Hyperresolution for Gödel Logic with Truth Constants

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Abstract

In the paper, we generalise the well-known hyperresolution principle to the general first-order Gödel logic with truth constants. We propose a hyperresolution calculus suitable for automated deduction in Gödel logic with explicit partial truth and solve the deduction problem of a formula from a countable theory. We expand Gödel logic by a countable set of intermediate truth constants of the form $\bar{c}$, $c \in (0,1)$. Our approach is based on translation of a formula to an equivalent satisfiable finite order clausal theory, consisting of order clauses. We introduce a notion of quantified atom: a formula a is a quantified atom if $a = Qx\, p(t_0, \dots, t_\tau)$ where Q is a quantifier ($\forall$, $\exists$); $p(t_0, \dots, t_\tau)$ is an atom; x is a variable occurring in $p(t_0, \dots, t_\tau)$; for all $i \leq \tau$, either $t_i = x$ or x does not occur in t_i. Then an order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is an atom or a quantified atom, and $\diamond$ is the connective $\eqcirc$ or $\prec$. $\eqcirc$ and $\prec$ are interpreted by the equality and standard strict linear order on $[0,1]$, respectively. We shall investigate the so-called canonical standard completeness, where the semantics of Gödel logic is given by the standard $\boldsymbol{G}$-algebra and truth constants are interpreted by 'themselves'. The hyperresolution calculus is refutation sound and complete for a countable order clausal theory under a certain condition for the set of truth constants occurring in the theory. As an interesting consequence, we get an affirmative solution to the open problem of recursive enumerability of unsatisfiable formulae in Gödel logic with truth constants.

Keywords: Gödel Logic, Resolution, Many-valued Logics, Automated Deduction

1. Introduction

Current research in many-valued logics is mainly concerned with left-continuous t-norm based logics including the three fundamental fuzzy logics: Gödel, Łukasiewicz, and Product ones. From a syntactical point of view, classical many-valued deduction calculi are widely studied, especially Hilbert-style ones. In addition, a perspective from automated deduction has received attractivity during the last two decades. A considerable effort has been made in development of *SAT* solvers for the problem of Boolean satisfiability. *SAT* solvers may exploit either complete solution methods (called complete or

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systematic *SAT* solvers) or incomplete or hybrid ones. Complete *SAT* solvers are mostly based on the Davis-Putnam-Logemann-Loveland procedure (*DPLL*) [1, 2] or resolution proof methods [3, 4, 5], improved by various features [6]. t-norm based logics are logics of comparative truth: the residuum of a t-norm satisfies, for all $x, y \in [0,1]$, $x \rightarrow y = 1$ if and only if $x \leq y$. Since implication is interpreted by a residuum, in the propositional case, a formula of the form $\phi \rightarrow \psi$ is a consequence of a theory if $\|\phi\|^{\mathfrak{A}} \leq \|\psi\|^{\mathfrak{A}}$ for every model $\mathfrak{A}$ of the theory. Most explorations of t-norm based logics are focused on tautologies and deduction calculi with the only distinguished truth value 1 [7]. However, in many real-world applications, one may be interested in representation and inference with explicit partial truth; besides the truth constants *0*, *1*, intermediate truth constants are involved in. In the literature, two main approaches to expansions with truth constants, are described. Historically, the first one has been introduced in [8], where the propositional Łukasiewicz logic is augmented by truth constants $\bar{r}$, $r \in [0,1]$, Pavelka's logic (*PL*). A formula of the form $\bar{r} \rightarrow \phi$ evaluated to 1 expresses that the truth value of ϕ is greater than or equal to r. In [9], further development of evaluated formulae, and in [7], Rational Pavelka's logic (*RPL*) - a simplification of *PL* exploiting book-keeping axioms, are described. Another approach relies on traditional algebraic semantics. Various completeness results for expansions of t-norm based logics with countably many truth constants are investigated, among others, in [10, 11, 12, 13, 14, 15, 16].

1.1. Motivation

Concerning the three fundamental first-order fuzzy logics, the set of logically valid formulae is Π_2-complete for Łukasiewicz logic, Π_2-hard for Product logic, and Σ_1-complete for Gödel logic, as with classical first-order logic. Among these fuzzy logics, only Gödel logic is recursively axiomatisable. Hence, it was necessary to provide a proof method suitable for automated deduction, as one has done for classical logic. In contrast to classical logic, we cannot make shifts of quantifiers arbitrarily and translate a formula to an equivalent (satisfiable) prenex form. In [17, 18, 19], the prenex fragment of Gödel logic in presence of the projection operator $\mathbf{\Delta} : [0,1] \longrightarrow [0,1]$,

$$\mathbf{\Delta}\, a = \begin{cases} 1 & \textit{if } a = 1, \\ 0 & \textit{else}, \end{cases}$$

is investigated, denoted as the prenex G_∞^Δ; and without $\mathbf{\Delta}$, as the prenex G_∞. Informally, the standard Skolemisation of a prenex formula $Q_1 x_1, \dots, Q_n x_n\, F$ (F is quantifier free) with respect to validity is the replacement of every universally bound variable y by a Skolem term $f(z_1, \dots, z_m)$ if y is in the scope of the existential quantifier occurrences $\exists z_1, \dots, \exists z_m$. Analogously, with respect to satisfiability, 'universal' is altered to 'existential', and vice-versa. A Skolemised formula by the standard way is an open formula. In classical logic, a Skolemised formula is valid | satisfiable if and only if so is the original formula. In case of the prenex G_∞ or G_∞^Δ, this duality does not hold. The standard Skolemisation with respect to validity is not sound for the prenex G_∞, but it is for the prenex G_∞^Δ. A variant of Herbrand's Theorem for the prenex G_∞^Δ is proved, which reduces the validity problem (*VAL*) of a formula in the prenex G_∞^Δ to the *VAL* problem of an open formula in G_∞^Δ. In case of the satisfiability problem (*SAT*), one cannot proceed as in classical logic. So, a novel extended form of Skolemisation has been introduced, which adds a new monadic predicate symbol besides replacing existential quantifier occurrences by Skolem terms. Both the Skolemisation methods are in polynomial time. Further, a meta-level logic of order clauses is defined, which is a fragment of classical one. An order clause is a finite set of inequalities of the form $a < b$ or $a \leq b$ where $<$, $\leq$ are meta-level binary predicate symbols and a, b are atoms of G_∞^Δ considered as meta-level terms. The semantics of the meta-level logic of order clauses is given by classical interpretations on $[0, 1]$, varying on assigned (truth) values to atoms of G_∞^Δ (meta-level terms), which are the strict dense linear order with endpoints on $[0, 1]$; $<$ is interpreted as the strict dense linear order with endpoints and $\leq$ as its reflexive closure on $[0, 1]$. A Skolemised open formula can be translated using a structural translation to order clauses in polynomial time. We conclude that a formula in the prenex G_∞^Δ is valid if and only if a translation (in polynomial time) of it to order clause form is unsatisfiable with respect to the semantics of the meta-level logic. Similarly, a formula in the prenex G_∞^Δ is (un)satisfiable if and only if a translation (in polynomial time) of it to order clause form is (un)satisfiable with respect to the semantics of the meta-level logic. The ordered chaining calculi [20, 21] (the irreflexivity resolution and factorized chaining rules) may be used for resolution-style deduction over order clauses. As an interesting consequence, we get that the set of unsatisfiable formulae in the prenex G_∞^Δ is recursively enumerable.

We believe that a solution via Skolemisation may be of limited use, if even feasible. However, we can avoid this way as follows. In [22, 23], we have generalised the well-known hyperresolution principle to the first-order Gödel logic for the general case. Our approach is based on translation of a formula of Gödel logic to an equivalent satisfiable finite order clausal theory, consisting of order clauses. We have introduced a notion of quantified atom: a formula a is a quantified atom if $a = Qx\, p(t_0, \dots, t_\tau)$ where Q is a quantifier ($\forall$, $\exists$); $p(t_0, \dots, t_\tau)$ is an atom; x is a variable occurring in $p(t_0, \dots, t_\tau)$; for all $i \leq \tau$, either $t_i = x$ or x does not occur in t_i (t_i is a free term in the quantified atom). The notion of quantified atom is all important. It permits us to extend classical unification to quantified atoms without any additional computational cost. Two quantified atoms $Qx\, p(t_0, \dots, t_\tau)$ and $Q'x'\, p'(t'_0, \dots, t'_\tau)$ are unifiable if $Q = Q'$, $x = x'$, $p = p'$, and the left-right sequence of free terms of $Qx\, p(t_0, \dots, t_\tau)$ is unifiable with the left-right sequence of free terms of $Q'x'\, p'(t'_0, \dots, t'_\tau)$ in the standard manner. An order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is an atom or a quantified atom, and $\diamond$ is the connective $=$ or $\prec$. $=$ and $\prec$ are interpreted by the equality and standard strict linear order on $[0, 1]$, respectively. On the basis of the hyperresolution principle, a calculus operating over order clausal theories, has been devised. The calculus is proved to be refutation sound and complete for the countable case with respect to the standard $\boldsymbol{G}$-algebra $\boldsymbol{G} = ([0, 1], \leq, \vee, \wedge, \Rightarrow, \overline{\ }, \boldsymbol{=}, \boldsymbol{\prec}, 0, 1)$ augmented by binary operators $\boldsymbol{=}$ and $\boldsymbol{\prec}$ for $=$ and $\prec$, respectively. As another step, one may incorporate a countable set of intermediate truth constants of the form $\bar{c}$, $c \in (0, 1)$, to get a modification of the hyperresolution calculus suitable for automated deduction with explicit partial truth [24]. We shall investigate the so-called canonical standard completeness, where the semantics of Gödel logic is given by the standard $\boldsymbol{G}$-algebra $\boldsymbol{G}$ and truth constants are interpreted by 'themselves'. Note that the Hilbert-style calculus for Gödel logic introduced in [7], is not suitable for expansion with intermediate truth constants. We have $\phi \vdash \psi$ if and only if $\phi \models \psi$ (wrt. $\boldsymbol{G}$). However, that cannot be preserved after adding intermediate truth constants. Let $c \in (0, 1)$ and a be an atom different from a constant. Then $\bar{c} \models a$ ($\bar{c}$ is unsatisfiable) but $\not\models \bar{c} \rightarrow a$, $\not\vdash \bar{c} \rightarrow a$, $\bar{c} \not\vdash a$ (from the soundness and the deduction-detachment theorem for this calculus). So, we cannot achieve a strict canonical standard completeness after expansion with intermediate truth constants. On the other side, such a completeness can be feasible for our hyperresolution calculus under a certain condition. We say that a set $\{\mathit{0}, \mathit{1}\} \subseteq X$ of truth constants is admissible with respect to suprema and infima if, for all $\emptyset \neq Y_1, Y_2 \subseteq X$ and $\bigvee Y_1 = \bigwedge Y_2$, $\bigvee Y_1 \in Y_1$, $\bigwedge Y_2 \in Y_2$ (truth constants are interpreted by 'themselves'). Then the hyperresolution calculus is refutation sound and complete for a countable order clausal theory if the set of truth constants occurring in the theory, is admissible with respect to suprema and infima. This condition obviously covers the case of finite order clausal theories. We solve the deduction problem of a formula from a countable theory. As an interesting consequence, we get an affirmative solution to the open problem of recursive enumerability of unsatisfiable formulae in Gödel logic with truth constants.

The paper is organised as follows. Section 2 gives the basic notions and notation concerning the first-order Gödel logic. Section 3 deals with clause form translation. In Section 4, we propose a hyperresolution calculus with truth constants and prove its refutational soundness, completeness. Section 5 provides some examples for translation and deduction. Section 6 brings conclusions.

2. First-order Gödel logic

Throughout the paper, we shall use the common notions and notation of first-order logic. $\mathbb{N}$ | $\mathbb{Z}$ designates the set of natural | integer numbers and $\leq$ | $<$ the standard order | strict order on $\mathbb{N}$ | $\mathbb{Z}$. By $\mathcal{L}$ we denote a first-order language. $\mathit{Var}_\mathcal{L}$ | $\mathit{Func}_\mathcal{L}$ | $\mathit{Pred}_\mathcal{L}$ | $\mathit{Term}_\mathcal{L}$ | $\mathit{GTerm}_\mathcal{L}$ | $\mathit{Atom}_\mathcal{L}$ | $\mathit{GAtom}_\mathcal{L}$ denotes the set of all variables | function symbols | predicate symbols | terms | ground terms | atoms | ground atoms of $\mathcal{L}$.[1] $\mathit{ar}_\mathcal{L} : \mathit{Func}_\mathcal{L} \cup \mathit{Pred}_\mathcal{L} \longrightarrow \mathbb{N}$[1] denotes the mapping assigning an arity to every function and predicate symbol of $\mathcal{L}$. We assume truth constants - nullary predicate symbols $\mathit{0}, \mathit{1} \in \mathit{Pred}_\mathcal{L}$, $\mathit{ar}_\mathcal{L}(\mathit{0}) = \mathit{ar}_\mathcal{L}(\mathit{1}) = 0$; $\mathit{0}$ denotes the false and $\mathit{1}$ the true in $\mathcal{L}$. Let $\mathbb{C}_\mathcal{L} \subseteq (0,1)$ be countable. In addition, we assume a countable set of nullary predicate symbols $\overline{C}_\mathcal{L} = \{\bar{c} \,|\, \bar{c} \in \mathit{Pred}_\mathcal{L}, \mathit{ar}_\mathcal{L}(\bar{c}) = 0, c \in \mathbb{C}_\mathcal{L}\} \subseteq \mathit{Pred}_\mathcal{L}$; $\{\mathit{0}\}$, $\{\mathit{1}\}$, $\overline{C}_\mathcal{L}$ are pairwise disjoint. $\mathit{0}$, $\mathit{1}$, $\bar{c} \in \overline{C}_\mathcal{L}$ are called truth constants. We denote $\mathit{Tcons}_\mathcal{L} = \{\mathit{0}, \mathit{1}\} \cup \overline{C}_\mathcal{L} \subseteq \mathit{Pred}_\mathcal{L}$. Let $X \subseteq \mathit{Tcons}_\mathcal{L}$. We denote $\overline{X} = \{0 \,|\, \mathit{0} \in X\} \cup \{1 \,|\, \mathit{1} \in X\} \cup \{c \,|\, \bar{c} \in X \cap \overline{C}_\mathcal{L}\} \subseteq [0,1]$. By $\mathit{Form}_\mathcal{L}$[1] we designate the set of all formulae of $\mathcal{L}$ built up from $\mathit{Atom}_\mathcal{L}$ and $\mathit{Var}_\mathcal{L}$ using the connectives: $\neg$, negation, $\wedge$, conjunction, $\vee$, disjunction, $\rightarrow$, implication, and the quantifiers: $\forall$, the universal quantifier, $\exists$, the existential one. In addition, we introduce new binary connectives $\eqcirc$, equality, and $\prec$, strict order. We denote $\mathit{Con} = \{\neg, \wedge, \vee, \rightarrow, \eqcirc, \prec\}$. By $\mathit{OrdForm}_\mathcal{L}$[1] we designate the set of all so-called order formulae of $\mathcal{L}$ built up from $\mathit{Atom}_\mathcal{L}$ and $\mathit{Var}_\mathcal{L}$ using the connectives in Con and the quantifiers: $\forall$, $\exists$.[2] Note that $\mathit{OrdForm}_\mathcal{L} \supseteq \mathit{Form}_\mathcal{L}$. In the paper, we shall assume that $\mathcal{L}$ is a countable first-order language; hence, all the above mentioned sets of symbols and expressions are countable. Let ε | ε_i, $1 \leq i \leq m$ | υ_i, $1 \leq i \leq n$, be either an expression or a set of expressions or a set of sets of expressions of $\mathcal{L}$, in general. By

[1] If the first-order language in question is not explicitly designated, we shall write denotations without index.

[2] We assume a decreasing connective and quantifier precedence: $\forall$, $\exists$, $\neg$, $\wedge$, $\rightarrow$, $\eqcirc$, $\prec$, $\vee$.

$vars(\varepsilon_1, \dots, \varepsilon_m) \subseteq Var_{\mathcal{L}} \mid freevars(\varepsilon_1, \dots, \varepsilon_m) \subseteq Var_{\mathcal{L}} \mid boundvars(\varepsilon_1, \dots, \varepsilon_m) \subseteq Var_{\mathcal{L}} \mid funcs(\varepsilon_1, \dots, \varepsilon_m) \subseteq Func_{\mathcal{L}} \mid preds(\varepsilon_1, \dots, \varepsilon_m) \subseteq Pred_{\mathcal{L}} \mid atoms(\varepsilon_1, \dots, \varepsilon_m) \subseteq Atom_{\mathcal{L}}$ we denote the set of all variables | free variables | bound variables | function symbols | predicate symbols | atoms of $\mathcal{L}$ occurring in $\varepsilon_1, \dots, \varepsilon_m$. ε is closed iff $freevars(\varepsilon) = \emptyset$. By ℓ we denote the empty sequence. By $|\varepsilon_1, \dots, \varepsilon_m| = m$ we denote the length of the sequence $\varepsilon_1, \dots, \varepsilon_m$. We define the concatenation of the sequences $\varepsilon_1, \dots, \varepsilon_m$ and $\upsilon_1, \dots, \upsilon_n$ as $(\varepsilon_1, \dots, \varepsilon_m), (\upsilon_1, \dots, \upsilon_n) = \varepsilon_1, \dots, \varepsilon_m, \upsilon_1, \dots, \upsilon_n$. Note that concatenation of sequences is associative.[3]

Let X, Y, Z be sets, $Z \subseteq X$; $f : X \longrightarrow Y$ be a mapping. By $\|X\|$ we denote the set-theoretic cardinality of X. X being a finite subset of Y is denoted as $X \subseteq_{\mathcal{F}} Y$. We designate $\mathcal{P}(X) = \{x \mid x \subseteq X\}$; $\mathcal{P}(X)$ is the power set of X; $\mathcal{P}_{\mathcal{F}}(X) = \{x \mid x \subseteq_{\mathcal{F}} X\}$; $\mathcal{P}_{\mathcal{F}}(X)$ is the set of all finite subsets of X; $f[Z] = \{f(z) \mid z \in Z\}$; $f[Z]$ is the image of Z under f; $f|_Z = \{(z, f(z)) \mid z \in Z\}$; $f|_Z$ is the restriction of f onto Z. Let $\gamma \leq \omega$. A sequence δ of X is a bijection $\delta : \gamma \longrightarrow X$. Recall that X is countable if and only if there exists a sequence of X. Let I be a set and $S_i \neq \emptyset$, $i \in I$, be sets. A selector $\mathcal{S}$ over $\{S_i \mid i \in I\}$ is a mapping $\mathcal{S} : I \longrightarrow \bigcup\{S_i \mid i \in I\}$ such that for all $i \in I$, $\mathcal{S}(i) \in S_i$. We denote $\mathcal{S}el(\{S_i \mid i \in I\}) = \{\mathcal{S} \mid \mathcal{S} \textit{ is a selector over } \{S_i \mid i \in I\}\}$. $\mathbb{R}$ designates the set of real numbers and $\leq$ | $<$ the standard order | strict order on $\mathbb{R}$. We denote $\mathbb{R}_0^+ = \{c \mid 0 \leq c \in \mathbb{R}\}$, $\mathbb{R}^+ = \{c \mid 0 < c \in \mathbb{R}\}$; $[0,1] = \{c \mid 0 \leq c \leq 1, c \in \mathbb{R}\}$; $[0,1]$ is the unit interval. Let $c \in \mathbb{R}^+$. $\log c$ denotes the binary logarithm of c. Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}_0^+$. f is of the order of g, in symbols $f \in O(g)$, iff there exist $n_0 \in \mathbb{N}$ and $c^* \in \mathbb{R}_0^+$ such that for all $n \geq n_0$, $f(n) \leq c^* \cdot g(n)$.

