$f(x_j, y_j)$. The function is defined by primitive recursion on i as a p.r. function:

$$M_0(x,y,t) = t$$
$$M_{i+1}(x,y,t) = U(M_i(x,y,t), i, x, y).$$

It has the following properties which will be needed in the sequel:

$$T \vdash i+1 \leq 2^{x+1} \to M_i\big(x+1, y, \langle z,l,r\rangle\big) = \big\langle z, M_i\big(x, \sigma_1[x,y], l\big), r\big\rangle \tag{1}$$

$$T \vdash i+1 \leq 2^{x+1} \wedge M_{2^{x+1} \dot- 1}\big(x, \sigma_1[x,y], l\big) = l_1 \to$$
$$M_{2^{x+1}\dot-1+i}\big(x+1, y, \langle z,l,r\rangle\big) = \big\langle z, l_1, M_i\big(x, \sigma_2[x,y,\pi_1(l_1)], r\big)\big\rangle. \tag{2}$$

Proof. (1): By induction on i. In the base case, clearly $0+1 \leq 2^{x+1}$ and thus

$$M_0(x+1, y, \langle z,l,r\rangle) = \langle z,l,r\rangle = \big\langle z, M_0(x, \sigma_1[x,y], l), r\big\rangle.$$

In the induction step, if $(i+1)+1 \leq 2^{x+1}$ then $i+1 \leq 2^{x+1}$ and therefore

$$M_{i+1}(x+1, y, \langle z,l,r\rangle) = U\big(M_i(x+1, y, \langle z,l,r\rangle), i, x+1, y\big) \overset{\text{IH}}{=}$$
$$= U\big(\big\langle z, M_i(x, \sigma_1[x,y], l), r\big\rangle, i, x+1, y\big) \overset{3.5.8(1)}{=}$$
$$= \Big\langle z, U\big(M_i(x, \sigma_1[x,y], l), i, x, \sigma_1[x,y]\big), r\Big\rangle = \big\langle z, M_{i+1}(x, \sigma_1[x,y], l), r\big\rangle.$$

(2): By induction on i. In the base case suppose that $M_{2^{x+1}\dot-1}(x, \sigma_1[x,y], l) = l_1$. We clearly have $0+1 \leq 2^{x+1}$ and thus

$$M_{2^{x+1}\dot-1+0}(x+1, y, \langle z,l,r\rangle) = M_{2^{x+1}\dot-1}(x+1, y, \langle z,l,r\rangle) \overset{(1)}{=}$$
$$= \big\langle z, M_{2^{x+1}\dot-1}(x, \sigma_1[x,y], l), r\big\rangle = \langle z, l_1, r\rangle = \big\langle z, l_1, M_0\big(x, \sigma_2[x,y,\pi_1(l_1)], r\big)\big\rangle.$$

In the induction step, assume $(i+1)+1 \leq 2^{x+1}$ and $M_{2^{x+1}\dot-1}(x, \sigma_1[x,y], l) = l_1$. Then $i+1 \leq 2^{x+1}$ and we obtain

$$M_{2^{x+1}\dot-1+(i+1)}(x+1, y, \langle z,l,r\rangle) = M_{2^{x+1}\dot-1+i+1}(x+1, y, \langle z,l,r\rangle) =$$
$$= U\big(M_{2^{x+1}\dot-1+i}(x+1, y, \langle z,l,r\rangle), 2^{x+1} \dot- 1 + i, x+1, y\big) \overset{\text{IH}}{=}$$
$$= U\Big(\big\langle z, l_1, M_i\big(x, \sigma_2[x,y,\pi_1(l_1)], r\big)\big\rangle, 2^{x+1} \dot- 1 + i, x+1, y\Big) \overset{3.5.8(2)}{=}$$
$$= \Big\langle z, l_1, U\Big(M_i\big(x, \sigma_2[x,y,\pi_1(l_1)], r\big), i, x, \sigma_2[x,y,\pi_1(l_1)]\Big)\Big\rangle =$$
$$= \big\langle z, l_1, M_{i+1}\big(x, \sigma_2[x,y,\pi_1(l_1)], r\big)\big\rangle.$$

