(d, \vec{x}))\ .$$

Proof. This is a straightforward, albeit technical, induction on the construction of f in $\mathcal{P}^-$. If $f(\vec{x}) = I_i^n(\vec{x})$ then define $\vec{f}(d, \vec{w}) = z_i, x_i$ where $\vec{w} \equiv z_1, x_1, \ldots, z_n, x_n$.

If $f(\vec{x}) = C_1(\vec{x})$ then define

$$\vec{f}(d, z, x) = \begin{cases} 1, 0 & \text{if } d = 0 \\ 0, 1 & \text{otherwise} \end{cases}$$

If $f(\vec{x}) = g(\vec{h}(\vec{x}))$ we obtain $\mathcal{P}^-$ functions $\vec{g}$ and $\vec{h}_1, \ldots \vec{h}_m$ by IH and define

$$\vec{f}(d, \vec{w}) = \vec{g}(d, \vec{h}_1(d, \vec{w}), \ldots, \vec{h}_m(d, \vec{w}))$$

where $\vec{w} \equiv z_1, x_1, \ldots, z_n, x_n$.

The most interesting case is when $f(0, \vec{y}) = g(\vec{y})$ and $f(x{+}1, \vec{y}) = h(x, \vec{y}, f(x, \vec{y}))$. We obtain $\mathcal{P}^-$ functions $\vec{g}$ and $\vec{h}$ by IH. We then define

$$\begin{aligned}
\vec{L}(d, v, w) &= \begin{cases} v, w & \text{if } v = 1 \wedge w = 0 \vee v = 0 \wedge w \leq d \\ 0, 0 & \text{otherwise} \end{cases} \\
\vec{F}(d, 0, \vec{w}) &= \vec{L}(d, \vec{g}(d, \vec{w})) \\
\vec{F}(d, x + 1, \vec{w}) &= \vec{L}(d, \vec{h}(d, 0, x, \vec{w}, \vec{F}(d, x, \vec{w})))
\end{aligned}$$

where $\vec{w} \equiv z_1, y_1, \ldots, z_n, y_n$. We have $\vec{F} \in \mathcal{P}^-$ by the Main Lemma. We then define

$$\vec{f}(d, z, x, \vec{w}) = \begin{cases} \vec{F}(d, x, \vec{w}) & \text{if } z = 0 \\ \vec{h}(d, 0, d, \vec{w}, \vec{F}(d, d, \vec{w})) & \text{otherwise} \end{cases}$$

□

2.6 Part II of the proof of the Main Theorem. In order to finish the proof of the Main Theorem 2.2 it remains to prove the property 2.2(1):

$$\mathcal{P}^-(\lambda x.d(x)+1)_\star \subseteq \mathcal{P}^-(d)_\star \ .$$

Thus take any detour function d and $f \in \mathcal{P}^-(\lambda x.d(x)+1)_\star$. There is a $g \in \mathcal{P}^-$ such that $f(\vec{x}) = g(d,\vec{x})$ where we set $d := d(\max(\vec{x}))+1$. We have $\vec{m}(d,d) = 1,0$ and, since $d(\max(\vec{x}))+1 \geq \max(\vec{x})+1$, also $\vec{m}(d,x_i) = 0,x_i$ for $i = 1,\ldots,n$. We get from Lemma 2.5 a pair of functions $\vec{g} = g_1,g_2 \in \mathcal{P}^-$ such that

$$\vec{m}(d,g(d,\vec{x})) = \vec{g}(d-1,1,0,0,x_1,\ldots,0,x_n) \ .$$

Note that, since f is nongrowing, so whenever $\max(\vec{x}) > 0$ we have $f(\vec{x}) \leq \max(\vec{x},1) = \max(\vec{x}) < d$. Hence

$$0,f(\vec{x}) = \vec{m}(d,f(\vec{x})) = \vec{m}(d,g(d,\vec{x})) = \vec{g}(d-1,1,0,0,x_1,\ldots,0,x_n) \ .$$

We thus explicitly define $G \in \mathcal{P}^-$ satisfying

$$G(d,x_1,\ldots,x_n) = \begin{cases} C_{f(0,\ldots,0)}(1) & \text{if } \max(x_1,\ldots,x_n) = 0 \\ g_2(d,1,0,0,x_1,\ldots,0,x_n) & \text{otherwise} \end{cases}$$

Since $f(\vec{x}) = G(d-1,\vec{x}) = G(d(\max(\vec{x})),\vec{x})$, we have $f \in \mathcal{P}^-(d)_\star$. □

References

1. Lars Kristiansen, Paul J. Voda. *The Structure of Detour Degrees*, submitted for publication, November 2007,