We define the size of term of $\mathcal{L}$ $|t| : Term_{\mathcal{L}} \longrightarrow \mathbb{N}$ by recursion on the structure of t:

$$|t| = \begin{cases} 1 & \textit{if } t \in Var_{\mathcal{L}}, \\ 1 + \sum_{i=1}^{\tau} |t_i| & \textit{if } t = f(t_1, \dots, t_\tau). \end{cases}$$

Subsequently, we define the size of order formula of $\mathcal{L}$ $|\phi| : OrdForm_{\mathcal{L}} \longrightarrow \mathbb{N}$ by recursion on the structure of ϕ:

$$|\phi| = \begin{cases} 1 + \sum_{i=1}^{\tau} |t_i| & \textit{if } \phi = p(t_1, \dots, t_\tau) \in Atom_{\mathcal{L}}, \\ 1 + |\phi_1| & \textit{if } \phi = \neg\phi_1, \\ 1 + |\phi_1| + |\phi_2| & \textit{if } \phi = \phi_1 \diamond \phi_2, \\ 2 + |\phi_1| & \textit{if } \phi = Qx\, \phi_1. \end{cases}$$

Let $T \subseteq_{\mathcal{F}} OrdForm_{\mathcal{L}}$. We define the size of T as $|T| = \sum_{\phi \in T} |\phi|$. By $varseq(\phi)$, $vars(varseq(\phi)) \subseteq Var_{\mathcal{L}}$, we denote the sequence of all variables of $\mathcal{L}$ occurring in ϕ which is built up via the left-right preorder traversal of ϕ. For example, $varseq(\exists w\, (\forall x\, p(x,x,z) \vee \exists y\, q(x,y,z))) = w, x, x, x, z, y, x, y, z$ and $|w, x, x, x, z, y, x, y, z| = 9$. A sequence of variables will often be denoted as $\bar{x}$, $\bar{y}$, $\bar{z}$, etc. Let $Q \in \{\forall, \exists\}$ and $\bar{x} = x_1, \dots, x_n$ be a sequence of variables of $\mathcal{L}$. By $Q\bar{x}\, \phi$ we denote $Qx_1 \dots Qx_n\, \phi$.

Gödel logic is interpreted by the standard $\boldsymbol{G}$-algebra augmented by binary operators $\boldsymbol{=}$ and $\boldsymbol{\prec}$ for $=$ and $\prec$, respectively.

$$\boldsymbol{G} = ([0,1], \leq, \vee, \wedge, \Rightarrow, \overline{}, \boldsymbol{=}, \boldsymbol{\prec}, 0, 1)$$

[3]Several simultaneous applications of concatenation will be written without parentheses.

where $\vee \mid \wedge$ denotes the supremum | infimum operator on $[0, 1]$;

$$a \Rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{else}; \end{cases} \qquad \overline{a} = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{else}; \end{cases}$$

$$a \eqcirc b = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{else}; \end{cases} \qquad a \prec b = \begin{cases} 1 & \text{if } a < b, \\ 0 & \text{else}. \end{cases}$$

Recall that $\boldsymbol{G}$ is a complete linearly ordered lattice algebra; $\vee \mid \wedge$ is commutative, associative, idempotent, monotone; $0 \mid 1$ is its neutral element;[4] the residuum operator $\Rightarrow$ of $\wedge$ satisfies the condition of residuation:

$$\text{for all } a, b, c \in \boldsymbol{G}, a \wedge b \leq c \Longleftrightarrow a \leq b \Rightarrow c; \tag{1}$$

Gödel negation $\overline{}$ satisfies the condition:

$$\text{for all } a \in \boldsymbol{G}, \overline{a} = a \Rightarrow 0; \tag{2}$$

the following properties, which will be exploited later, hold:[5]

for all $a, b, c \in \boldsymbol{G}$,

$$a \vee b \wedge c = (a \vee b) \wedge (a \vee c), \qquad \text{(distributivity of } \vee \text{ over } \wedge) \tag{3}$$

$$a \wedge (b \vee c) = a \wedge b \vee a \wedge c, \qquad \text{(distributivity of } \wedge \text{ over } \vee) \tag{4}$$

$$a \Rightarrow (b \vee c) = a \Rightarrow b \vee a \Rightarrow c, \tag{5}$$

$$a \Rightarrow b \wedge c = (a \Rightarrow b) \wedge (a \Rightarrow c), \tag{6}$$

$$(a \vee b) \Rightarrow c = (a \Rightarrow c) \wedge (b \Rightarrow c), \tag{7}$$

$$a \wedge b \Rightarrow c = a \Rightarrow c \vee b \Rightarrow c, \tag{8}$$

$$a \Rightarrow (b \Rightarrow c) = a \wedge b \Rightarrow c, \tag{9}$$

$$((a \Rightarrow b) \Rightarrow b) \Rightarrow b = a \Rightarrow b, \tag{10}$$

$$(a \Rightarrow b) \Rightarrow c = ((a \Rightarrow b) \Rightarrow b) \wedge (b \Rightarrow c) \vee c, \tag{11}$$

$$(a \Rightarrow b) \Rightarrow 0 = ((a \Rightarrow 0) \Rightarrow 0) \wedge (b \Rightarrow 0). \tag{12}$$

An interpretation $\mathcal{I}$ for $\mathcal{L}$ is a triple $\big(\mathcal{U}_\mathcal{I}, \{f^\mathcal{I} \,|\, f \in \mathit{Func}_\mathcal{L}\}, \{p^\mathcal{I} \,|\, p \in \mathit{Pred}_\mathcal{L}\}\big)$ defined as follows: $\mathcal{U}_\mathcal{I} \neq \emptyset$ is the universum of $\mathcal{I}$; every $f \in \mathit{Func}_\mathcal{L}$ is interpreted as a function $f^\mathcal{I} : \mathcal{U}_\mathcal{I}^{ar_\mathcal{L}(f)} \longrightarrow \mathcal{U}_\mathcal{I}$; every $p \in \mathit{Pred}_\mathcal{L}$ is interpreted as a $[0, 1]$-relation $p^\mathcal{I} : \mathcal{U}_\mathcal{I}^{ar_\mathcal{L}(p)} \longrightarrow [0, 1]$. A variable assignment in $\mathcal{I}$ is a mapping $\mathit{Var}_\mathcal{L} \longrightarrow \mathcal{U}_\mathcal{I}$. We denote the set of all variable assignments in $\mathcal{I}$ as $\mathcal{S}_\mathcal{I}$. Let $e \in \mathcal{S}_\mathcal{I}$ and $u \in \mathcal{U}_\mathcal{I}$. A variant $e[x/u] \in \mathcal{S}_\mathcal{I}$ of e with respect to x and u is defined as

$$e[x/u](z) = \begin{cases} u & \text{if } z = x, \\ e(z) & \text{else}. \end{cases}$$

[4]Using the commutativity, associativity, idempotence, monotonicity, neutral element of $\vee \mid \wedge$ will not explicitly be referred to.

[5]We assume a decreasing operator precedence: $\overline{}$, $\wedge$, $\Rightarrow$, $\eqcirc$, $\prec$, $\vee$.

Let $t \in \mathit{Term}_{\mathcal{L}}$, $\bar{x}$ be a sequence of variables of $\mathcal{L}$, $\phi \in \mathit{OrdForm}_{\mathcal{L}}$. In $\mathcal{I}$ with respect to e, we define the value $\|t\|_e^{\mathcal{I}} \in \mathcal{U}_{\mathcal{I}}$ of t by recursion on the structure of t, the value $\|\bar{x}\|_e^{\mathcal{I}} \in \mathcal{U}_{\mathcal{I}}^{|\bar{x}|}$ of $\bar{x}$, the truth value $\|\phi\|_e^{\mathcal{I}} \in [0,1]$ of ϕ by recursion on the structure of ϕ, as follows:

$$
\begin{array}{ll}
t \in \mathit{Var}_{\mathcal{L}}, & \|t\|_e^{\mathcal{I}} = e(t); \\
t = f(t_1, \dots, t_\tau), & \|t\|_e^{\mathcal{I}} = f^{\mathcal{I}}(\|t_1\|_e^{\mathcal{I}}, \dots, \|t_\tau\|_e^{\mathcal{I}}); \\
\bar{x} = x_1, \dots, x_{|\bar{x}|}, & \|\bar{x}\|_e^{\mathcal{I}} = e(x_1), \dots, e(x_{|\bar{x}|}); \\
\phi = \mathit{0}, & \|\phi\|_e^{\mathcal{I}} = 0; \\
\phi = \mathit{1}, & \|\phi\|_e^{\mathcal{I}} = 1; \\
\phi = \bar{c}, & \|\phi\|_e^{\mathcal{I}} = c; \\
\phi = p(t_1, \dots, t_\tau), & \|\phi\|_e^{\mathcal{I}} = p^{\mathcal{I}}(\|t_1\|_e^{\mathcal{I}}, \dots, \|t_\tau\|_e^{\mathcal{I}}); \\
\phi = \neg \phi_1, & \|\phi\|_e^{\mathcal{I}} = \overline{\|\phi_1\|_e^{\mathcal{I}}}; \\
\phi = \phi_1 \wedge \phi_2, & \|\phi\|_e^{\mathcal{I}} = \|\phi_1\|_e^{\mathcal{I}} \bigwedge \|\phi_2\|_e^{\mathcal{I}}; \\
\phi = \phi_1 \vee \phi_2, & \|\phi\|_e^{\mathcal{I}} = \|\phi_1\|_e^{\mathcal{I}} \bigvee \|\phi_2\|_e^{\mathcal{I}}; \\
\phi = \phi_1 \rightarrow \phi_2, & \|\phi\|_e^{\mathcal{I}} = \|\phi_1\|_e^{\mathcal{I}} \Rightarrow \|\phi_2\|_e^{\mathcal{I}}; \\
\phi = \phi_1 \eqcirc \phi_2, & \|\phi\|_e^{\mathcal{I}} = \|\phi_1\|_e^{\mathcal{I}} \eqcirc \|\phi_2\|_e^{\mathcal{I}}; \\
\phi = \phi_1 \prec \phi_2, & \|\phi\|_e^{\mathcal{I}} = \|\phi_1\|_e^{\mathcal{I}} \prec \|\phi_2\|_e^{\mathcal{I}}; \\
\phi = \forall x\, \phi_1, & \|\phi\|_e^{\mathcal{I}} = \bigwedge_{u \in \mathcal{U}_{\mathcal{I}}} \|\phi_1\|_{e[x/u]}^{\mathcal{I}}; \\
\phi = \exists x\, \phi_1, & \|\phi\|_e^{\mathcal{I}} = \bigvee_{u \in \mathcal{U}_{\mathcal{I}}} \|\phi_1\|_{e[x/u]}^{\mathcal{I}}.
\end{array}
$$

Let ϕ be closed. Then, for all $e, e' \in \mathcal{S}_{\mathcal{I}}$, $\|\phi\|_e^{\mathcal{I}} = \|\phi\|_{e'}^{\mathcal{I}}$. Let $e \in \mathcal{S}_{\mathcal{I}} \neq \emptyset$. We denote $\|\phi\|^{\mathcal{I}} = \|\phi\|_e^{\mathcal{I}}$.

Let $\mathcal{L} \mid \mathcal{L}'$ be a first-order language and $\mathcal{I} \mid \mathcal{I}'$ be an interpretation for $\mathcal{L} \mid \mathcal{L}'$. $\mathcal{L}'$ is an expansion of $\mathcal{L}$ iff $\mathit{Func}_{\mathcal{L}'} \supseteq \mathit{Func}_{\mathcal{L}}$ and $\mathit{Pred}_{\mathcal{L}'} \supseteq \mathit{Pred}_{\mathcal{L}}$; on the other side, we say $\mathcal{L}$ is a reduct of $\mathcal{L}'$. $\mathcal{I}'$ is an expansion of $\mathcal{I}$ to $\mathcal{L}'$ iff $\mathcal{L}'$ is an expansion of $\mathcal{L}$, $\mathcal{U}_{\mathcal{I}'} = \mathcal{U}_{\mathcal{I}}$, for all $f \in \mathit{Func}_{\mathcal{L}}$, $f^{\mathcal{I}'} = f^{\mathcal{I}}$, for all $p \in \mathit{Pred}_{\mathcal{L}}$, $p^{\mathcal{I}'} = p^{\mathcal{I}}$; on the other side, we say $\mathcal{I}$ is a reduct of $\mathcal{I}'$ to $\mathcal{L}$, in symbols $\mathcal{I} = \mathcal{I}'|_{\mathcal{L}}$.

A theory of $\mathcal{L}$ is a set of formulae of $\mathcal{L}$. An order theory of $\mathcal{L}$ is a set of order formulae of $\mathcal{L}$. Let $\phi, \phi' \in \mathit{OrdForm}_{\mathcal{L}}$, $T \subseteq \mathit{OrdForm}_{\mathcal{L}}$, $e \in \mathcal{S}_{\mathcal{I}}$. ϕ is true in $\mathcal{I}$ with respect to e, written as $\mathcal{I} \models_e \phi$, iff $\|\phi\|_e^{\mathcal{I}} = 1$. $\mathcal{I}$ is a model of ϕ, in symbols $\mathcal{I} \models \phi$, iff, for all $e \in \mathcal{S}_{\mathcal{I}}$, $\mathcal{I} \models_e \phi$. $\mathcal{I}$ is a model of T, in symbols $\mathcal{I} \models T$, iff, for all $\phi \in T$, $\mathcal{I} \models \phi$. ϕ is a logically valid formula iff, for every interpretation $\mathcal{I}$ for $\mathcal{L}$, $\mathcal{I} \models \phi$. ϕ is equivalent to ϕ', in symbols $\phi \equiv \phi'$, iff, for every interpretation $\mathcal{I}$ for $\mathcal{L}$ and $e \in \mathcal{S}_{\mathcal{I}}$, $\|\phi\|_e^{\mathcal{I}} = \|\phi'\|_e^{\mathcal{I}}$. We denote $\mathit{tcons}(\phi) = \{\mathit{0}, \mathit{1}\} \cup (\mathit{preds}(\phi) \cap \overline{C}_{\mathcal{L}}) \subseteq \mathit{Tcons}_{\mathcal{L}}$ and $\mathit{tcons}(T) = \{\mathit{0}, \mathit{1}\} \cup (\mathit{preds}(T) \cap \overline{C}_{\mathcal{L}}) \subseteq \mathit{Tcons}_{\mathcal{L}}$.

3. Translation to clausal form

In the propositional case [25], we have proposed some translation of a formula to an equivalent *CNF* containing literals of the form either a or $a \rightarrow b$ or $(a \rightarrow b) \rightarrow b$ where a is a propositional atom and b is either a propositional atom or the propositional constant $\mathit{0}$. An output equivalent *CNF* may be of exponential size with respect to the input formula; we had laid no restrictions on use of the distributivity law (3) during translation to conjunctive normal form. To avoid this disadvantage, we have devised translation to *CNF* via interpolation using new atoms, which produces an output *CNF* of linear size at the cost of being only equisatisfiable to the input formula. A similar approach exploiting the renaming subformulae technique can be found in [26, 27, 28, 29, 30]. A *CNF* is further translated to a finite set of order clauses. An order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is either a propositional atom or a propositional constant, $\mathit{0}$, $\mathit{1}$, and $\diamond \in \{\eqcirc, \prec\}$.

We now describe some generalisation of the mentioned translation to the first-order case. At first, we introduce a notion of quantified atom. Let $a \in \mathit{Form}_{\mathcal{L}}$. a is a quantified atom of $\mathcal{L}$ iff $a = Qx\, p(t_0, \dots, t_\tau)$ where $p(t_0, \dots, t_\tau) \in \mathit{Atom}_{\mathcal{L}}$, $x \in \mathit{vars}(p(t_0, \dots, t_\tau))$, either $t_i = x$ or $x \notin \mathit{vars}(t_i)$. $\mathit{QAtom}_{\mathcal{L}} \subseteq \mathit{Form}_{\mathcal{L}}$ denotes the set of all quantified atoms of $\mathcal{L}$. $\mathit{QAtom}_{\mathcal{L}}^{Q} \subseteq \mathit{QAtom}_{\mathcal{L}}$, $Q \in \{\forall, \exists\}$, denotes the set of all quantified atoms of $\mathcal{L}$ of the form $Qx\, a$. Let ε_i, $1 \leq i \leq m$, be either an expression or a set of expressions or a set of sets of expressions of $\mathcal{L}$, in general. By $\mathit{qatoms}(\varepsilon_1, \dots, \varepsilon_m) \subseteq \mathit{QAtom}_{\mathcal{L}}$ we denote the set of all quantified atoms of $\mathcal{L}$ occurring in $\varepsilon_1, \dots, \varepsilon_m$. We denote $\mathit{qatoms}^{Q}(\varepsilon_1, \dots, \varepsilon_m) = \mathit{qatoms}(\varepsilon_1, \dots, \varepsilon_m) \cap \mathit{QAtom}_{\mathcal{L}}^{Q}$, $Q \in \{\forall, \exists\}$. Let $Qx\, p(t_0, \dots, t_\tau) \in \mathit{QAtom}_{\mathcal{L}}$ and $p(t'_0, \dots, t'_\tau) \in \mathit{Atom}_{\mathcal{L}}$. We denote

$$\mathit{boundindset}(Qx\, p(t_0, \dots, t_\tau)) = \{i \mid i \leq \tau, t_i = x\} \neq \emptyset.$$

Let $I = \{i \mid i \leq \tau, x \notin \mathit{vars}(t_i)\}$ and $r_1, \dots, r_k$, $r_i \leq \tau$, $k \leq \tau$, for all $1 \leq i < i' \leq k$, $r_i < r_{i'}$, be a sequence such that $\{r_i \mid 1 \leq i \leq k\} = I$. We denote

$$\mathit{freetermseq}(Qx\, p(t_0, \dots, t_\tau)) = t_{r_1}, \dots, t_{r_k},$$
$$\mathit{freetermseq}(p(t'_0, \dots, t'_\tau)) = t'_0, \dots, t'_\tau.$$

We further introduce conjunctive normal form (*CNF*) in Gödel logic. In contrast to two-valued logic, we have to consider an augmented set of literals appearing in *CNF* formulae. Let $l, \phi \in \mathit{Form}_{\mathcal{L}}$. l is a literal of $\mathcal{L}$ iff either $l = a$ or $l = b \rightarrow c$ or $l = (a \rightarrow d) \rightarrow d$ or $l = a \rightarrow e$ or $l = e \rightarrow a$, $a \in \mathit{Atom}_{\mathcal{L}} - \mathit{Tcons}_{\mathcal{L}}$, $b \in \mathit{Atom}_{\mathcal{L}} - \{\mathit{0}, \mathit{1}\}$, $c \in \mathit{Atom}_{\mathcal{L}} - \{\mathit{1}\}$, $d \in (\mathit{Atom}_{\mathcal{L}} - \mathit{Tcons}_{\mathcal{L}}) \cup \{\mathit{0}\}$, $e \in \mathit{QAtom}_{\mathcal{L}}$, $\{b, c\} \nsubseteq \mathit{Tcons}_{\mathcal{L}}$. The set of all literals of $\mathcal{L}$ is designated as $\mathit{Lit}_{\mathcal{L}} \subseteq \mathit{Form}_{\mathcal{L}}$. ϕ is a conjunctive | disjunctive normal form of $\mathcal{L}$, in symbols *CNF* | *DNF*, iff either $\phi \in \mathit{Tcons}_{\mathcal{L}}$ or $\phi = \bigwedge_{i \leq n} \bigvee_{j \leq m_i} l_j^i$ | $\phi = \bigvee_{i \leq n} \bigwedge_{j \leq m_i} l_j^i$, $l_j^i \in \mathit{Lit}_{\mathcal{L}}$.[6] Let $D = l_1 \vee \dots \vee l_n \in \mathit{Form}_{\mathcal{L}}$, $l_i \in \mathit{Lit}_{\mathcal{L}}$. We denote $\mathit{lits}(D) = \{l_1, \dots, l_n\} \subseteq \mathit{Lit}_{\mathcal{L}}$. D is a factor iff, for all $1 \leq i < i' \leq n$, $l_i \neq l_{i'}$.