□

3.5.10 Course of values function. The binary function $\overline{f}(x, y)$ returns the computation tree for $f(x, y)$. The course of values function for f satisfies

$$T \vdash \overline{f}(0, y) = \big\langle \rho[y], 0, 0 \big\rangle \tag{1}$$

$$T \vdash \overline{f}\big(x, \sigma_1[x, y]\big) = l \wedge \overline{f}\big(x, \sigma_2[x, y, \pi_1(l)]\big) = r \rightarrow \tag{2}$$
$$\overline{f}(x + 1, y) = \big\langle \theta[x, \pi_1(l), \pi_1(r), y], l, r \big\rangle$$

and it is defined explicitly as a p.r. function by

$$\overline{f}(x, y) = M_{2^{x+1} \dot{-} 1}\big(x, y, \mathit{Full}(x + 1)\big).$$

Proof. (1): It follows from

$$\begin{aligned}
\overline{f}(0, y) &= M_{2^{0+1} \dot{-} 1}\big(0, y, \mathit{Full}(0 + 1)\big) = M_1\big(0, y, \mathit{Full}(1)\big) = M_1\big(0, y, \langle 0, 0, 0 \rangle\big) = \\
&= U\Big(M_0\big(0, y, \langle 0, 0, 0 \rangle\big), 0, 0, y\Big) = U\big(\langle 0, 0, 0 \rangle, 0, 0, y\big) \stackrel{3.5.8(3)}{=} \\
&= \big\langle V(0, y, 0, 0), 0, 0 \big\rangle = \big\langle \rho[y], 0, 0 \big\rangle.
\end{aligned}$$

(2): Suppose that

$$\overline{f}\big(x, \sigma_1[x, y]\big) = l$$
$$\overline{f}\big(x, \sigma_2[x, y, \pi_1(l)]\big) = r.$$

Then, by definition, we have

$$M_{2^{x+1} \dot{-} 1}\big(x, \sigma_1[x, y], \mathit{Full}(x + 1)\big) = l \tag{$\dagger_1$}$$

$$M_{2^{x+1} \dot{-} 1}\big(x, \sigma_2[x, y, \pi_1(l)], \mathit{Full}(x + 1)\big) = r \tag{$\dagger_2$}$$

and therefore

$$\begin{aligned}
\overline{f}(x + 1, y) &= M_{2^{x+1+1} \dot{-} 1}\big(x + 1, y, \mathit{Full}(x + 1 + 1)\big) = \\
&= M_{2^{x+2} \dot{-} 2 + 1}\big(x + 1, y, \big\langle 0, \mathit{Full}(x + 1), \mathit{Full}(x + 1)\big\rangle\big) = \\
&= U\Big(M_{2^{x+2} \dot{-} 2}\big(x + 1, y, \big\langle 0, \mathit{Full}(x + 1), \mathit{Full}(x + 1)\big\rangle\big), 2^{x+2} \dot{-} 2, x + 1, y\Big) = \\
&= U\Big(M_{2^{x+1} \dot{-} 1 + (2^{x+1} \dot{-} 1)}\big(x + 1, y, \big\langle 0, \mathit{Full}(x + 1), \mathit{Full}(x + 1)\big\rangle\big), \\
&\qquad 2^{x+2} \dot{-} 2, x + 1, y\Big) \stackrel{(\dagger_1),\ 3.5.9(2)}{=} \\
&= U\Big(\big\langle 0, l, M_{2^{x+1} \dot{-} 1}\big(x, \sigma_2[x, y, \pi_1(l)], \mathit{Full}(x + 1)\big)\big\rangle, 2^{x+2} \dot{-} 2, x + 1, y\Big) \stackrel{(\dagger_2)}{=}
\end{aligned}$$