We finally introduce order clauses in Gödel logic. Let $l \in \mathit{OrdForm}_{\mathcal{L}}$. l is an order literal of $\mathcal{L}$ iff $l = \varepsilon_1 \diamond \varepsilon_2$, $\varepsilon_i \in \mathit{Atom}_{\mathcal{L}} \cup \mathit{QAtom}_{\mathcal{L}}$, $\diamond \in \{\eqcirc, \prec\}$. The set of all order literals

[6]Associativity of $\wedge$, $\vee$ is not explicitly referred to, and hence, $\bigwedge_{i \leq n} \phi_i, \bigvee_{i \leq n} \phi_i \in \mathit{OrdForm}_{\mathcal{L}}$ are written without parentheses.

of $\mathcal{L}$ is designated as $\mathit{OrdLit}_{\mathcal{L}} \subseteq \mathit{OrdForm}_{\mathcal{L}}$. An order clause of $\mathcal{L}$ is a finite set of order literals of $\mathcal{L}$; since $\eqcirc$ is commutative, for all $\varepsilon_1 \eqcirc \varepsilon_2 \in \mathit{OrdLit}_{\mathcal{L}}$, we identify $\varepsilon_1 \eqcirc \varepsilon_2$ and $\varepsilon_2 \eqcirc \varepsilon_1 \in \mathit{OrdLit}_{\mathcal{L}}$ with respect to order clauses. An order clause $\{l_1, \dots, l_n\}$ is written in the form $l_1 \vee \cdots \vee l_n$. The order clause $\emptyset$ is called the empty order clause and denoted as $\square$. An order clause $\{l\}$ is called a unit order clause and denoted as l; if it does not cause the ambiguity with the denotation of the single order literal l in given context. We designate the set of all order clauses of $\mathcal{L}$ as $\mathit{OrdCl}_{\mathcal{L}}$.[7] Let $l, l_0, \dots, l_n \in \mathit{OrdLit}_{\mathcal{L}}$ and $C, C' \in \mathit{OrdCl}_{\mathcal{L}}$. We define the size of C as $|C| = \sum_{l \in C} |l|$. By $l \vee C$ we denote $\{l\} \cup C$ where $l \notin C$. Analogously, by $l_0 \vee \cdots \vee l_n \vee C$ we denote $\{l_0\} \cup \cdots \cup \{l_n\} \cup C$ where, for all $i, i' \leq n$, $i \neq i'$, $l_i \notin C$ and $l_i \neq l_{i'}$. By $C \vee C'$ we denote $C \cup C'$. C is a subclause of C', in symbols $C \sqsubseteq C'$, iff $C \subseteq C'$. An order clausal theory of $\mathcal{L}$ is a set of order clauses of $\mathcal{L}$. A unit order clausal theory is a set of unit order clauses.

Let $\phi, \phi' \in \mathit{OrdForm}_{\mathcal{L}}$, $T, T' \subseteq \mathit{OrdForm}_{\mathcal{L}}$, $S, S' \subseteq \mathit{OrdCl}_{\mathcal{L}}$, $\mathcal{I}$ be an interpretation for $\mathcal{L}$, $e \in \mathcal{S}_{\mathcal{I}}$. Note that $\mathcal{I} \models_e l$ if and only if either $l = \varepsilon_1 \eqcirc \varepsilon_2$, $\|\varepsilon_1 \eqcirc \varepsilon_2\|^{\mathcal{I}}_e = 1$, $\|\varepsilon_1\|^{\mathcal{I}}_e = \|\varepsilon_2\|^{\mathcal{I}}_e$; or $l = \varepsilon_1 \prec \varepsilon_2$, $\|\varepsilon_1 \prec \varepsilon_2\|^{\mathcal{I}}_e = 1$, $\|\varepsilon_1\|^{\mathcal{I}}_e < \|\varepsilon_2\|^{\mathcal{I}}_e$. C is true in $\mathcal{I}$ with respect to e, written as $\mathcal{I} \models_e C$, iff there exists $l^* \in C$ such that $\mathcal{I} \models_e l^*$. $\mathcal{I}$ is a model of C, in symbols $\mathcal{I} \models C$, iff, for all $e \in \mathcal{S}_{\mathcal{I}}$, $\mathcal{I} \models_e C$. $\mathcal{I}$ is a model of S, in symbols $\mathcal{I} \models S$, iff, for all $C \in S$, $\mathcal{I} \models C$. $\phi' \mid T' \mid C' \mid S'$ is a logical consequence of $\phi \mid T \mid C \mid S$, in symbols $\phi \,|\, T \,|\, C \,|\, S \models \phi' \,|\, T' \,|\, C' \,|\, S'$, iff, for every model $\mathcal{I}$ of $\phi \mid T \mid C \mid S$ for $\mathcal{L}$, $\mathcal{I} \models \phi' \,|\, T' \,|\, C' \,|\, S'$. $\phi \mid T \mid C \mid S$ is satisfiable iff there exists a model of $\phi \mid T \mid C \mid S$ for $\mathcal{L}$. Note that both $\square$ and $\square \in S$ are unsatisfiable. $\phi \mid T \mid C \mid S$ is equisatisfiable to $\phi' \mid T' \mid C' \mid S'$ iff $\phi \mid T \mid C \mid S$ is satisfiable if and only if $\phi' \mid T' \mid C' \mid S'$ is satisfiable. We denote $\mathit{tcons}(S) = \{\mathit{0}, \mathit{1}\} \cup (\mathit{preds}(S) \cap \overline{C}_{\mathcal{L}}) \subseteq \mathit{Tcons}_{\mathcal{L}}$. Let $S \subseteq_{\mathcal{F}} \mathit{OrdCl}_{\mathcal{L}}$. We define the size of S as $|S| = \sum_{C \in S} |C|$. l is a simplified order literal of $\mathcal{L}$ iff $l = \varepsilon_1 \diamond \varepsilon_2$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq \mathit{Tcons}_{\mathcal{L}}$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq \mathit{QAtom}_{\mathcal{L}}$. The set of all simplified order literals of $\mathcal{L}$ is designated as $\mathit{SimOrdLit}_{\mathcal{L}} \subseteq \mathit{OrdLit}_{\mathcal{L}}$. We denote $\mathit{SimOrdCl}_{\mathcal{L}} = \{C \,|\, C \in \mathit{OrdCl}_{\mathcal{L}}, C \subseteq \mathit{SimOrdLit}_{\mathcal{L}}\} \subseteq \mathit{OrdCl}_{\mathcal{L}}$. Let $\tilde{f}_0 \notin \mathit{Func}_{\mathcal{L}}$; $\tilde{f}_0$ is a new function symbol. Let $\mathbb{I} = \mathbb{N} \times \mathbb{N}$; $\mathbb{I}$ is an infinite countable set of indices. Let $\tilde{\mathbb{P}} = \{\tilde{p}_{\mathbb{i}} \,|\, \mathbb{i} \in \mathbb{I}\}$ such that $\tilde{\mathbb{P}} \cap \mathit{Pred}_{\mathcal{L}} = \emptyset$; $\tilde{\mathbb{P}}$ is an infinite countable set of new predicate symbols.

3.1. A computational point of view

From a computational point of view, the worst case time and space complexity will be estimated using the logarithmic cost measurement. Let $n_s \in \mathbb{N}$ and E, $\bar{E}$ be either a term or an order formula or an order clause or a finite order theory or a finite order clausal theory. E can be represented by a tree-like data structure $\mathcal{D}(E)$ having nodes data records. A data record of $\mathcal{D}(E)$ represents either a subexpression or a suitable subset of E, depending on the form of E. In Table 1, we introduce all possible forms of data record. Concerning Table 1, variable, function, predicate symbols occurring in E can be indexed by indices of the form $(n_s, j) \in \mathbb{I}$. The value of an index may be written in the binary number representation. We shall assume that $p_{index} \in \mathit{Tcons}$ if and only if $index$ starts with the digit 1 in the binary number representation. The number of indices occurring in $\mathcal{D}(E)$ is in $O(|E|)$ and the length of an index in $O(\log(1+n_s)+\log(1+|E|))$. Concerning Table 1, a data record in $\mathcal{D}(E)$ may contain pointers which reference other data records

[7] If the first-order language in question is not explicitly designated, we shall write *OrdCl*, without index.

Table 1: Forms of data record

Expression	Data record
$x_{index} \in Var$	$(x, index)$
$f_{index}(t_1, \dots, t_\tau) \in Term$	$(f, index, pointer_{t_1}, \dots, pointer_{t_\tau})$
$p_{index}(t_1, \dots, t_\tau) \in Atom$	$(p, index, pointer_{t_1}, \dots, pointer_{t_\tau})$
$\neg\phi_1 \in OrdForm$	$(\neg, pointer_{\phi_1})$
$\phi_1 \wedge \phi_2 \in OrdForm$	$(\wedge, pointer_{\phi_1}, pointer_{\phi_2})$
$\phi_1 \vee \phi_2 \in OrdForm$	$(\vee, pointer_{\phi_1}, pointer_{\phi_2})$
$\phi_1 \rightarrow \phi_2 \in OrdForm$	$(\rightarrow, pointer_{\phi_1}, pointer_{\phi_2})$
$\phi_1 \eqcirc \phi_2 \in OrdForm$	$(\eqcirc, pointer_{\phi_1}, pointer_{\phi_2})$
$\phi_1 \prec \phi_2 \in OrdForm$	$(\prec, pointer_{\phi_1}, pointer_{\phi_2})$
$\forall x_{index}\, \phi_1 \in OrdForm$	$(\forall, pointer_{x_{index}}, pointer_{\phi_1})$
$\exists x_{index}\, \phi_1 \in OrdForm$	$(\exists, pointer_{x_{index}}, pointer_{\phi_1})$
$\square \in OrdCl$	$(\square)$
$l \vee C \in OrdCl$	$(\mid, pointer_l, pointer_C)$
$\emptyset \subseteq OrdForm, OrdCl$	$(\emptyset)$
$\{\phi\} \cup T \subseteq_{\mathcal{F}} OrdForm, \phi \notin T$	$(\&, pointer_\phi, pointer_T)$
$\{C\} \cup S \subseteq_{\mathcal{F}} OrdCl, C \notin S$	$(\&, pointer_C, pointer_S)$

$pointer_{\bar{E}}$ denotes a pointer which references $\mathcal{D}(\bar{E})$.

in $\mathcal{D}(E)$. The value of a pointer may be written in the binary number representation. Since $\mathcal{D}(E)$ is a tree-like data structure, no two different pointers reference the same data record. The number of pointers occurring in $\mathcal{D}(E)$ is in $O(|E|)$ and the length of a pointer in $O(\log(1 + n_s \cdot |E|)) = O(\log(1 + n_s) + \log(1 + |E|))$. For every expression E, there exists a tree-like data structure $\mathcal{D}(E)$; the proof is by induction on the structure of E using Table 1. The time and space complexity of an elementary operation on an index | a pointer, is of the order of its length, in $O(\log(1+n_s)+\log(1+|E|))$. Concerning Table 1, a data record consists of a field of length in $O(1)$ (of constant length with respect to input) and of a finite number of indices, pointers. The time and space complexity of an elementary operation on a field, is of the order of its length, in $O(1)$. $\mathcal{D}(E)$ consists of a finite number of data records. Therefore, it suffices to consider only elementary operations on fields, indices, pointers in data records of $\mathcal{D}(E)$. For simplicity, we shall call them elementary operations on $\mathcal{D}(E)$. We get the time and space complexity of an elementary operation on $\mathcal{D}(E)$, is in $O(\log(1+n_s)+\log(1+|E|))$. An upper bound on the size of $\mathcal{D}(E)$ can be calculated as the sum of upper bounds on the total lengths of fields, indices, pointers in data records of $\mathcal{D}(E)$. The total length of fields in data records of $\mathcal{D}(E)$, is in $O(|E|)$. The total length of indices | pointers in data records of $\mathcal{D}(E)$, is in $O(|E| \cdot (\log(1+n_s)+\log(1+|E|)))$. We obtain the size of $\mathcal{D}(E)$ is in $O(|E| \cdot (\log(1+n_s)+\log(1+|E|)))$.

Let $n_s \in \mathbb{N}$ and $\mathcal{A}$ be an algorithm with inputs E_0, E_1 which uses only E_j, $j = 0, \dots, q$, represented by a tree-like data structure $\mathcal{D}(E_j)$ such that E_j is either a term or an order formula or an order clause or a finite order theory or a finite order clausal theory; $q \geq 1$ is a constant with respect to input; there exists a constant $r \geq 1$ with respect to input satisfying, for all $j \leq q$, $|E_j| \in O(|E_0|^r + |E_1|^r)$. $\#\mathcal{O}_{\mathcal{A}}(E_0, E_1) \geq 1$ denotes the number of all elementary operations executed by $\mathcal{A}$;[8] we assume that $\mathcal{A}$ executes at least one elementary operation. The number of indices | pointers | fields occurring in $\mathcal{D}(E_j)$, $j = 0, \dots, q$, is in $O(|E_0|^r + |E_1|^r)$ and the length of an index | a pointer in $O(\log(1+n_s)+\log(1+|E_0|+|E_1|))$. The time and space complexity of an elementary operation on $\mathcal{D}(E_j)$ executed by $\mathcal{A}$, is in $O(\log(1+n_s)+\log(1+|E_0|+|E_1|))$. The size of $\mathcal{D}(E_j)$, $j = 0, \dots, q$, *area of data records*, is in $O((|E_0|^r+|E_1|^r) \cdot (\log(1+n_s)+\log(1+|E_0|+|E_1|)))$.

$\mathcal{A}$ also uses several auxiliary data structures: *stack*, *index generator*, *addressing unit*. *stack* consists of a finite number of frames. A frame is of the form (*field*, *pointer*) where *field* is of length in $O(1)$ and *pointer* is a copy of that occurring in $\mathcal{D}(E_j)$ of length in $O(\log(1+n_s)+\log(1+|E_0|+|E_1|))$. The length of a frame of *stack* is in $O(\log(1+n_s)+\log(1+|E_0|+|E_1|))$. The time and space complexity of an elementary operation on *stack* executed by $\mathcal{A}$, is of the order of the length of a frame of *stack*, in $O(\log(1+n_s)+\log(1+|E_0|+|E_1|))$. The size of *stack* is in $O(\#\mathcal{O}_{\mathcal{A}}(E_0, E_1) \cdot (\log(1+n_s)+\log(1+|E_0|+|E_1|)))$.

index generator serves for generating new predicate symbols of the form $\tilde{p}_{\mathbb{i}} \in \tilde{\mathbb{P}}$. It may consist of a constant number of indices, of length in $O(\log(1+n_s)+\log(1+|E_0|+|E_1|))$. The size of *index generator* is in $O(\log(1+n_s)+\log(1+|E_0|+|E_1|))$. The time and space complexity of an elementary operation on *index generator* executed by $\mathcal{A}$, is of the order of its size, in $O(\log(1+n_s)+\log(1+|E_0|+|E_1|))$.

addressing unit serves for addressing in the memory used by $\mathcal{A}$. It may consist of a constant number of address registers. The memory can be arranged as follows:

$$\big(\textit{stack} \qquad \textit{index generator} \qquad \textit{addressing unit} \qquad \textit{area of data records}\big).$$

An address register has to be able to reference an arbitrary address in the memory. The total size of *stack*, *index generator*, *area of data records* is in

$$\begin{aligned} &O(\#\mathcal{O}_{\mathcal{A}}(E_0, E_1) \cdot (\log(1+n_s)+\log(1+|E_0|+|E_1|)) + \\ &\quad \log(1+n_s)+\log(1+|E_0|+|E_1|) + \\ &\quad (|E_0|^r+|E_1|^r) \cdot (\log(1+n_s)+\log(1+|E_0|+|E_1|))) = \\ &O((\#\mathcal{O}_{\mathcal{A}}(E_0, E_1)+|E_0|^r+|E_1|^r) \cdot (\log(1+n_s)+\log(1+|E_0|+|E_1|))). \end{aligned}$$

[8] If the algorithm in question is not explicitly designated, we shall only write $\#\mathcal{O}(E_0, E_1)$.

The length of an address register is in

$$
\begin{aligned}
&O(\log((\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|^r+|E_1|^r)\cdot(1+\log(1+n_s)+\log(1+|E_0|+|E_1|))))= \\
&O(\log(\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|+|E_1|)+\log(1+\log(1+n_s)+\log(1+|E_0|+|E_1|)))= \\
&O(\log(\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|+|E_1|)+\log\log(2+n_s)+\log\log(2+|E_0|+|E_1|))= \\
&O(\log(\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|+|E_1|)+\log\log(2+n_s)).
\end{aligned}
$$

The size of *addressing unit* is in $O(\log(\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|+|E_1|)+\log\log(2+n_s))$. The time and space complexity of an elementary operation on *addressing unit* executed by $\mathcal{A}$, is of the order of its size, in $O(\log(\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|+|E_1|)+\log\log(2+n_s))$. The size of the memory is in

$$
\begin{aligned}
&O((\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|^r+|E_1|^r)\cdot(\log(1+n_s)+\log(1+|E_0|+|E_1|))+ \\
&\quad \log(\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|+|E_1|)+\log\log(2+n_s))= \\
&O((\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|^r+|E_1|^r)\cdot(\log(1+n_s)+\log(1+|E_0|+|E_1|))),
\end{aligned}
$$

as the total size of *stack*, *index generator*, *area of data records*.

We may assume that $\mathcal{A}$ executes only elementary operations on *stack*, *index generator*, *addressing unit*, *area of data records*. We get the time and space complexity of an elementary operation executed by $\mathcal{A}$, is in

$$
\begin{aligned}
&O(\mathit{max}(\log(1+n_s)+\log(1+|E_0|+|E_1|), \\
&\qquad \log(\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|+|E_1|)+\log\log(2+n_s)))= \\
&O(\log(1+n_s)+\log(1+|E_0|+|E_1|)+ \\
&\quad \log(\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|+|E_1|)+\log\log(2+n_s))= \\
&O(\log(1+n_s)+\log(\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|+|E_1|)).
\end{aligned}
$$

We conclude that

the time complexity of $\mathcal{A}$ on E_0 and E_1, is in $O(\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)\cdot(\log(1+n_s)+\log(\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|+|E_1|)))$; (13)

the space complexity of $\mathcal{A}$ on E_0 and E_1, is in $O((\#\mathcal{O}_{\mathcal{A}}(E_0,E_1)+|E_0|^r+|E_1|^r)\cdot(\log(1+n_s)+\log(1+|E_0|+|E_1|)))$. (14)

3.2. Substitutions

We assume the reader to be familiar with the standard notions and notation of substitutions. Cf. Appendix, Subsection 7.1, 41. We introduce a few definitions and denotations; some of them are slightly different from the standard ones, but found to be more convenient. Let $X=\{x_i\,|\,1\leq i\leq n\}\subseteq \mathit{Var}_{\mathcal{L}}$. A substitution ϑ of $\mathcal{L}$ is a mapping $\vartheta: X\longrightarrow \mathit{Term}_{\mathcal{L}}$. ϑ may be written in the form $x_1/\vartheta(x_1),\dots,x_n/\vartheta(x_n)$. We denote $\mathit{dom}(\vartheta)=X\subseteq_{\mathcal{F}}\mathit{Var}_{\mathcal{L}}$ and $\mathit{range}(\vartheta)=\bigcup_{x\in X}\mathit{vars}(\vartheta(x))\subseteq_{\mathcal{F}}\mathit{Var}_{\mathcal{L}}$. The set of all substitutions of $\mathcal{L}$ is designated as $\mathit{Subst}_{\mathcal{L}}$. Let $Qx\,a\in \mathit{QAtom}_{\mathcal{L}}$. ϑ is applicable to $Qx\,a$ iff $\mathit{dom}(\vartheta)\supseteq \mathit{freevars}(Qx\,a)$ and $x\notin \mathit{range}(\vartheta|_{\mathit{freevars}(Qx\,a)})$. We define the application of ϑ to $Qx\,a$ as $(Qx\,a)\vartheta=Qx\,a(\vartheta|_{\mathit{freevars}(Qx\,a)}\cup x/x)\in \mathit{QAtom}_{\mathcal{L}}$.

Let ε and ε' be expressions. ε' is an instance of ε of $\mathcal{L}$ iff there exists $\vartheta^* \in \mathit{Subst}_{\mathcal{L}}$ such that $\varepsilon' = \varepsilon\vartheta^*$. ε' is a variant of ε of $\mathcal{L}$ iff there exists a variable renaming $\rho^* \in \mathit{Subst}_{\mathcal{L}}$ such that $\varepsilon' = \varepsilon\rho^*$. Let $C \in \mathit{OrdCl}_{\mathcal{L}}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L}}$. C is an instance | a variant of S of $\mathcal{L}$ iff there exists $C^* \in S$ such that C is an instance | a variant of C^* of $\mathcal{L}$. We denote $\mathit{Inst}_{\mathcal{L}}(S) = \{C \,|\, C \text{ is an instance of } S \text{ of } \mathcal{L}\} \subseteq \mathit{OrdCl}_{\mathcal{L}}$ and $\mathit{Vrnt}_{\mathcal{L}}(S) = \{C \,|\, C \text{ is a variant of } S \text{ of } \mathcal{L}\} \subseteq \mathit{OrdCl}_{\mathcal{L}}$.

Let E be a set of expressions. ϑ is a unifier of $\mathcal{L}$ for E iff $E\vartheta$ is a singleton set. Let $\theta \in \mathit{Subst}_{\mathcal{L}}$. θ is a most general unifier of $\mathcal{L}$ for E iff θ is a unifier of $\mathcal{L}$ for E, and for every unifier ϑ of $\mathcal{L}$ for E, there exists $\gamma^* \in \mathit{Subst}_{\mathcal{L}}$ such that $\vartheta|_{\mathit{freevars}(E)} = \theta|_{\mathit{freevars}(E)} \circ \gamma^*$. By $\mathit{mgu}_{\mathcal{L}}(E) \subseteq \mathit{Subst}_{\mathcal{L}}$ we denote the set of all most general unifiers of $\mathcal{L}$ for E. Let $\overline{E} = E_0, \dots, E_n$, $E_i \subseteq \mathsf{A}_i$, either $\mathsf{A}_i = \mathit{Term}_{\mathcal{L}}$ or $\mathsf{A}_i = \mathit{Atom}_{\mathcal{L}}$ or $\mathsf{A}_i = \mathit{QAtom}_{\mathcal{L}}$ or $\mathsf{A}_i = \mathit{OrdLit}_{\mathcal{L}}$. ϑ is a unifier of $\mathcal{L}$ for $\overline{E}$ iff, for all $i \leq n$, ϑ is a unifier of $\mathcal{L}$ for E_i. θ is a most general unifier of $\mathcal{L}$ for $\overline{E}$ iff θ is a unifier of $\mathcal{L}$ for $\overline{E}$, and for every unifier ϑ of $\mathcal{L}$ for $\overline{E}$, there exists $\gamma^* \in \mathit{Subst}_{\mathcal{L}}$ such that $\vartheta|_{\mathit{freevars}(\overline{E})} = \theta|_{\mathit{freevars}(\overline{E})} \circ \gamma^*$. By $\mathit{mgu}_{\mathcal{L}}(\overline{E}) \subseteq \mathit{Subst}_{\mathcal{L}}$ we denote the set of all most general unifiers of $\mathcal{L}$ for $\overline{E}$.

Theorem 1 (Unification Theorem). *Let $\overline{E} = E_0, \dots, E_n$, either $E_i \subseteq_{\mathcal{F}} \mathit{Term}_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} \mathit{Atom}_{\mathcal{L}}$. If there exists a unifier of $\mathcal{L}$ for $\overline{E}$, then there exists $\theta^* \in \mathit{mgu}_{\mathcal{L}}(\overline{E})$ such that $\mathit{range}(\theta^*|_{\mathit{vars}(\overline{E})}) \subseteq \mathit{vars}(\overline{E})$.*

Proof. By induction on $\|\mathit{vars}(\overline{E})\|$; a modification of the proof of Theorem 2.3 (Unification Theorem) in [31], Section 2.4, pp. 5–6. $\square$

Theorem 2 (Extended Unification Theorem). *Let $\overline{E} = E_0, \dots, E_n$, either $E_i \subseteq_{\mathcal{F}} \mathit{Term}_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} \mathit{Atom}_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} \mathit{QAtom}_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} \mathit{OrdLit}_{\mathcal{L}}$, and $\mathit{boundvars}(\overline{E}) \subseteq V \subseteq_{\mathcal{F}} \mathit{Var}_{\mathcal{L}}$. If there exists a unifier of $\mathcal{L}$ for $\overline{E}$, then there exists $\theta^* \in \mathit{mgu}_{\mathcal{L}}(\overline{E})$ such that $\mathit{range}(\theta^*|_{\mathit{freevars}(\overline{E})}) \cap V = \emptyset$.*

Proof. A straightforward consequence of Theorem 1. $\square$

3.3. A formal treatment

Translation of a formula or a theory to *CNF* and clausal form, is based on the following lemma:

Lemma 3. *Let $n_\phi, n_0 \in \mathbb{N}$, $\phi \in \mathit{Form}_{\mathcal{L}}$, $T \subseteq \mathit{Form}_{\mathcal{L}}$.*

(I) *There exist either $J_\phi = \emptyset$ or $J_\phi = \{(n_\phi, j) \,|\, j \leq n_{J_\phi}\}$, $J_\phi \subseteq \{(n_\phi, j) \,|\, j \in \mathbb{N}\}$, a CNF $\psi \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}}$, $S_\phi \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}}$ such that*

(a) $\|J_\phi\| \leq 2 \cdot |\phi|$;

(b) *either $J_\phi = \emptyset$, $S_\phi = \{\square\}$ or $J_\phi = S_\phi = \emptyset$ or $J_\phi \neq \emptyset$, $\square \notin S_\phi \neq \emptyset$;*

(c) *there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models \phi$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}$ and $\mathfrak{A}' \models \psi$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;*

(d) *there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models \phi$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}$ and $\mathfrak{A}' \models S_\phi$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;*

(e) *$|\psi| \in O(|\phi|^2)$; the number of all elementary operations of the translation of ϕ to ψ, is in $O(|\phi|^2)$; the time and space complexity of the translation of ϕ to ψ, is in $O(|\phi|^2 \cdot (\log(1 + n_\phi) + \log|\phi|))$;*

(f) $|S_\phi| \in O(|\phi|^2)$; *the number of all elementary operations of the translation of* ϕ *to* S_ϕ, *is in* $O(|\phi|^2)$; *the time and space complexity of the translation of* ϕ *to* S_ϕ, *is in* $O(|\phi|^2 \cdot (\log(1+n_\phi) + \log|\phi|))$;

(g) *if* $\psi \not\in Tcons_{\mathcal{L}}$, *then* $\psi = \bigwedge_{i \leq n_\psi} D_i$, D_i *is a factor,* $J_\phi \neq \emptyset$, *for all* $i \leq n_\psi$, $\emptyset \neq preds(D_i) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}$, *for all* $i < i' \leq n_\psi$, $lits(D_i) \neq lits(D_{i'})$;

(h) *if* $S_\phi \neq \emptyset, \{\Box\}$, *then* $J_\phi \neq \emptyset$, *for all* $C \in S_\phi$, $\emptyset \neq preds(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}$;

(i) *for all* $a \in qatoms(\psi)$, *there exists* $\mathbb{j}^* \in J_\phi$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;

(j) *for all* $\mathbb{j} \in J_\phi$, *there exists a sequence* $\bar{x}$ *of variables of* $\mathcal{L}$ *and* $\tilde{p}_{\mathbb{j}}(\bar{x}) \in atoms(\psi)$ *satisfying, for all* $a \in atoms(\psi)$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; *if there exists* $a^* \in qatoms(\psi)$ *and* $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, *then there exists* $Qx\,\tilde{p}_{\mathbb{j}}(\bar{x}) \in qatoms(\psi)$ *satisfying, for all* $a \in qatoms(\psi)$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\,\tilde{p}_{\mathbb{j}}(\bar{x})$;

(k) *for all* $a \in qatoms(S_\phi)$, *there exists* $\mathbb{j}^* \in J_\phi$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;

(l) *for all* $\mathbb{j} \in J_\phi$, *there exists a sequence* $\bar{x}$ *of variables of* $\mathcal{L}$ *and* $\tilde{p}_{\mathbb{j}}(\bar{x}) \in atoms(S_\phi)$ *satisfying, for all* $a \in atoms(S_\phi)$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; *if there exists* $a^* \in qatoms(S_\phi)$ *and* $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, *then there exists* $Qx\,\tilde{p}_{\mathbb{j}}(\bar{x}) \in qatoms(S_\phi)$ *satisfying, for all* $a \in qatoms(S_\phi)$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\,\tilde{p}_{\mathbb{j}}(\bar{x})$;

(m) $tcons(\psi) = tcons(S_\phi) \subseteq tcons(\phi)$.

(II) *There exist* $J_T \subseteq \{(i,j) \,|\, i \geq n_0\}$ *and* $S_T \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$ *such that*

(a) *either* $J_T = \emptyset$, $S_T = \{\Box\}$ *or* $J_T = S_T = \emptyset$ *or* $J_T \neq \emptyset$, $\Box \not\in S_T \neq \emptyset$;

(b) *there exists an interpretation* $\mathfrak{A}$ *for* $\mathcal{L}$ *and* $\mathfrak{A} \models T$ *if and only if there exists an interpretation* $\mathfrak{A}'$ *for* $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}$ *and* $\mathfrak{A}' \models S_T$, *satisfying* $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;

(c) *if* $T \subseteq_{\mathcal{F}} Form_{\mathcal{L}}$, *then* $J_T \subseteq_{\mathcal{F}} \{(i,j) \,|\, i \geq n_0\}$, $\|J_T\| \leq 2 \cdot |T|$, $S_T \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$, $|S_T| \in O(|T|^2)$; *the number of all elementary operations of the translation of* T *to* S_T, *is in* $O(|T|^2)$; *the time and space complexity of the translation of* T *to* S_T, *is in* $O(|T|^2 \cdot \log(1 + n_0 + |T|))$;

(d) *if* $S_T \neq \emptyset, \{\Box\}$, *then* $J_T \neq \emptyset$, *for all* $C \in S_T$, $\emptyset \neq preds(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}$;

(e) *for all* $a \in qatoms(S_T)$, *there exists* $\mathbb{j}^* \in J_T$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;

(f) *for all* $\mathbb{j} \in J_T$, *there exists a sequence* $\bar{x}$ *of variables of* $\mathcal{L}$ *and* $\tilde{p}_{\mathbb{j}}(\bar{x}) \in atoms(S_T)$ *satisfying, for all* $a \in atoms(S_T)$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; *if there exists* $a^* \in qatoms(S_T)$ *and* $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, *then there exists* $Qx\,\tilde{p}_{\mathbb{j}}(\bar{x}) \in qatoms(S_T)$ *satisfying, for all* $a \in qatoms(S_T)$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\,\tilde{p}_{\mathbb{j}}(\bar{x})$;

(g) $tcons(S_T) \subseteq tcons(T)$.

Proof. Technical, using interpolation. It is straightforward to prove the following statements:

Let $n_\theta \in \mathbb{N}$ and $\theta \in \mathit{Form}_\mathcal{L}$. There exists $\theta' \in \mathit{Form}_\mathcal{L}$ such that (15)

(a) $\theta' \equiv \theta$;

(b) $|\theta'| \leq 2 \cdot |\theta|$; θ' can be built up from θ via a postorder traversal of θ with $\#\mathcal{O}(\theta) \in O(|\theta|)$ and the time, space complexity in $O(|\theta| \cdot (\log(1 + n_\theta) + \log |\theta|))$;

(c) θ' does not contain $\neg$;

(d) $\theta' \in \mathit{Tcons}_\mathcal{L}$; or $\mathit{1}$ is not a subformula of θ'; for every subformula of θ' of the form $\varepsilon_1 \diamond \varepsilon_2$, $\diamond \in \{\wedge, \vee\}$, $\varepsilon_i \neq \mathit{0}, \mathit{1}$, $\{\varepsilon_1, \varepsilon_2\} \nsubseteq \mathit{Tcons}_\mathcal{L}$; for every subformula of θ' of the form $\varepsilon_1 \rightarrow \varepsilon_2$, $\varepsilon_1 \neq \mathit{0}, \mathit{1}$, $\varepsilon_2 \neq \mathit{1}$, $\{\varepsilon_1, \varepsilon_2\} \nsubseteq \mathit{Tcons}_\mathcal{L}$; for every subformula of θ' of the form $Qx\, \varepsilon_1$, $Q \in \{\forall, \exists\}$, $\varepsilon_1 \notin \mathit{Tcons}_\mathcal{L}$;

(e) $\mathit{tcons}(\theta') \subseteq \mathit{tcons}(\theta)$.

The proof is by induction on the structure of θ.

Let $l \in \mathit{Lit}_\mathcal{L}$. There exists $C \in \mathit{SimOrdCl}_\mathcal{L}$ such that (16)

(a) for every interpretation $\mathfrak{A}$ for $\mathcal{L}$, for all $e \in \mathcal{S}_\mathfrak{A}$, $\mathfrak{A} \models_e l$ if and only if $\mathfrak{A} \models_e C$;

(b) $|C| \leq 3 \cdot |l|$, C can be built up from l with $\#\mathcal{O}(l) \in O(|l|)$.

In Table 2, for every form of l, C is assigned so that for every interpretation $\mathfrak{A}$ for $\mathcal{L}$, for all $e \in \mathcal{S}_\mathfrak{A}$, $\mathfrak{A} \models_e l$ if and only if $\mathfrak{A} \models_e C$.

Let $n_\theta \in \mathbb{N}$, $\theta \in \mathit{Form}_\mathcal{L} - \{\mathit{0}, \mathit{1}\}$, (15c,d) hold for θ; $\bar{x}$ be a sequence of variables, $\mathit{vars}(\theta) \subseteq \mathit{vars}(\bar{x}) \subseteq \mathit{Var}_\mathcal{L}$; $\mathbb{i} = (n_\theta, j_\mathbb{i}) \in \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$, $\tilde{p}_\mathbb{i} \in \tilde{\mathbb{P}}$, $\mathit{ar}(\tilde{p}_\mathbb{i}) = |\bar{x}|$. There exist $J = \{(n_\theta, j) \,|\, j_\mathbb{i} + 1 \leq j \leq n_J\} \subseteq \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$, $j_\mathbb{i} \leq n_J$, $\mathbb{i} \notin J$, a CNF $\psi^s \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}}$, $S^s \subseteq_\mathcal{F} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}}$, $s = +, -$, such that for both s, (17)

(a) $\|J\| \leq |\theta| - 1$;

(b) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}$ and $\mathfrak{A} \models \tilde{p}_\mathbb{i}(\bar{x}) \rightarrow \theta \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models \psi^+$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$;

(c) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}$ and $\mathfrak{A} \models \theta \rightarrow \tilde{p}_\mathbb{i}(\bar{x}) \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models \psi^-$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$;

(d) for every interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}$, $\mathfrak{A} \models \psi^s$ if and only if $\mathfrak{A} \models S^s$;

(e) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}$ and $\mathfrak{A} \models \tilde{p}_\mathbb{i}(\bar{x}) \rightarrow \theta \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models S^+$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$;

Table 2: Translation of l to C

Case	l	C	$\|l\|$	$\|C\|$
1	a	$a \eqcirc \mathit{1}$	$\|a\|$	$\|a\| + 2 \leq 3 \cdot \|l\|$
2	$a \rightarrow \mathit{0}$	$a \eqcirc \mathit{0}$	$\|a\| + 2$	$\|a\| + 2 \leq 3 \cdot \|l\|$
3	$c \rightarrow b$	$c \prec b \vee c \eqcirc b$	$\|b\| + 2$	$2 \cdot \|b\| + 4 \leq 3 \cdot \|l\|$
4	$a \rightarrow c$	$a \prec c \vee a \eqcirc c$	$\|a\| + 2$	$2 \cdot \|a\| + 4 \leq 3 \cdot \|l\|$
5	$a \rightarrow b$	$a \prec b \vee a \eqcirc b$	$\|a\| + \|b\| + 1$	$2 \cdot \|a\| + 2 \cdot \|b\| + 2 \leq 3 \cdot \|l\|$
6	$(a \rightarrow \mathit{0}) \rightarrow \mathit{0}$	$\mathit{0} \prec a$	$\|a\| + 4$	$\|a\| + 2 \leq 3 \cdot \|l\|$
7	$(a \rightarrow b) \rightarrow b$	$b \prec a \vee b \eqcirc \mathit{1}$	$\|a\| + 2 \cdot \|b\| + 2$	$\|a\| + 2 \cdot \|b\| + 3 \leq 3 \cdot \|l\|$
8	$a \rightarrow d$	$a \prec d \vee a \eqcirc d$	$\|a\| + \|d\| + 1$	$2 \cdot \|a\| + 2 \cdot \|d\| + 2 \leq 3 \cdot \|l\|$
9	$d \rightarrow a$	$d \prec a \vee d \eqcirc a$	$\|a\| + \|d\| + 1$	$2 \cdot \|a\| + 2 \cdot \|d\| + 2 \leq 3 \cdot \|l\|$

$a, b \in \mathit{Atom}_{\mathcal{L}} - \mathit{Tcons}_{\mathcal{L}}$, $c \in \overline{C}_{\mathcal{L}}$, $d \in \mathit{QAtom}_{\mathcal{L}}$.

(f) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models S^-$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$;

(g) $|\psi^s| \leq 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$, ψ^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ with $\#\mathcal{O}(\theta, \tilde{f}_0(\bar{x})) \in O(|\theta| \cdot (1 + |\bar{x}|))$;

(h) $|S^s| \leq 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$, S^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ with $\#\mathcal{O}(\theta, \tilde{f}_0(\bar{x})) \in O(|\theta| \cdot (1 + |\bar{x}|))$;

(i) $\psi^s = \bigwedge_{i \leq n_{\psi^s}} D_i^s$, $D_i^s \neq \tilde{p}_{\mathbb{i}}(\bar{x})$ is a factor, for all $i \leq n_{\psi^s}$, $\emptyset \neq \mathit{preds}(D_i^s) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$, for all $i < i' \leq n_{\psi^s}$, $\mathit{lits}(D_i^s) \neq \mathit{lits}(D_{i'}^s)$;

(j) for all $C \in S^s$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$, $\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1} \notin S^s$;

(k) for all $a \in \mathit{qatoms}(\psi^s)$, there exists $\mathbb{j}^* \in J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;

(l) for all $\mathbb{j} \in \{\mathbb{i}\} \cup J$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\psi^s)$ satisfying, for all $a \in \mathit{atoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\mathit{qatoms}(\psi^s))$, for all $\mathbb{j} \in J$, if there exists $a^* \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(\psi^s)$ satisfying, for all $a \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$;

(m) for all $a \in \mathit{qatoms}(S^s)$, there exists $\mathbb{j}^* \in J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;

(n) for all $\mathbb{j} \in \{\mathbb{i}\} \cup J$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(S^s)$ satisfying, for all $a \in \mathit{atoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(\mathit{qatoms}(S^s))$, for all $\mathbb{j} \in J$, if there exists $a^* \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\,\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S^s)$ satisfying, for all $a \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\,\tilde{p}_{\mathbb{j}}(\bar{x})$;

(o) $\mathit{tcons}(\psi^s) = \mathit{tcons}(S^s) = \mathit{tcons}(\theta)$.

The proof is by induction on the structure of θ using the interpolation rules in Tables 3–6.

(I) By (15) for n_ϕ, ϕ, there exists $\phi' \in \mathit{Form}_{\mathcal{L}}$ such that (15a–e) hold for n_ϕ, ϕ, ϕ'. We distinguish three cases for ϕ'. Case 1: $\phi' \in \mathit{Tcons}_{\mathcal{L}} - \{\mathit{1}\}$. We put $J_\phi = \emptyset \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, $\psi = \mathit{0} \in \mathit{Form}_{\mathcal{L}}$, $S_\phi = \{\Box\} \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}}$. Case 2: $\phi' = \mathit{1}$. We put $J_\phi = \emptyset \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, $\psi = \mathit{1} \in \mathit{Form}_{\mathcal{L}}$, $S_\phi = \emptyset \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}}$. Case 3: $\phi' \not\in \mathit{Tcons}_{\mathcal{L}}$. We put $\bar{x} = \mathit{varseq}(\phi')$, $j_{\mathbb{i}} = 0$, $\mathbb{i} = (n_\phi, j_{\mathbb{i}})$, $\mathit{ar}(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$. We get by (17) for n_ϕ, ϕ', $\bar{x}$, $\mathbb{i}$, $\tilde{p}_{\mathbb{i}}$ that there exist $J = \{(n_\phi, j) \mid 1 \leq j \leq n_J\} \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, $j_{\mathbb{i}} \leq n_J$, $\mathbb{i} \not\in J$, a *CNF* $\psi^+ \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, $S^+ \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, and (17a,b,e,g–o) hold for ϕ', $\bar{x}$, $\tilde{p}_{\mathbb{i}}$, J, ψ^+, S^+. We put $n_{J_\phi} = n_J$, $J_\phi = \{(n_\phi, j) \mid j \leq n_{J_\phi}\} \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, $\psi = \tilde{p}_{\mathbb{i}}(\bar{x}) \wedge \psi^+ \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}}$, $S_\phi = \{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\} \cup S^+ \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}}$. (II) straightforwardly follows from (I). The lemma is proved. $\Box$

The described translation produces order clausal theories in some restrictive form, which will be utilised in inference using our order hyperresolution calculus to get shorter deductions in average case, cf. Section 5. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L} \cup P}$. S is admissible iff

(a) for all $a \in \mathit{qatoms}(S)$, $\mathit{preds}(a) \subseteq P$;

(b) for all $\tilde{p} \in P$, there exists a sequence $\bar{x}$ of variables of $\mathcal{L}$ and $\tilde{p}(\bar{x}) \in \mathit{atoms}(S)$ satisfying, for all $a \in \mathit{atoms}(S)$ and $\mathit{preds}(a) = \{\tilde{p}\}$, a is an instance of $\tilde{p}(\bar{x})$ of $\mathcal{L} \cup P$; if there exists $a^* \in \mathit{qatoms}(S)$ and $\mathit{preds}(a^*) = \{\tilde{p}\}$, then there exists $Qx\,\tilde{p}(\bar{x}) \in \mathit{qatoms}(S)$ satisfying, for all $a \in \mathit{qatoms}(S)$ and $\mathit{preds}(a) = \{\tilde{p}\}$, a is an instance of $Qx\,\tilde{p}(\bar{x})$ of $\mathcal{L} \cup P$.

(a) and (b) imply that for all $Qx\,a, Q'x'\,a' \in \mathit{qatoms}(S)$, if $\mathit{preds}(a) = \mathit{preds}(a')$, then $Q = Q'$, $x = x'$, $\mathit{boundindset}(Qx\,a) = \mathit{boundindset}(Q'x'\,a')$.

Theorem 4. *Let* $n_0 \in \mathbb{N}$, $\phi \in \mathit{Form}_{\mathcal{L}}$, $T \subseteq \mathit{Form}_{\mathcal{L}}$. *There exist* $J_T^\phi \subseteq \{(i, j) \mid i \geq n_0\}$ *and* $S_T^\phi \subseteq \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$ *such that*

(i) *there exists an interpretation* $\mathfrak{A}$ *for* $\mathcal{L}$ *and* $\mathfrak{A} \models T$, $\mathfrak{A} \not\models \phi$ *if and only if there exists an interpretation* $\mathfrak{A}'$ *for* $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}$ *and* $\mathfrak{A}' \models S_T^\phi$, *satisfying* $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;

(ii) *if* $T \subseteq_{\mathcal{F}} \mathit{Form}_{\mathcal{L}}$, *then* $J_T^\phi \subseteq_{\mathcal{F}} \{(i, j) \mid i \geq n_0\}$, $\|J_T^\phi\| \in O(|T| + |\phi|)$, $S_T^\phi \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$, $|S_T^\phi| \in O(|T|^2 + |\phi|^2)$; *the number of all elementary operations of the translation of* T *and* ϕ *to* S_T^ϕ, *is in* $O(|T|^2 + |\phi|^2)$; *the time and space complexity of the translation of* T *and* ϕ *to* S_T^ϕ, *is in* $O(|T|^2 \cdot \log(1 + n_0 + |T|) + |\phi|^2 \cdot (\log(1 + n_0) + \log|\phi|))$;

(iii) S_T^ϕ *is admissible;*

(iv) $tcons(S_T^\phi) \subseteq tcons(\phi) \cup tcons(T)$.

Proof. Similar to that of Lemma 3(I). We get by Lemma 3(II) for n_0+1, T that there exist $J_T \subseteq \{(i,j) \mid i \geq n_0+1\}$, $S_T \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}}$, and Lemma 3(II a–g) hold for n_0+1, T, J_T, S_T. By (15) for n_0, ϕ, there exists $\phi' \in Form_{\mathcal{L}}$ such that (15a–e) hold for n_0, ϕ, ϕ'. We distinguish three cases for ϕ'. Case 1: $\phi' \in Tcons_{\mathcal{L}} - \{1\}$. We put $J_T^\phi = J_T \subseteq \{(i,j) \mid i \geq n_0+1\} \subseteq \{(i,j) \mid i \geq n_0\}$ and $S_T^\phi = S_T \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$. Case 2: $\phi' = 1$. We put $J_T^\phi = \emptyset \subseteq \{(i,j) \mid i \geq n_0\}$ and $S_T^\phi = \{\Box\} \subseteq SimOrdCl_{\mathcal{L}}$. Case 3: $\phi' \notin Tcons_{\mathcal{L}}$. We put $\bar{x} = varseq(\phi')$, $j_{\mathbb{i}} = 0$, $\mathbb{i} = (n_0, j_{\mathbb{i}})$, $ar(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$. We get by (17) for n_0, $\forall \bar{x}\, \phi'$, $\bar{x}$, $\mathbb{i}$, $\tilde{p}_{\mathbb{i}}$ that there exist $J = \{(n_0,j) \mid 1 \leq j \leq n_J\} \subseteq \{(n_0,j) \mid j \in \mathbb{N}\}$, $j_{\mathbb{i}} \leq n_J$, $\mathbb{i} \notin J$, $S^- \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, and (17f,h,j,m–o) hold for $\forall \bar{x}\, \phi'$, $\bar{x}$, $\tilde{p}_{\mathbb{i}}$, J, S^-. We put $J_T^\phi = J_T \cup \{\mathbb{i}\} \cup J \subseteq \{(i,j) \mid i \geq n_0\}$ and $S_T^\phi = S_T \cup \{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\} \cup S^- \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$. The theorem is proved. □

Corollary 5. *Let $n_0 \in \mathbb{N}$, $\phi \in Form_{\mathcal{L}}$, $T \subseteq Form_{\mathcal{L}}$. There exist $J_T^\phi \subseteq \{(i,j) \mid i \geq n_0\}$ and $S_T^\phi \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$ such that*

(i) $T \models \phi$ *if and only if* S_T^ϕ *is unsatisfiable;*

(ii) *if $T \subseteq_{\mathcal{F}} Form_{\mathcal{L}}$, then $J_T^\phi \subseteq_{\mathcal{F}} \{(i,j) \mid i \geq n_0\}$, $\|J_T^\phi\| \in O(|T| + |\phi|)$, $S_T^\phi \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$, $|S_T^\phi| \in O(|T|^2 + |\phi|^2)$; the number of all elementary operations of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 + |\phi|^2)$; the time and space complexity of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 \cdot \log(1 + n_0 + |T|) + |\phi|^2 \cdot (\log(1+n_0) + \log|\phi|))$;*

(iii) S_T^ϕ *is admissible;*

(iv) $tcons(S_T^\phi) \subseteq tcons(\phi) \cup tcons(T)$.

Proof. Let $T \models \phi$. Then, for every interpretation $\mathfrak{A}$ for $\mathcal{L}$, $\mathfrak{A} \not\models T$ or $\mathfrak{A} \models \phi$; by Theorem 4(i), there does not exist an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}$ and $\mathfrak{A}' \models S_T^\phi$; S_T^ϕ is unsatisfiable.

Let S_T^ϕ is unsatisfiable. Then, for every interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}$, $\mathfrak{A}' \not\models S_T^\phi$; by Theorem 4(i), there does not exist an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models T$, $\mathfrak{A} \not\models \phi$; for every interpretation $\mathfrak{A}$ for $\mathcal{L}$, $\mathfrak{A} \not\models T$ or $\mathfrak{A} \models \phi$; $T \models \phi$; (i) holds.

(ii–iv) are the same as Theorem 4(ii–iv); (ii–iv) hold. The corollary is proved. □

4. Hyperresolution over order clauses

In this section, we propose an order hyperresolution calculus with truth constants operating over order clausal theories, and prove its refutational soundness, completeness.

4.1. Order hyperresolution rules

At first, we introduce some basic notions and notation concerning chains of order literals. A chain Ξ of $\mathcal{L}$ is a sequence $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$, $\varepsilon_i \diamond_i \upsilon_i \in \mathit{OrdLit}_{\mathcal{L}}$, such that for all $i < n$, $\upsilon_i = \varepsilon_{i+1}$. ε_0 is the beginning element of Ξ and υ_n the ending element of Ξ. $\varepsilon_0 \,\Xi\, \upsilon_n$ denotes Ξ together with its respective beginning and ending element. Let $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ be a chain of $\mathcal{L}$. Ξ is an equality chain of $\mathcal{L}$ iff, for all $i \leq n$, $\diamond_i =\eqcirc$. Ξ is an increasing chain of $\mathcal{L}$ iff there exists $i^* \leq n$ such that $\diamond_{i^*} =\prec$. Ξ is a contradiction of $\mathcal{L}$ iff Ξ is an increasing chain of $\mathcal{L}$ of the form $\varepsilon_0 \,\Xi\, \mathit{0}$ or $\mathit{1} \,\Xi\, \upsilon_n$ or $\varepsilon_0 \,\Xi\, \varepsilon_0$. Let $S \subseteq \mathit{OrdCl}_{\mathcal{L}}$ be unit and $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ be a chain | an equality chain | an increasing chain | a contradiction of $\mathcal{L}$. Ξ is a chain | an equality chain | an increasing chain | a contradiction of S iff, for all $i \leq n$, $\varepsilon_i \diamond_i \upsilon_i \in S$.

Let $\tilde{\mathbb{W}} = \{\tilde{w}_{\mathbb{i}} \,|\, \mathbb{i} \in \mathbb{I}\}$ such that $\tilde{\mathbb{W}} \cap (\mathit{Func}_{\mathcal{L}} \cup \{\tilde{f}_0\}) = \emptyset$; $\tilde{\mathbb{W}}$ is an infinite countable set of new function symbols. Let $\mathcal{L}$ contain a constant (nullary function) symbol. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L} \cup P}$. We denote $\mathit{GOrdCl}_{\mathcal{L}} = \{C \,|\, C \in \mathit{OrdCl}_{\mathcal{L}} \text{ is closed}\} \subseteq \mathit{OrdCl}_{\mathcal{L}}$, $\mathit{GInst}_{\mathcal{L}}(S) = \{C \,|\, C \in \mathit{GOrdCl}_{\mathcal{L}} \text{ is an instance of } S \text{ of } \mathcal{L}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}}$, $\mathit{ordtcons}(S) = \{\mathit{0} \prec \mathit{1}\} \cup \{\mathit{0} \prec \bar{c} \,|\, \bar{c} \in \mathit{tcons}(S) \cap \overline{C}_{\mathcal{L}}\} \cup \{\bar{c} \prec \mathit{1} \,|\, \bar{c} \in \mathit{tcons}(S) \cap \overline{C}_{\mathcal{L}}\} \cup \{\bar{c}_1 \prec \bar{c}_2 \,|\, \bar{c}_1, \bar{c}_2 \in \mathit{tcons}(S) \cap \overline{C}_{\mathcal{L}}, c_1 < c_2\} \subseteq \mathit{GOrdCl}_{\mathcal{L}}$. A basic order hyperresolution calculus is defined as follows. The first rule is a central order hyperresolution one with obvious intuition.

(*Basic order hyperresolution rule*) (42)

$$\frac{l_0 \vee C_0, \dots, l_n \vee C_n \in S_{\kappa-1}}{\bigvee_{i=0}^{n} C_i \in S_\kappa};$$

$$l_0, \dots, l_n \text{ is a contradiction of } \mathcal{L}_{\kappa-1}.$$

We say that $\bigvee_{i=0}^{n} C_i$ is a basic order hyperresolvent of $l_0 \vee C_0, \dots, l_n \vee C_n$. The second and third rules are auxiliary ones that order derived atoms in both the cases $\mathit{qatoms}(S) = \emptyset$ and $\mathit{qatoms}(S) \neq \emptyset$, which is exploited in the proof of the completeness of the calculus, Theorem 9.

(*Basic order trichotomy rule*) (43)

$$\frac{a, b \in \mathit{atoms}(S_{\kappa-1}), a \in \overline{C}_{\mathcal{L}}, b \notin \mathit{Tcons}_{\mathcal{L}}, \mathit{qatoms}(S) = \emptyset}{a \prec b \vee a \eqcirc b \vee b \prec a \in S_\kappa}.$$

(*Basic order trichotomy rule*) (44)

$$\frac{a, b \in \mathit{atoms}(S_{\kappa-1}) - \{\mathit{0}, \mathit{1}\}, \{a, b\} \not\subseteq \mathit{Tcons}_{\mathcal{L}}, \mathit{qatoms}(S) \neq \emptyset}{a \prec b \vee a \eqcirc b \vee b \prec a \in S_\kappa}.$$

$a \prec b \vee a \eqcirc b \vee b \prec a$ is a basic order trichotomy resolvent of a and b. The next two rules order a quantified atom and its ground instances.

(*Basic order $\forall$-quantification rule*) (45)

$$\frac{\forall x\, a \in \mathit{qatoms}^{\forall}(S_{\kappa-1})}{\forall x\, a \prec a\gamma \vee \forall x\, a \eqcirc a\gamma \in S_\kappa};$$

$$t \in \mathit{GTerm}_{\mathcal{L}_{\kappa-1}}, \gamma = x/t \in \mathit{Subst}_{\mathcal{L}_{\kappa-1}}, \mathit{dom}(\gamma) = \{x\} = \mathit{vars}(a).$$

$\forall x\, a \prec a\gamma \vee \forall x\, a \eqcirc a\gamma$ is a basic order $\forall$-quantification resolvent of $\forall x\, a$.

(*Basic order $\exists$-quantification rule*) (46)

$$\frac{\exists x\, a \in \mathit{qatoms}^{\exists}(S_{\kappa-1})}{a\gamma \prec \exists x\, a \vee a\gamma \eqcirc \exists x\, a \in S_\kappa};$$

$$t \in \mathit{GTerm}_{\mathcal{L}_{\kappa-1}}, \gamma = x/t \in \mathit{Subst}_{\mathcal{L}_{\kappa-1}}, \mathit{dom}(\gamma) = \{x\} = \mathit{vars}(a).$$

$a\gamma \prec \exists x\, a \vee a\gamma \eqcirc \exists x\, a$ is a basic order $\exists$-quantification resolvent of $\exists x\, a$. The last two rules introduce a witness with respect to infimum | supremum, as a ground term with a new function symbol, between a derived quantified atom and an atom | a quantified atom. They also ensure a total order over a derived quantified atom and atoms | quantified atoms together with Rules (45) and (46), which is exploited in the proof of the completeness.

(*Basic order $\forall$-witnessing rule*) (47)

$$\frac{\forall x\, a \in \mathit{qatoms}^{\forall}(S_{\kappa-1}), b \in \mathit{atoms}(S_{\kappa-1}) \cup \mathit{qatoms}(S_{\kappa-1})}{a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a \in S_\kappa};$$

$$\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_{\kappa-1}}, \mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b)|,$$
$$\gamma = x/\tilde{w}(\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b)) \in \mathit{Subst}_{\mathcal{L}_\kappa}, \mathit{dom}(\gamma) = \{x\} = \mathit{vars}(a).$$

$a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a$ is a basic order $\forall$-witnessing resolvent of $\forall x\, a$ and b.

(*Basic order $\exists$-witnessing rule*) (48)

$$\frac{\exists x\, a \in \mathit{qatoms}^{\exists}(S_{\kappa-1}), b \in \mathit{atoms}(S_{\kappa-1}) \cup \mathit{qatoms}(S_{\kappa-1})}{b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b \in S_\kappa};$$

$$\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_{\kappa-1}}, \mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b)|,$$
$$\gamma = x/\tilde{w}(\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b)) \in \mathit{Subst}_{\mathcal{L}_\kappa}, \mathit{dom}(\gamma) = \{x\} = \mathit{vars}(a).$$

$b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b$ is a basic order $\exists$-witnessing resolvent of $\exists x\, a$ and b.

The basic order hyperresolution calculus can be generalised to an order hyperresolution one. Intuition behind rules is similar to that in the basic case.

(*Order hyperresolution rule*) (49)

$$\frac{\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0 \vee \bigvee_{j=1}^{m_0} l_j^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n \vee \bigvee_{j=1}^{m_n} l_j^n \in S_{\kappa-1}^{\mathit{Vr}}}{\Big(\bigvee_{i=0}^{n} \bigvee_{j=1}^{m_i} l_j^i\Big)\theta \in S_\kappa};$$

for all $i < i' \leq n$,
$\mathit{freevars}(\bigvee_{j=0}^{k_i} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee \bigvee_{j=1}^{m_i} l_j^i) \cap \mathit{freevars}(\bigvee_{j=0}^{k_{i'}} \varepsilon_j^{i'} \diamond_j^{i'} \upsilon_j^{i'} \vee \bigvee_{j=1}^{m_{i'}} l_j^{i'}) = \emptyset$,
$\theta \in \mathit{mgu}_{\mathcal{L}_{\kappa-1}}\Big(\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0, l_1^0, \dots, l_{m_0}^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n, l_1^n, \dots, l_{m_n}^n,$
$\{\upsilon_0^0, \varepsilon_0^1\}, \dots, \{\upsilon_0^{n-1}, \varepsilon_0^n\}, \{a, b\}\Big)$,
$\mathit{dom}(\theta) = \mathit{freevars}\big(\{\varepsilon_j^i \diamond_j^i \upsilon_j^i \mid j \leq k_i, i \leq n\}, \{l_j^i \mid 1 \leq j \leq m_i, i \leq n\}\big)$,
$a = \varepsilon_0^0, b = 1$ *or* $a = \upsilon_0^n, b = 0$ *or* $a = \varepsilon_0^0, b = \upsilon_0^n$,
there exists $i^* \leq n$ *such that* $\diamond_0^{i^*} = \prec$.

$(\bigvee_{i=0}^{n}\bigvee_{j=1}^{m_i} l_j^i)\theta$ is an order hyperresolvent of $\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0 \vee \bigvee_{j=1}^{m_0} l_j^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n \vee \bigvee_{j=1}^{m_n} l_j^n$.

(*Order trichotomy rule*) (50)

$$\frac{a, b \in atoms(S_{\kappa-1}), a \in \overline{C}_{\mathcal{L}}, b \notin Tcons_{\mathcal{L}}, qatoms(S) = \emptyset}{a \prec b \vee a \eqcirc b \vee b \prec a \in S_\kappa}.$$

(*Order trichotomy rule*) (51)

$$\frac{a, b \in atoms(S_{\kappa-1}^{Vr}) - \{0, 1\}, \{a, b\} \not\subseteq Tcons_{\mathcal{L}}, qatoms(S) \neq \emptyset}{a \prec b \vee a \eqcirc b \vee b \prec a \in S_\kappa};$$

$vars(a) \cap vars(b) = \emptyset$.

$a \prec b \vee a \eqcirc b \vee b \prec a$ is an order trichotomy resolvent of a and b.

(*Order* $\forall$*-quantification rule*) (52)

$$\frac{\forall x\, a \in qatoms^{\forall}(S_{\kappa-1})}{\forall x\, a \prec a \vee \forall x\, a \eqcirc a \in S_\kappa}.$$

$\forall x\, a \prec a \vee \forall x\, a \eqcirc a$ is an order $\forall$-quantification resolvent of $\forall x\, a$.

(*Order* $\exists$*-quantification rule*) (53)

$$\frac{\exists x\, a \in qatoms^{\exists}(S_{\kappa-1})}{a \prec \exists x\, a \vee a \eqcirc \exists x\, a \in S_\kappa}.$$

$a \prec \exists x\, a \vee a \eqcirc \exists x\, a$ is an order $\exists$-quantification resolvent of $\exists x\, a$.

(*Order* $\forall$*-witnessing rule*) (54)

$$\frac{\forall x\, a \in qatoms^{\forall}(S_{\kappa-1}^{Vr}), b \in atoms(S_{\kappa-1}^{Vr}) \cup qatoms(S_{\kappa-1}^{Vr})}{a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a \in S_\kappa};$$

$freevars(\forall x\, a) \cap freevars(b) = \emptyset$,
$\tilde{w} \in \tilde{\mathbb{W}} - Func_{\mathcal{L}_{\kappa-1}}, ar(\tilde{w}) = |freetermseq(\forall x\, a), freetermseq(b)|$,
$\gamma = x/\tilde{w}(freetermseq(\forall x\, a), freetermseq(b)) \cup id|_{vars(a)-\{x\}} \in Subst_{\mathcal{L}_\kappa}$,
$dom(\gamma) = \{x\} \cup (vars(a) - \{x\}) = vars(a)$.

$a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a$ is an order $\forall$-witnessing resolvent of $\forall x\, a$ and b.

(*Order* $\exists$*-witnessing rule*) (55)

$$\frac{\exists x\, a \in qatoms^{\exists}(S_{\kappa-1}^{Vr}), b \in atoms(S_{\kappa-1}^{Vr}) \cup qatoms(S_{\kappa-1}^{Vr})}{b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b \in S_\kappa};$$

$freevars(\exists x\, a) \cap freevars(b) = \emptyset$,
$\tilde{w} \in \tilde{\mathbb{W}} - Func_{\mathcal{L}_{\kappa-1}}, ar(\tilde{w}) = |freetermseq(\exists x\, a), freetermseq(b)|$,
$\gamma = x/\tilde{w}(freetermseq(\exists x\, a), freetermseq(b)) \cup id|_{vars(a)-\{x\}} \in Subst_{\mathcal{L}_\kappa}$,
$dom(\gamma) = \{x\} \cup (vars(a) - \{x\}) = vars(a)$.

$b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b$ is an order $\exists$-witnessing resolvent of $\exists x\, a$ and b.

Let $\mathcal{L}_0 = \mathcal{L} \cup P$, a reduct of $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$, and $S_0 = \emptyset \subseteq \mathit{GOrdCl}_{\mathcal{L}_0} \mid \mathit{OrdCl}_{\mathcal{L}_0}$. Let $\mathcal{D} = C_1, \dots, C_n$, $C_\kappa \in \mathit{GOrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P} \mid \mathit{OrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $n \geq 1$. $\mathcal{D}$ is a deduction of C_n from S by basic order hyperresolution iff, for all $1 \leq \kappa \leq n$, $C_\kappa \in \mathit{ordtcons}(S) \cup \mathit{GInst}_{\mathcal{L}_{\kappa-1}}(S)$, or there exist $1 \leq j_k^* \leq \kappa - 1$, $k = 1, \dots, m$, such that C_κ is a basic order resolvent of $C_{j_1^*}, \dots, C_{j_m^*} \in S_{\kappa-1}$ using Rule (42)–(48) with respect to $\mathcal{L}_{\kappa-1}$ and $S_{\kappa-1}$; $\mathcal{D}$ is a deduction of C_n from S by order hyperresolution iff, for all $1 \leq \kappa \leq n$, $C_\kappa \in \mathit{ordtcons}(S) \cup S$, or there exist $1 \leq j_k^* \leq \kappa - 1$, $k = 1, \dots, m$, such that C_κ is an order resolvent of $C'_{j_1^*}, \dots, C'_{j_m^*} \in S_{\kappa-1}^{Vr}$ using Rule (49)–(55) with respect to $\mathcal{L}_{\kappa-1}$ and $S_{\kappa-1}$ where $C'_{j_k^*}$ is a variant of $C_{j_k^*} \in S_{\kappa-1}$ of $\mathcal{L}_{\kappa-1}$; $\mathcal{L}_\kappa$ and S_κ are defined by recursion on $1 \leq \kappa \leq n$ as follows:

$$\mathcal{L}_\kappa = \begin{cases} \mathcal{L}_{\kappa-1} \cup \{\tilde{w}\} & \textit{in case of Rule } (47), (48) \mid (54), (55), \\ \mathcal{L}_{\kappa-1} & \textit{else}, \end{cases} \qquad \textit{a reduct of } \mathcal{L} \cup \tilde{\mathbb{W}} \cup P;$$

$$S_\kappa = S_{\kappa-1} \cup \{C_\kappa\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_\kappa} \mid \mathit{OrdCl}_{\mathcal{L}_\kappa},$$

$$S_\kappa^{Vr} = \mathit{Vrnt}_{\mathcal{L}_\kappa}(S_\kappa) \subseteq \mathit{OrdCl}_{\mathcal{L}_\kappa}.$$

$\mathcal{D}$ is a refutation of S iff $C_n = \square$. We denote

$$\mathit{clo}^{\mathcal{BH}}(S) = \{C \mid \textit{there exists a deduction of } C \textit{ from } S \textit{ by basic order hyperresolution}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P},$$

$$\mathit{clo}^{\mathcal{H}}(S) = \{C \mid \textit{there exists a deduction of } C \textit{ from } S \textit{ by order hyperresolution}\} \subseteq \mathit{OrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}.$$

4.2. Refutational soundness and completeness

We are in position to prove the refutational soundness and completeness of the order hyperresolution calculus. At first, we list some auxiliary lemmata.

Lemma 6 (Lifting Lemma). *Let $\mathcal{L}$ contain a constant symbol. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L}\cup P}$. Let $C \in \mathit{clo}^{\mathcal{BH}}(S)$. There exists $C^* \in \mathit{clo}^{\mathcal{H}}(S)$ such that C is an instance of C^* of $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$.*

Proof. Technical, analogous to the standard one. $\square$

Lemma 7 (Reduction Lemma). *Let $\mathcal{L}$ contain a constant symbol. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L}\cup P}$. Let $\{\bigvee_{j=0}^{k_i} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee C_i \mid i \leq n\} \subseteq \mathit{clo}^{\mathcal{BH}}(S)$ such that for all $\mathcal{S} \in \mathcal{S}el(\{\{j \mid j \leq k_i\}_i \mid i \leq n\})$, there exists a contradiction of $\{\varepsilon_{\mathcal{S}(i)}^i \diamond_{\mathcal{S}(i)}^i \upsilon_{\mathcal{S}(i)}^i \mid i \leq n\} \subseteq \mathit{GOrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. There exists $\emptyset \neq I^* \subseteq \{i \mid i \leq n\}$ such that $\bigvee_{i \in I^*} C_i \in \mathit{clo}^{\mathcal{BH}}(S)$.*

Proof. Technical, analogous to the one of Proposition 2, [32]. $\square$

Lemma 8 (Unit Lemma). *Let $\mathcal{L}$ contain a constant symbol. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L}\cup P}$. Let $\square \notin \mathit{clo}^{\mathcal{BH}}(S) = \{\bigvee_{j=0}^{k_\iota} \varepsilon_j^\iota \diamond_j^\iota \upsilon_j^\iota \mid \iota < \gamma\}$, $\gamma \leq \omega$. There exists $\mathcal{S}^* \in \mathcal{S}el(\{\{j \mid j \leq k_\iota\}_\iota \mid \iota < \gamma\})$ such that there does not exist a contradiction of $\{\varepsilon_{\mathcal{S}^*(\iota)}^\iota \diamond_{\mathcal{S}^*(\iota)}^\iota \upsilon_{\mathcal{S}^*(\iota)}^\iota \mid \iota < \gamma\} \subseteq \mathit{GOrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$.*

PROOF. Technical, a straightforward consequence of König's Lemma and Lemma 7. $\square$

Let $\{0,1\} \subseteq X \subseteq [0,1]$. X is admissible with respect to suprema and infima iff, for all $\emptyset \neq Y_1, Y_2 \subseteq X$ and $\bigvee Y_1 = \bigwedge Y_2$, $\bigvee Y_1 \in Y_1$, $\bigwedge Y_2 \in Y_2$. Let $\{\mathit{0},\mathit{1}\} \subseteq Tc \subseteq Tcons_{\mathcal{L}}$. Tc is admissible with respect to suprema and infima iff $\{0,1\} \subseteq \overline{Tc} \subseteq [0,1]$ is admissible with respect to suprema and infima.

Theorem 9 (Refutational Soundness and Completeness). *Let $\mathcal{L}$ contain a constant symbol. Let $P \subseteq \tilde{\mathbb{P}}$, $S \subseteq OrdCl_{\mathcal{L}\cup P}$, $tcons(S)$ be admissible with respect to suprema and infima. $\square \in clo^{\mathcal{H}}(S)$ if and only if S is unsatisfiable.*

PROOF. ($\Longrightarrow$) Let $\mathfrak{A}$ be a model of S for $\mathcal{L}\cup P$ and $C \in clo^{\mathcal{H}}(S) \subseteq OrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. Then there exists an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathcal{L}\cup\tilde{\mathbb{W}}\cup P$ such that $\mathfrak{A}' \models C$. The proof is by complete induction on the length of a deduction of C from S by order hyperresolution. Let $\square \in clo^{\mathcal{H}}(S)$ and $\mathfrak{A}$ be a model of S for $\mathcal{L}\cup P$. Hence, there exists an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathcal{L}\cup\tilde{\mathbb{W}}\cup P$ such that $\mathfrak{A}' \models \square$, which is a contradiction; S is unsatisfiable.

($\Longleftarrow$) Let $\square \notin clo^{\mathcal{H}}(S)$. Then, by Lemma 6 for S, $\square$, $\square \notin clo^{\mathcal{BH}}(S)$; we have $\mathcal{L}$, $\tilde{\mathbb{P}}$, $\tilde{\mathbb{W}}$ are countable, $P \subseteq \tilde{\mathbb{P}}$, $S \subseteq OrdCl_{\mathcal{L}\cup P}$, $clo^{\mathcal{BH}}(S) \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$; P, $\mathcal{L}\cup P$, $OrdCl_{\mathcal{L}\cup P}$, S, $\mathcal{L}\cup\tilde{\mathbb{W}}\cup P$, $GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $clo^{\mathcal{BH}}(S)$ are countable; there exists $\gamma_1 \leq \omega$ and $\square \notin clo^{\mathcal{BH}}(S) = \{\bigvee_{j=0}^{k_\iota} \varepsilon_j^\iota \diamond_j^\iota \upsilon_j^\iota \,|\, \iota < \gamma_1\}$; by Lemma 8 for S, there exists $\mathcal{S}^* \in \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota < \gamma_1\})$ and there does not exist a contradiction of $\{\varepsilon^\iota_{\mathcal{S}^*(\iota)} \diamond^\iota_{\mathcal{S}^*(\iota)} \upsilon^\iota_{\mathcal{S}^*(\iota)} \,|\, \iota < \gamma_1\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. We put $\mathbb{S} = \{\varepsilon^\iota_{\mathcal{S}^*(\iota)} \diamond^\iota_{\mathcal{S}^*(\iota)} \upsilon^\iota_{\mathcal{S}^*(\iota)} \,|\, \iota < \gamma_1\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. Then $ordtcons(S) \subseteq clo^{\mathcal{BH}}(S)$, $\mathbb{S} \supseteq ordtcons(S)$ is countable, unit, $(q)atoms(\mathbb{S}) \subseteq (q)atoms(clo^{\mathcal{BH}}(S))$; there does not exist a contradiction of $\mathbb{S}$. We have $\mathcal{L}$ contains a constant symbol. Hence, there exists $cn^* \in Func_{\mathcal{L}}$, $ar_{\mathcal{L}}(cn^*) = 0$. We put $\tilde{\mathbb{W}}^* = funcs(\mathbb{S}) \cap \tilde{\mathbb{W}} \subseteq \tilde{\mathbb{W}}$, $\tilde{\mathbb{W}}^* \cap (Func_{\mathcal{L}} \cup \{\tilde{f}_0\}) \subseteq \tilde{\mathbb{W}} \cap (Func_{\mathcal{L}} \cup \{\tilde{f}_0\}) = \emptyset$,

$$\mathcal{U}_{\mathfrak{A}} = GTerm_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}, cn^* \in \mathcal{U}_{\mathfrak{A}} \neq \emptyset,$$
$$\mathcal{B} = atoms(\mathbb{S}) \cup qatoms(\mathbb{S}) \subseteq GAtom_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P} \cup QAtom_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}.$$

We have $\mathbb{S}$ is countable. Then $tcons(S) = atoms(ordtcons(S)) \subseteq atoms(\mathbb{S}) \subseteq \mathcal{B}$, $\mathcal{B} = tcons(S) \cup (\mathcal{B} - tcons(S))$, $tcons(S) \cap (\mathcal{B} - tcons(S)) = \emptyset$, $atoms(\mathbb{S})$, $qatoms(\mathbb{S})$, $\mathcal{B}$, $tcons(S)$, $\mathcal{B} - tcons(S)$ are countable; there exist $\gamma_2 \leq \omega$ and a sequence $\delta_2 : \gamma_2 \longrightarrow \mathcal{B} - tcons(S)$ of $\mathcal{B} - tcons(S)$. Let $\varepsilon_1, \varepsilon_2 \in \mathcal{B}$. $\varepsilon_1 \triangleq \varepsilon_2$ iff there exists an equality chain $\varepsilon_1 \,\Xi\, \varepsilon_2$ of $\mathbb{S}$. Note that $\triangleq$ is a binary symmetric transitive relation on $\mathcal{B}$. $\varepsilon_1 \lhd \varepsilon_2$ iff there exists an increasing chain $\varepsilon_1 \,\Xi\, \varepsilon_2$ of $\mathbb{S}$. Note that $\lhd$ is a binary transitive relation on $\mathcal{B}$.

$$\mathit{0} \not\triangleq \mathit{1}, \mathit{1} \not\triangleq \mathit{0}, \mathit{0} \lhd \mathit{1}, \mathit{1} \ntriangleleft \mathit{0}, \text{ for all } \varepsilon \in \mathcal{B}, \varepsilon \ntriangleleft \mathit{0}, \mathit{1} \ntriangleleft \varepsilon, \varepsilon \ntriangleleft \varepsilon. \tag{56}$$

The proof is straightforward; we have that there does not exist a contradiction of $\mathbb{S}$. Note that $\lhd$ is also irreflexive and a partial strict order on $\mathcal{B}$.

Let $tcons(S) \subseteq X \subseteq \mathcal{B}$. A partial valuation $\mathcal{V}$ is a mapping $\mathcal{V} : X \longrightarrow [0,1]$ such that $\mathcal{V}(\mathit{0}) = 0$, $\mathcal{V}(\mathit{1}) = 1$, for all $\bar{c} \in tcons(S) \cap \overline{C}_{\mathcal{L}}$, $\mathcal{V}(\bar{c}) = c$. We denote $dom(\mathcal{V}) = X$, $tcons(S) \subseteq dom(\mathcal{V}) \subseteq \mathcal{B}$. We define a partial valuation $\mathcal{V}_\alpha$ by recursion on $\alpha \leq \gamma_2$ as follows:

$$\mathcal{V}_0 = \{(\mathit{0},0),(\mathit{1},1)\} \cup \{(\bar{c},c) \,|\, \bar{c} \in \mathit{tcons}(S) \cap \overline{C}_{\mathcal{L}}\};$$

$$\mathcal{V}_\alpha = \mathcal{V}_{\alpha-1} \cup \{(\delta_2(\alpha-1),\lambda_{\alpha-1})\} \quad (1 \leq \alpha \leq \gamma_2 \textit{ is a successor ordinal}),$$

$$\mathbb{E}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, a \triangleq \delta_2(\alpha-1), a \in \mathit{dom}(\mathcal{V}_{\alpha-1})\},$$

$$\mathbb{D}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, a \lhd \delta_2(\alpha-1), a \in \mathit{dom}(\mathcal{V}_{\alpha-1})\},$$

$$\mathbb{U}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, \delta_2(\alpha-1) \lhd a, a \in \mathit{dom}(\mathcal{V}_{\alpha-1})\},$$

$$\lambda_{\alpha-1} = \begin{cases} \dfrac{\bigvee \mathbb{D}_{\alpha-1} + \bigwedge \mathbb{U}_{\alpha-1}}{2} & \textit{if } \mathbb{E}_{\alpha-1} = \emptyset, \\ \bigvee \mathbb{E}_{\alpha-1} & \textit{else}; \end{cases}$$

$$\mathcal{V}_{\gamma_2} = \bigcup_{\alpha<\gamma_2} \mathcal{V}_\alpha \quad (\gamma_2 \textit{ is a limit ordinal}).$$

For all $\alpha \leq \alpha' \leq \gamma_2$, $\mathcal{V}_\alpha$ is a partial valuation, $\mathit{dom}(\mathcal{V}_\alpha) = \mathit{tcons}(S) \cup \delta_2[\alpha]$, $\mathcal{V}_\alpha \subseteq \mathcal{V}_{\alpha'}$. (57)

The proof is by induction on $\alpha \leq \gamma_2$.

We list some auxiliary statements without proofs:

If $\mathit{qatoms}(S) = \emptyset$, then $\mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S)) = \emptyset$. (58)

$$\mathit{tcons}(S) = \mathit{tcons}(\mathit{clo}^{\mathcal{BH}}(S)). \quad (59)$$

For all $a, b \in \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S)) \cup \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$, there exist a deduction $C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq \mathit{GOrdCl}_{\mathcal{L}_n}$, such that $a, b \in \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n)$. (60)

For all $\emptyset \neq A \subseteq_{\mathcal{F}} \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S)) \cup \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$, there exist a deduction $C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq \mathit{GOrdCl}_{\mathcal{L}_n}$, such that $A \subseteq \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n)$. (61)

For all $a \in \mathit{tcons}(S) \cap \overline{C}_{\mathcal{L}}$, $b \in \mathcal{B} - \mathit{tcons}(S)$, either $a \lhd b$ or $a \triangleq b$ or $b \lhd a$. (62)

Let $\mathit{qatoms}(S) \neq \emptyset$. For all $a, b \in \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $a \lhd b$ or ($a = b$ or $a \triangleq b$) or $b \lhd a$. (63)

For all $\alpha \leq \gamma_2$, for all $a, b \in \mathit{dom}(\mathcal{V}_\alpha)$, (64)
if $a \triangleq b$, then $\mathcal{V}_\alpha(a) = \mathcal{V}_\alpha(b)$;
if $a \lhd b$, then $\mathcal{V}_\alpha(a) < \mathcal{V}_\alpha(b)$;
if $\mathcal{V}_\alpha(a) = 0$, then $a = \mathit{0}$ or $a \triangleq \mathit{0}$;
if $\mathcal{V}_\alpha(a) = 1$, then $a = \mathit{1}$ or $a \triangleq \mathit{1}$;
for all $\alpha < \gamma_2$,
$\mathcal{V}_\alpha[\mathit{dom}(\mathcal{V}_\alpha)]$ is admissible with respect to suprema and infima.

The proof is by induction on $\alpha \leq \gamma_2$ using the assumption that $\overline{tcons(S)}$ is admissible with respect to suprema and infima.

We put $\mathcal{V} = \mathcal{V}_{\gamma_2}$, $dom(\mathcal{V}) \overset{(57)}{=} tcons(S) \cup \delta[\gamma_2] = tcons(S) \cup (\mathcal{B} - tcons(S)) = \mathcal{B}$. We further list some other auxiliary statements without proofs:

$$\begin{aligned}&\text{For all } a, b \in \mathcal{B},\\ &\quad \text{if } a \triangleq b, \text{ then } \mathcal{V}(a) = \mathcal{V}(b);\\ &\quad \text{if } a \lhd b, \text{ then } \mathcal{V}(a) < \mathcal{V}(b).\end{aligned} \tag{65}$$

$$\text{For all } Qx\,a \in qatoms(clo^{\mathcal{BH}}(S)) \text{ and } u \in \mathcal{U}_{\mathfrak{A}},\ a(x/u) \in atoms(clo^{\mathcal{BH}}(S)). \tag{66}$$

$$\begin{aligned}&\text{For all } a \in \mathcal{B},\\ &\quad \text{if } a = \forall x b, \text{ then } \mathcal{V}(a) = \textstyle\bigwedge_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u));\\ &\quad \text{if } a = \exists x b, \text{ then } \mathcal{V}(a) = \textstyle\bigvee_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u)).\end{aligned} \tag{67}$$

We put

$$f^{\mathfrak{A}}(u_1, \dots, u_\tau) = \begin{cases} f(u_1, \dots, u_\tau) & \textit{if } f \in \mathit{Func}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}, \\ cn^* & \textit{else}, \end{cases} \qquad f \in \mathit{Func}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}, u_i \in \mathcal{U}_{\mathfrak{A}};$$

$$p^{\mathfrak{A}}(u_1, \dots, u_\tau) = \begin{cases} \mathcal{V}(p(u_1, \dots, u_\tau)) & \textit{if } p(u_1, \dots, u_\tau) \in \mathcal{B}, \\ 0 & \textit{else}, \end{cases} \qquad p \in \mathit{Pred}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}, u_i \in \mathcal{U}_{\mathfrak{A}};$$

$$\begin{aligned}\mathfrak{A} = \big(&\mathcal{U}_{\mathfrak{A}}, \{f^{\mathfrak{A}} \,|\, f \in \mathit{Func}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}\}, \{p^{\mathfrak{A}} \,|\, p \in \mathit{Pred}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}\}\big),\\ &\text{an interpretation for } \mathcal{L} \cup \tilde{\mathbb{W}} \cup P.\end{aligned}$$

$$\text{For all } C \in S \text{ and } e \in \mathcal{S}_{\mathfrak{A}},\ C(e|_{\mathit{freevars}(C)}) \in clo^{\mathcal{BH}}(S). \tag{68}$$

It is straightforward to prove that for all $a \in \mathcal{B}$ and $e \in \mathcal{S}_{\mathfrak{A}}$, $\|a\|_e^{\mathfrak{A}} = \mathcal{V}(a)$. Let $l = \varepsilon_1 \eqcirc \varepsilon_2 \in \mathbb{S}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $\varepsilon_1, \varepsilon_2 \in \mathcal{B}$, $\varepsilon_1 \triangleq \varepsilon_2$, by (65) for ε_1, ε_2, $\mathcal{V}(\varepsilon_1) = \mathcal{V}(\varepsilon_2)$, $\|l\|_e^{\mathfrak{A}} = \|\varepsilon_1 \eqcirc \varepsilon_2\|_e^{\mathfrak{A}} = \|\varepsilon_1\|_e^{\mathfrak{A}} \eqcirc \|\varepsilon_2\|_e^{\mathfrak{A}} = \mathcal{V}(\varepsilon_1) \eqcirc \mathcal{V}(\varepsilon_2) = 1$. Let $l = \varepsilon_1 \prec \varepsilon_2 \in \mathbb{S}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $\varepsilon_1, \varepsilon_2 \in \mathcal{B}$, $\varepsilon_1 \lhd \varepsilon_2$, by (65) for ε_1, ε_2, $\mathcal{V}(\varepsilon_1) < \mathcal{V}(\varepsilon_2)$, $\|l\|_e^{\mathfrak{A}} = \|\varepsilon_1 \prec \varepsilon_2\|_e^{\mathfrak{A}} = \|\varepsilon_1\|_e^{\mathfrak{A}} \prec \|\varepsilon_2\|_e^{\mathfrak{A}} = \mathcal{V}(\varepsilon_1) \prec \mathcal{V}(\varepsilon_2) = 1$. So, for all $l \in \mathbb{S}$ and $e \in \mathcal{S}_{\mathfrak{A}}$, for both the cases $l = \varepsilon_1 \eqcirc \varepsilon_2 \in \mathbb{S}$ and $l = \varepsilon_1 \prec \varepsilon_2 \in \mathbb{S}$, $\|l\|_e^{\mathfrak{A}} = 1$; $\|l\|_e^{\mathfrak{A}} = 1$. Let $C \in S \subseteq \mathit{OrdCl}_{\mathcal{L} \cup P}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $e : \mathit{Var}_{\mathcal{L}} \longrightarrow \mathcal{U}_{\mathfrak{A}}$, $\mathit{freevars}(C) \subseteq_{\mathcal{F}} \mathit{Var}_{\mathcal{L}}$, $e|_{\mathit{freevars}(C)} \in \mathit{Subst}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $dom(e|_{\mathit{freevars}(C)}) = \mathit{freevars}(C)$, $range(e|_{\mathit{freevars}(C)}) = \emptyset$; $e|_{\mathit{freevars}(C)}$ is applicable to C; by (68) for C, e, $C(e|_{\mathit{freevars}(C)}) \in clo^{\mathcal{BH}}(S)$, there exists $l^* \in C(e|_{\mathit{freevars}(C)})$ and $l^* \in \mathbb{S}$, $\|l^*\|_e^{\mathfrak{A}} = 1$; there exists $l^{**} \in C \in \mathit{OrdCl}_{\mathcal{L} \cup P}$ and $l^{**} \in \mathit{OrdLit}_{\mathcal{L} \cup P} \subseteq \mathit{OrdLit}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $\mathit{freevars}(l^{**}) \subseteq \mathit{freevars}(C)$; $e|_{\mathit{freevars}(l^{**})}$ is applicable to l^{**}, $l^{**}(e|_{\mathit{freevars}(l^{**})}) = l^*$; for all $t \in \mathit{Term}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $a \in \mathit{Atom}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P} \cup \mathit{QAtom}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $l \in \mathit{OrdLit}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $\|t\|_e^{\mathfrak{A}} = t(e|_{\mathit{vars}(t)}) = \|t(e|_{\mathit{vars}(t)})\|_e^{\mathfrak{A}}$, $\|a\|_e^{\mathfrak{A}} = \|a(e|_{\mathit{freevars}(a)})\|_e^{\mathfrak{A}}$, $\|l\|_e^{\mathfrak{A}} = \|l(e|_{\mathit{freevars}(l)})\|_e^{\mathfrak{A}}$; the proof is by induction on t and by definition; $\|l^{**}\|_e^{\mathfrak{A}} = \|l^{**}(e|_{\mathit{freevars}(l^{**})})\|_e^{\mathfrak{A}} = \|l^*\|_e^{\mathfrak{A}} = 1$; $\mathfrak{A} \models_e C$; $\mathfrak{A} \models S$, $\mathfrak{A}|_{\mathcal{L} \cup P} \models S$; S is satisfiable. The theorem is proved. $\square$

Consider $S = \{\mathit{0} \prec a\} \cup \{a \prec \overline{\tfrac{1}{n}} \,|\, n \geq 2\} \subseteq \mathit{OrdCl}_{\mathcal{L}}$, $a \in \mathit{Pred}_{\mathcal{L}} - \mathit{Tcons}_{\mathcal{L}}$, $ar_{\mathcal{L}}(a) = 0$. $tcons(S)$ is not admissible with respect to suprema and infima; for $\{\mathit{0}\}$ and $\{\overline{\tfrac{1}{n}} \,|\, n \geq 2\}$,

$\bigvee\{0\} = \bigwedge\{\frac{1}{n} \,|\, n \geq 2\} = 0$, $0 \notin \{\frac{1}{n} \,|\, n \geq 2\}$. S is unsatisfiable; both the cases $\|a\|^{\mathfrak{A}} = 0$ and $\|a\|^{\mathfrak{A}} > 0$ lead to $\mathfrak{A} \not\models S$ for every interpretation $\mathfrak{A}$ for $\mathcal{L}$. However, $\square \notin clo^{\mathcal{H}}(S) = S \cup \{\mathit{0} \prec \mathit{1}\} \cup \{\mathit{0} \prec \overline{\frac{1}{n}} \,|\, n \geq 2\} \cup \{\overline{\frac{1}{n}} \prec \mathit{1} \,|\, n \geq 2\} \cup \{\overline{\frac{1}{n_1}} \prec \overline{\frac{1}{n_2}} \,|\, n_1 > n_2 \geq 2\} \cup \{\overline{\frac{1}{n}} \prec a \vee \overline{\frac{1}{n}} \eqcirc a \vee a \prec \overline{\frac{1}{n}} \,|\, n \geq 2\} \cup \{\overline{\frac{1}{n}} \eqcirc a \vee a \prec \overline{\frac{1}{n}} \,|\, n \geq 2\} \cup \{\overline{\frac{1}{n}} \prec a \vee a \prec \overline{\frac{1}{n}} \,|\, n \geq 2\}$, using Rules (50) and (49); $clo^{\mathcal{H}}(S)$ contains the order clauses from S, from $ordtcons(S)$, and some superclauses of them. So, the condition on $tcons(S)$ being admissible with respect to suprema and infima, is necessary.

The deduction problem of a formula from a theory can be solved as follows:

Corollary 10. *Let $\mathcal{L}$ contain a constant symbol. Let $n_0 \in \mathbb{N}$, $\phi \in Form_{\mathcal{L}}$, $T \subseteq Form_{\mathcal{L}}$, $tcons(T)$ be admissible with respect to suprema and infima. There exist $J_T^\phi \subseteq \{(i,j) \,|\, i \geq n_0\}$ and $S_T^\phi \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^\phi\}}$ such that $tcons(S_T^\phi)$ is admissible with respect to suprema and infima; $T \models \phi$ if and only if $\square \in clo^{\mathcal{H}}(S_T^\phi)$.*

PROOF. By Corollary 5 for n_0, ϕ, T, there exist

$$J_T^\phi \subseteq \{(i,j) \,|\, i \geq n_0\}, S_T^\phi \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^\phi\}}$$

and Corollary 5(i,iv) hold for ϕ, T, S_T^ϕ; we have $tcons(T)$ is admissible with respect to suprema and infima, $tcons(S_T^\phi) \subseteq tcons(\phi) \cup tcons(T)$; $tcons(\phi) \subseteq_{\mathcal{F}} Tcons_{\mathcal{L}}$, $tcons(S_T^\phi)$ is admissible with respect to suprema and infima; we have $T \models \phi$ if and only if S_T^ϕ is unsatisfiable; by Theorem 9 for $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^\phi\}$, S_T^ϕ, S_T^ϕ is unsatisfiable if and only if $\square \in clo^{\mathcal{H}}(S_T^\phi)$; $T \models \phi$ if and only if $\square \in clo^{\mathcal{H}}(S_T^\phi)$. The corollary is proved. $\square$

5. Examples

In this section, we illustrate the solution to the deduction problem with several examples. We show that $\phi = \forall x\,(q_1(x) \rightarrow \overline{0.3}) \rightarrow (\exists x\, q_1(x) \rightarrow \overline{0.5}) \in Form_{\mathcal{L}}$ is logically valid using the proposed translation to clausal form and the order hyperresolution calculus.

$\phi = \forall x\,(q_1(x) \rightarrow \overline{0.3}) \rightarrow (\exists x\, q_1(x) \rightarrow \overline{0.5})$

$$\Big\{\tilde{p}_0(x) \prec \mathit{1},\, \big(\underbrace{\forall x\,(q_1(x) \rightarrow \overline{0.3})}_{\tilde{p}_1(x)} \rightarrow \underbrace{(\exists x\, q_1(x) \rightarrow \overline{0.5})}_{\tilde{p}_2(x)}\big) \rightarrow \tilde{p}_0(x)\Big\} \quad (29)$$

$$\Big\{\tilde{p}_0(x) \prec \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \tilde{p}_0(x) = \mathit{1},$$
$$\tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) = \tilde{p}_0(x),$$
$$\tilde{p}_1(x) \rightarrow \forall x\,(\underbrace{q_1(x) \rightarrow \overline{0.3}}_{\tilde{p}_3(x)}), (\underbrace{\exists x\, q_1(x)}_{\tilde{p}_4(x)} \rightarrow \underbrace{\overline{0.5}}_{\tilde{p}_5(x)}) \rightarrow \tilde{p}_2(x)\Big\} \quad (35), (29)$$

$$\Big\{\tilde{p}_0(x) \prec \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \tilde{p}_0(x) = \mathit{1},$$
$$\tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) = \tilde{p}_0(x),$$
$$\tilde{p}_1(x) \prec \forall x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) = \forall x\, \tilde{p}_3(x), \tilde{p}_3(x) \rightarrow (\underbrace{q_1(x)}_{\tilde{p}_6(x)} \rightarrow \underbrace{\overline{0.3}}_{\tilde{p}_7(x)}),$$
$$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) = \mathit{1} \vee \tilde{p}_2(x) = \mathit{1}, \tilde{p}_5(x) \prec \tilde{p}_2(x) \vee \tilde{p}_5(x) = \tilde{p}_2(x),$$
$$\tilde{p}_4(x) \rightarrow \exists x\, \underbrace{q_1(x)}_{\tilde{p}_8(x)}, \overline{0.5} \prec \tilde{p}_5(x) \vee \overline{0.5} = \tilde{p}_5(x)\Big\} \quad (27), (39)$$

$$S^{\phi} = \Big\{\; \boxed{\tilde{p}_0(x) \prec \mathit{1}} \quad [1]$$
$$\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \boxed{\tilde{p}_0(x) = \mathit{1}} \quad [2]$$
$$\boxed{\tilde{p}_2(x) \prec \tilde{p}_0(x)} \vee \tilde{p}_2(x) = \tilde{p}_0(x) \quad [3]$$
$$\boxed{\tilde{p}_1(x) \prec \forall x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) = \forall x\, \tilde{p}_3(x)} \quad [4]$$
$$\boxed{\tilde{p}_3(x) \prec \tilde{p}_7(x) \vee \tilde{p}_3(x) = \tilde{p}_7(x)} \vee \tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) = \tilde{p}_7(x) \quad [5]$$
$$\boxed{q_1(x) \prec \tilde{p}_6(x) \vee q_1(x) = \tilde{p}_6(x)} \quad [6]$$
$$\boxed{\tilde{p}_7(x) \prec \overline{0.3} \vee \tilde{p}_7(x) = \overline{0.3}} \quad [7]$$
$$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) = \mathit{1} \vee \boxed{\tilde{p}_2(x) = \mathit{1}} \quad [8]$$
$$\boxed{\boxed{\tilde{p}_5(x) \prec \tilde{p}_2(x)} \vee \tilde{p}_5(x) = \tilde{p}_2(x)} \quad [9]$$
$$\boxed{\tilde{p}_4(x) \prec \exists x\, \tilde{p}_8(x) \vee \tilde{p}_4(x) = \exists x\, \tilde{p}_8(x)} \quad [10]$$
$$\boxed{\tilde{p}_8(x) \prec q_1(x) \vee \tilde{p}_8(x) = q_1(x)} \quad [11]$$
$$\boxed{\overline{0.5} \prec \tilde{p}_5(x) \vee \overline{0.5} = \tilde{p}_5(x)} \Big\} \quad [12]$$

Rule (49) : [1][2] :

$\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \boxed{\tilde{p}_2(x) = \mathit{1}}$ [13]

Rule (49) : [3][13] :

$\boxed{\tilde{p}_2(x) = \tilde{p}_0(x)} \vee \tilde{p}_2(x) \prec \tilde{p}_1(x)$ [14]

Rule (49) : [1][13][14] :

$\boxed{\tilde{p}_2(x) \prec \tilde{p}_1(x)}$ [15]

Rule (49) : [8][15] :

$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \boxed{\tilde{p}_5(x) = \mathit{1}}$ [16]

Rule (49) : [9][16] :

$\boxed{\tilde{p}_5(x) = \tilde{p}_2(x)} \vee \tilde{p}_5(x) \prec \tilde{p}_4(x)$ [17]

Rule (49) : [15][16][17] :

$\boxed{\tilde{p}_5(x) \prec \tilde{p}_4(x)}$ [18]

Rule (52) : $\forall x\, \tilde{p}_3(x)$:

$\boxed{\forall x\, \tilde{p}_3(x) \prec \tilde{p}_3(x) \vee \forall x\, \tilde{p}_3(x) = \tilde{p}_3(x)}$ [19]

$\overline{\mathbf{0.3}} \prec \overline{\mathbf{0.5}} \in \mathit{ordtcons}(S^{\phi})$

$\boxed{\overline{0.3} \prec \overline{0.5}}$ [20]

repeatedly **Rule (49)** : [4][5][7][9][12][15][19][20] :

$\boxed{\tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) = \tilde{p}_7(x)}$ [21]

Rule (55) : $\exists x\, \tilde{p}_8(x), \overline{0.5}$:

$\overline{0.5} \prec \tilde{p}_8(\tilde{w}_{(0,0)}) \vee \boxed{\exists x\, \tilde{p}_8(x) \prec \overline{0.5} \vee \exists x\, \tilde{p}_8(x) = \overline{0.5}}$ [22]

repeatedly **Rule (49)** : [10][12][18][22] :

$\boxed{\overline{0.5} \prec \tilde{p}_8(\tilde{w}_{(0,0)})}$ [23]

repeatedly **Rule (49)** : [6][7][11][21]; $x/\tilde{w}_{(0,0)}$: [20][23] :

$\square$ [24]

We next show that $\phi = \forall x\,(\overline{0.5} \to q_2(x)) \to (\overline{0.3} \to \forall x\, q_2(x)) \in \mathit{Form}_{\mathcal{L}}$ is logically valid.

$$\phi = \forall x\,(\overline{0.5} \to q_2(x)) \to (\overline{0.3} \to \forall x\, q_2(x))$$

$$\Big\{\tilde{p}_0(x) \prec \mathit{1}, \big(\underbrace{\forall x\,(\overline{0.5} \to q_2(x))}_{\tilde{p}_1(x)} \to (\underbrace{\overline{0.3} \to \forall x\, q_2(x)}_{\tilde{p}_2(x)})\big) \to \tilde{p}_0(x)\Big\} \qquad (29)$$

$$\begin{aligned}
&\Big\{\tilde{p}_0(x) \prec \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \tilde{p}_0(x) = \mathit{1},\\
&\tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) = \tilde{p}_0(x),\\
&\tilde{p}_1(x) \to \forall x\,(\underbrace{\overline{0.5} \to q_2(x)}_{\tilde{p}_3(x)}), (\underbrace{\overline{0.3}}_{\tilde{p}_4(x)} \to \underbrace{\forall x\, q_2(x)}_{\tilde{p}_5(x)}) \to \tilde{p}_2(x)\Big\} \qquad (35), (29)
\end{aligned}$$

$$\begin{aligned}
&\Big\{\tilde{p}_0(x) \prec \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \tilde{p}_0(x) = \mathit{1},\\
&\tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) = \tilde{p}_0(x),\\
&\tilde{p}_1(x) \prec \forall x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) = \forall x\, \tilde{p}_3(x), \tilde{p}_3(x) \to (\underbrace{\overline{0.5}}_{\tilde{p}_6(x)} \to \underbrace{q_2(x)}_{\tilde{p}_7(x)}),\\
&\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) = \mathit{1} \vee \tilde{p}_2(x) = \mathit{1}, \tilde{p}_5(x) \prec \tilde{p}_2(x) \vee \tilde{p}_5(x) = \tilde{p}_2(x),\\
&\tilde{p}_4(x) \prec \overline{0.3} \vee \tilde{p}_4(x) = \overline{0.3}, \forall x\, \underbrace{q_2(x)}_{\tilde{p}_8(x)} \to \tilde{p}_5(x)\Big\} \qquad (27), (37)
\end{aligned}$$

$$S^\phi = \left\{\begin{array}{ll}
\boxed{\tilde{p}_0(x) \prec \mathit{1}} & [1]\\
\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \boxed{\tilde{p}_0(x) = \mathit{1}} & [2]\\
\boxed{\tilde{p}_2(x) \prec \tilde{p}_0(x)} \vee \tilde{p}_2(x) = \tilde{p}_0(x) & [3]\\
\boxed{\tilde{p}_1(x) \prec \forall x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) = \forall x\, \tilde{p}_3(x)} & [4]\\
\boxed{\tilde{p}_3(x) \prec \tilde{p}_7(x) \vee \tilde{p}_3(x) = \tilde{p}_7(x)} \vee \boxed{\tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) = \tilde{p}_7(x)} & [5]\\
\boxed{\overline{0.5} \prec \tilde{p}_6(x) \vee \overline{0.5} = \tilde{p}_6(x)} & [6]\\
\boxed{\tilde{p}_7(x) \prec q_2(x) \vee \tilde{p}_7(x) = q_2(x)} & [7]\\
\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) = \mathit{1} \vee \boxed{\tilde{p}_2(x) = \mathit{1}} & [8]\\
\boxed{\boxed{\tilde{p}_5(x) \prec \tilde{p}_2(x)} \vee \tilde{p}_5(x) = \tilde{p}_2(x)} & [9]\\
\boxed{\tilde{p}_4(x) \prec \overline{0.3} \vee \tilde{p}_4(x) = \overline{0.3}} & [10]\\
\boxed{\forall x\, \tilde{p}_8(x) \prec \tilde{p}_5(x) \vee \forall x\, \tilde{p}_8(x) = \tilde{p}_5(x)} & [11]\\
\boxed{q_2(x) \prec \tilde{p}_8(x) \vee q_2(x) = \tilde{p}_8(x)} & [12]
\end{array}\right\}$$

Rule (49) : [1][2] :

$\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \boxed{\tilde{p}_2(x) = \mathit{1}}$ [13]

Rule (49) : [3][13] :

$\boxed{\tilde{p}_2(x) = \tilde{p}_0(x)} \vee \tilde{p}_2(x) \prec \tilde{p}_1(x)$ [14]

Rule (49) : [1][13][14] :

$\boxed{\tilde{p}_2(x) \prec \tilde{p}_1(x)}$ [15]

Rule (49) : [8][15] :

$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \boxed{\tilde{p}_5(x) = \mathit{1}}$ [16]

Rule (49) : [9][16] :

$\boxed{\tilde{p}_5(x) = \tilde{p}_2(x)} \vee \tilde{p}_5(x) \prec \tilde{p}_4(x)$ [17]

Rule (49) : [15][16][17] :

$\boxed{\tilde{p}_5(x) \prec \tilde{p}_4(x)}$ [18]

Rule (52) : $\forall x\, \tilde{p}_3(x)$:

$\boxed{\forall x\, \tilde{p}_3(x) \prec \tilde{p}_3(x) \vee \forall x\, \tilde{p}_3(x) = \tilde{p}_3(x)}$ [19]

Rule (54) : $\forall x\, \tilde{p}_8(x), \overline{0.3}$:

$\tilde{p}_8(\tilde{w}_{(0,0)}) \prec \overline{0.3} \vee \boxed{\overline{0.3} \prec \forall x\, \tilde{p}_8(x) \vee \overline{0.3} = \forall x\, \tilde{p}_8(x)}$ [20]

repeatedly **Rule (49)** : [10][11][18][20] :

$\boxed{\tilde{p}_8(\tilde{w}_{(0,0)}) \prec \overline{0.3}}$ [21]

Rule (54) : $\forall x\, \tilde{p}_8(x), \forall x\, \tilde{p}_3(x)$:

$\tilde{p}_8(\tilde{w}_{(1,1)}) \prec \forall x\, \tilde{p}_3(x) \vee \boxed{\forall x\, \tilde{p}_3(x) \prec \forall x\, \tilde{p}_8(x) \vee \forall x\, \tilde{p}_3(x) = \forall x\, \tilde{p}_8(x)}$ [22]

repeatedly **Rule (49)** : [4][9][11][15][22] :

$\boxed{\tilde{p}_8(\tilde{w}_{(1,1)}) \prec \forall x\, \tilde{p}_3(x)}$ [23]

$\overline{\mathbf{0.3}} \prec \overline{\mathbf{0.5}} \in \mathit{ordtcons}(S^{\phi})$

$\boxed{\overline{0.3} \prec \overline{0.5}}$ [24]

repeatedly **Rule (49)** : [5][6][7][12]; $x/\tilde{w}_{(0,0)}$: [21][24] :

$\boxed{\tilde{p}_3(\tilde{w}_{(0,0)}) \prec \tilde{p}_7(\tilde{w}_{(0,0)}) \vee \tilde{p}_3(\tilde{w}_{(0,0)}) = \tilde{p}_7(\tilde{w}_{(0,0)})}$ [25]

repeatedly **Rule (49)** : [5][7][12][19]; $x/\tilde{w}_{(1,1)}$: [23] :

$\boxed{\tilde{p}_6(\tilde{w}_{(1,1)}) \prec \tilde{p}_7(\tilde{w}_{(1,1)}) \vee \tilde{p}_6(\tilde{w}_{(1,1)}) = \tilde{p}_7(\tilde{w}_{(1,1)})}$ [26]

repeatedly **Rule (49)** : [7][12][19]; $x/\tilde{w}_{(0,0)}$: [21][24][25] : [6][7][12]; $x/\tilde{w}_{(1,1)}$: [23][26] :

$\square$ [27]

We further show that $\phi = \exists x\,(q_1(x) \to \overline{0.3}) \to (\forall x\, q_1(x) \to \overline{0.5}) \in \mathit{Form}_{\mathcal{L}}$ is logically valid.

$$\phi = \exists x\,(q_1(x) \to \overline{0.3}) \to (\forall x\, q_1(x) \to \overline{0.5})$$

$$\Big\{\tilde{p}_0(x) \prec \mathit{1}, \big(\underbrace{\exists x\,(q_1(x) \to \overline{0.3})}_{\tilde{p}_1(x)} \to (\underbrace{\forall x\, q_1(x) \to \overline{0.5}}_{\tilde{p}_2(x)})\big) \to \tilde{p}_0(x)\Big\} \tag{29}$$

$$\begin{aligned}
&\Big\{\tilde{p}_0(x) \prec \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \tilde{p}_0(x) = \mathit{1},\\
&\tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) = \tilde{p}_0(x),\\
&\tilde{p}_1(x) \to \exists x\,(\underbrace{q_1(x) \to \overline{0.3}}_{\tilde{p}_3(x)}), (\underbrace{\forall x\, q_1(x)}_{\tilde{p}_4(x)} \to \underbrace{\overline{0.5}}_{\tilde{p}_5(x)}) \to \tilde{p}_2(x)\Big\}
\end{aligned} \qquad (39), (29)$$

$$\begin{aligned}
&\Big\{\tilde{p}_0(x) \prec \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \tilde{p}_0(x) = \mathit{1},\\
&\tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) = \tilde{p}_0(x),\\
&\tilde{p}_1(x) \prec \exists x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) = \exists x\, \tilde{p}_3(x), \tilde{p}_3(x) \to (\underbrace{q_1(x)}_{\tilde{p}_6(x)} \to \underbrace{\overline{0.3}}_{\tilde{p}_7(x)}),\\
&\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) = \mathit{1} \vee \tilde{p}_2(x) = \mathit{1}, \tilde{p}_5(x) \prec \tilde{p}_2(x) \vee \tilde{p}_5(x) = \tilde{p}_2(x),\\
&\tilde{p}_4(x) \to \forall x\, \underbrace{q_1(x)}_{\tilde{p}_8(x)}, \overline{0.5} \prec \tilde{p}_5(x) \vee \overline{0.5} = \tilde{p}_5(x)\Big\}
\end{aligned} \qquad (27), (35)$$

$$S^{\phi} = \left\{\begin{array}{ll}
\boxed{\tilde{p}_0(x) \prec \mathit{1}} & [1]\\
\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \boxed{\tilde{p}_0(x) = \mathit{1}} & [2]\\
\boxed{\tilde{p}_2(x) \prec \tilde{p}_0(x)} \vee \tilde{p}_2(x) = \tilde{p}_0(x) & [3]\\
\boxed{\tilde{p}_1(x) \prec \exists x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) = \exists x\, \tilde{p}_3(x)} & [4]\\
\tilde{p}_3(x) \prec \tilde{p}_7(x) \vee \tilde{p}_3(x) = \tilde{p}_7(x) \vee \boxed{\tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) = \tilde{p}_7(x)} & [5]\\
\boxed{q_1(x) \prec \tilde{p}_6(x) \vee q_1(x) = \tilde{p}_6(x)} & [6]\\
\boxed{\tilde{p}_7(x) \prec \overline{0.3} \vee \tilde{p}_7(x) = \overline{0.3}} & [7]\\
\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) = \mathit{1} \vee \boxed{\tilde{p}_2(x) = \mathit{1}} & [8]\\
\boxed{\boxed{\tilde{p}_5(x) \prec \tilde{p}_2(x)} \vee \tilde{p}_5(x) = \tilde{p}_2(x)} & [9]\\
\boxed{\tilde{p}_4(x) \prec \forall x\, \tilde{p}_8(x) \vee \tilde{p}_4(x) = \forall x\, \tilde{p}_8(x)} & [10]\\
\boxed{\tilde{p}_8(x) \prec q_1(x) \vee \tilde{p}_8(x) = q_1(x)} & [11]\\
\boxed{\overline{0.5} \prec \tilde{p}_5(x) \vee \overline{0.5} = \tilde{p}_5(x)} & [12]
\end{array}\right\}$$

Rule (49) : [1][2] :

$\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \boxed{\tilde{p}_2(x) = \mathit{1}}$ [13]

Rule (49) : [3][13] :

$\boxed{\tilde{p}_2(x) = \tilde{p}_0(x)} \vee \tilde{p}_2(x) \prec \tilde{p}_1(x)$ [14]

Rule (49) : [1][13][14] :

$\boxed{\tilde{p}_2(x) \prec \tilde{p}_1(x)}$ [15]

Rule (49) : [8][15] :

$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \boxed{\tilde{p}_5(x) = \mathit{1}}$ [16]

Rule (49) : [9][16] :

$\boxed{\tilde{p}_5(x) = \tilde{p}_2(x)} \vee \tilde{p}_5(x) \prec \tilde{p}_4(x)$ [17]

Rule (49) : [15][16][17] :

$\boxed{\tilde{p}_5(x) \prec \tilde{p}_4(x)}$ [18]

Rule (52) : $\forall x\, \tilde{p}_8(x)$:

$\boxed{\forall x\, \tilde{p}_8(x) \prec \tilde{p}_8(x) \vee \forall x\, \tilde{p}_8(x) = \tilde{p}_8(x)}$ [19]

$\overline{\mathbf{0.3}} \prec \overline{\mathbf{0.5}} \in \mathit{ordtcons}(S^{\phi})$

$\boxed{\overline{0.3} \prec \overline{0.5}}$ [20]

Rule (55) : $\exists x\, \tilde{p}_3(x), \overline{0.3}$:

$\overline{0.3} \prec \tilde{p}_3(\tilde{w}_{(0,0)}) \vee \boxed{\exists x\, \tilde{p}_3(x) \prec \overline{0.3} \vee \exists x\, \tilde{p}_3(x) = \overline{0.3}}$ [21]

repeatedly **Rule (49)** : [4][9][12][15][20][21] :

$\boxed{\overline{0.3} \prec \tilde{p}_3(\tilde{w}_{(0,0)})}$ [22]

repeatedly **Rule (49)** : [5][6][7][10][11][12][18][19][20] :

$\boxed{\tilde{p}_3(x) \prec \tilde{p}_7(x) \vee \tilde{p}_3(x) = \tilde{p}_7(x)}$ [23]

repeatedly **Rule (49)** : [7][23]; $x/\tilde{w}_{(0,0)}$: [22] :

$\square$ [24]

We finally show that $\phi = \exists x\,(\overline{0.5} \to q_2(x)) \to (\overline{0.3} \to \exists x\, q_2(x)) \in Form_{\mathcal{L}}$ is logically valid.

$$\phi = \exists x\,(\overline{0.5} \to q_2(x)) \to (\overline{0.3} \to \exists x\, q_2(x))$$

$$\Big\{\tilde{p}_0(x) \prec \mathit{1}, \big(\underbrace{\exists x\,(\overline{0.5} \to q_2(x))}_{\tilde{p}_1(x)} \to \underbrace{(\overline{0.3} \to \exists x\, q_2(x))}_{\tilde{p}_2(x)}\big) \to \tilde{p}_0(x)\Big\} \qquad (29)$$

$$\Big\{\tilde{p}_0(x) \prec \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \tilde{p}_0(x) = \mathit{1},$$
$$\tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) = \tilde{p}_0(x),$$
$$\tilde{p}_1(x) \to \exists x\,\underbrace{(\overline{0.5} \to q_2(x))}_{\tilde{p}_3(x)}, (\underbrace{\overline{0.3}}_{\tilde{p}_4(x)} \to \underbrace{\exists x\, q_2(x)}_{\tilde{p}_5(x)}) \to \tilde{p}_2(x)\Big\} \qquad (39), (29)$$

$$\Big\{\tilde{p}_0(x) \prec \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \tilde{p}_0(x) = \mathit{1},$$
$$\tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) = \tilde{p}_0(x),$$
$$\tilde{p}_1(x) \prec \exists x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) = \exists x\, \tilde{p}_3(x), \tilde{p}_3(x) \to (\underbrace{\overline{0.5}}_{\tilde{p}_6(x)} \to \underbrace{q_2(x)}_{\tilde{p}_7(x)}),$$
$$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) = \mathit{1} \vee \tilde{p}_2(x) = \mathit{1}, \tilde{p}_5(x) \prec \tilde{p}_2(x) \vee \tilde{p}_5(x) = \tilde{p}_2(x),$$
$$\tilde{p}_4(x) \prec \overline{0.3} \vee \tilde{p}_4(x) = \overline{0.3}, \exists x\, \underbrace{q_2(x)}_{\tilde{p}_8(x)} \to \tilde{p}_5(x)\Big\} \qquad (27), (41)$$

$$S^\phi = \Big\{ \boxed{\tilde{p}_0(x) \prec \mathit{1}} \qquad [1]$$
$$\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) = \mathit{1} \vee \boxed{\tilde{p}_0(x) = \mathit{1}} \qquad [2]$$
$$\boxed{\tilde{p}_2(x) \prec \tilde{p}_0(x)} \vee \tilde{p}_2(x) = \tilde{p}_0(x) \qquad [3]$$
$$\boxed{\tilde{p}_1(x) \prec \exists x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) = \exists x\, \tilde{p}_3(x)} \qquad [4]$$
$$\tilde{p}_3(x) \prec \tilde{p}_7(x) \vee \tilde{p}_3(x) = \tilde{p}_7(x) \vee \boxed{\tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) = \tilde{p}_7(x)} \qquad [5]$$
$$\boxed{\overline{0.5} \prec \tilde{p}_6(x) \vee \overline{0.5} = \tilde{p}_6(x)} \qquad [6]$$
$$\boxed{\tilde{p}_7(x) \prec q_2(x) \vee \tilde{p}_7(x) = q_2(x)} \qquad [7]$$
$$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) = \mathit{1} \vee \boxed{\tilde{p}_2(x) = \mathit{1}} \qquad [8]$$
$$\boxed{\boxed{\tilde{p}_5(x) \prec \tilde{p}_2(x)} \vee \tilde{p}_5(x) = \tilde{p}_2(x)} \qquad [9]$$
$$\boxed{\tilde{p}_4(x) \prec \overline{0.3} \vee \tilde{p}_4(x) = \overline{0.3}} \qquad [10]$$
$$\boxed{\exists x\, \tilde{p}_8(x) \prec \tilde{p}_5(x) \vee \exists x\, \tilde{p}_8(x) = \tilde{p}_5(x)} \qquad [11]$$
$$\boxed{q_2(x) \prec \tilde{p}_8(x) \vee q_2(x) = \tilde{p}_8(x)} \Big\} \qquad [12]$$

Rule (49) : [1][2] :

$\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \boxed{\tilde{p}_2(x) \eqcirc \mathit{1}}$ [13]

Rule (49) : [3][13] :

$\boxed{\tilde{p}_2(x) \eqcirc \tilde{p}_0(x)} \vee \tilde{p}_2(x) \prec \tilde{p}_1(x)$ [14]

Rule (49) : [1][13][14] :

$\boxed{\tilde{p}_2(x) \prec \tilde{p}_1(x)}$ [15]

Rule (49) : [8][15] :

$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \boxed{\tilde{p}_5(x) \eqcirc \mathit{1}}$ [16]

Rule (49) : [9][16] :

$\boxed{\tilde{p}_5(x) \eqcirc \tilde{p}_2(x)} \vee \tilde{p}_5(x) \prec \tilde{p}_4(x)$ [17]

Rule (49) : [15][16][17] :

$\boxed{\tilde{p}_5(x) \prec \tilde{p}_4(x)}$ [18]

Rule (53) : $\exists x\, \tilde{p}_8(x)$:

$\boxed{\tilde{p}_8(x) \prec \exists x\, \tilde{p}_8(x) \vee \tilde{p}_8(x) \eqcirc \exists x\, \tilde{p}_8(x)}$ [19]

Rule (55) : $\exists x\, \tilde{p}_3(x), \exists x\, \tilde{p}_8(x)$:

$\exists x\, \tilde{p}_8(x) \prec \tilde{p}_3(\tilde{w}_{(0,0)}) \vee \boxed{\exists x\, \tilde{p}_3(x) \prec \exists x\, \tilde{p}_8(x) \vee \exists x\, \tilde{p}_3(x) \eqcirc \exists x\, \tilde{p}_8(x)}$ [20]

repeatedly **Rule (49)** : [4][9][11][15][20] :

$\boxed{\exists x\, \tilde{p}_8(x) \prec \tilde{p}_3(\tilde{w}_{(0,0)})}$ [21]

$\overline{\mathbf{0.3}} \prec \overline{\mathbf{0.5}} \in \mathit{ordtcons}(S^\phi)$

$\boxed{\overline{0.3} \prec \overline{0.5}}$ [22]

repeatedly **Rule (49)** : [5][6][7][10][11][12][18][19][22] :

$\boxed{\tilde{p}_3(x) \prec \tilde{p}_7(x) \vee \tilde{p}_3(x) \eqcirc \tilde{p}_7(x)}$ [23]

repeatedly **Rule (49)** : [7][12][19][23]; $x/\tilde{w}_{(0,0)}$: [21] :

$\square$ [24]

6. Conclusions

In the paper, we have proposed a modification of the hyperresolution calculus from [22, 23], which is suitable for automated deduction in the first-order Gödel logic with explicit partial truth. Gödel logic is expanded by a countable set of intermediate truth constants of the form $\bar{c}$, $c \in (0,1)$. We have modified translation of a formula to an equivalent satisfiable finite order clausal theory, consisting of order clauses. An order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is an atom or a quantified atom, and $\diamond$ is the connective $\eqcirc$ or $\prec$. $\eqcirc$ and $\prec$ are interpreted by the equality and standard strict linear order on $[0,1]$, respectively. We have investigated the so-called canonical standard completeness, where the semantics of Gödel logic is given by the standard $\boldsymbol{G}$-algebra and truth constants are interpreted by 'themselves'. The modified hyperresolution calculus is refutation sound and complete for a countable order clausal theory if the set of truth constants occurring in the theory, is admissible with respect to suprema and infima. This condition covers the case of finite order clausal theories. We have solved the deduction problem of a formula from a countable theory. As an interesting consequence, we get an affirmative solution to the open problem of recursive enumerability of unsatisfiable formulae in Gödel logic with truth constants.

Corollary 11. *The set of unsatisfiable formulae of $\mathcal{L}$ is recursively enumerable.*

Proof. Without loss of generality, we may assume that $\mathcal{L}$ contains a constant symbol. Let $\phi \in \mathit{Form}_{\mathcal{L}}$. Then ϕ contains a finite number of truth constants and $\mathit{tcons}(\{\phi\})$ is admissible with respect to suprema and infima. ϕ is unsatisfiable if and only if $\{\phi\} \models \mathit{0}$. Hence, the problem that ϕ is unsatisfiable can be reduced to the deduction problem $\{\phi\} \models \mathit{0}$ after a constant number of steps. Let $n_0 \in \mathbb{N}$. By Corollary 10 for n_0, $\mathit{0}$, $\{\phi\}$, there exist $J^{\mathit{0}}_{\{\phi\}} \subseteq \{(i,j) \,|\, i \geq n_0\}$, $S^{\mathit{0}}_{\{\phi\}} \subseteq \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J^{\mathit{0}}_{\{\phi\}}\}}$ and $\mathit{tcons}(S^{\mathit{0}}_{\{\phi\}})$ is admissible with respect to suprema and infima, $\{\phi\} \models \mathit{0}$ if and only if $\square \in \mathit{clo}^{\mathcal{H}}(S^{\mathit{0}}_{\{\phi\}})$; if $\{\phi\} \models \mathit{0}$, then $\square \in \mathit{clo}^{\mathcal{H}}(S^{\mathit{0}}_{\{\phi\}})$ and we can decide it after a finite number of steps. This straightforwardly implies that the set of unsatisfiable formulae of $\mathcal{L}$ is recursively enumerable. The corollary is proved. $\square$

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Table 3: Binary interpolation rules for $\wedge$ and $\vee$

Case		Laws	

$\theta = \theta_1 \wedge \theta_2$

Positive interpolation

$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to \theta_1 \wedge \theta_2}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \to \tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}}(\bar{x}) \to \tilde{p}_{\mathbb{i}_2}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2)} \quad (6) \qquad (18)$$

$$|\text{Consequent}| = 9 + 4 \cdot |\bar{x}| + |\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1| + |\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2| \leq 13 \cdot (1 + |\bar{x}|) + |\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1| + |\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2|$$

Positive interpolation

$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to \theta_1 \wedge \theta_2}{\left\{\begin{matrix} \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}), \\ \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2 \end{matrix}\right\}} \qquad (19)$$

$$|\text{Consequent}| = 12 + 8 \cdot |\bar{x}| + |\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1| + |\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2| \leq 15 \cdot (1 + |\bar{x}|) + |\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1| + |\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2|$$

Negative interpolation

$$\frac{\theta_1 \wedge \theta_2 \to \tilde{p}_{\mathbb{i}}(\bar{x})}{(\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x}))} \quad (8) \qquad (20)$$

$$|\text{Consequent}| = 9 + 4 \cdot |\bar{x}| + |\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})| + |\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})| \leq 13 \cdot (1 + |\bar{x}|) + |\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})| + |\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})|$$

Negative interpolation

$$\frac{\theta_1 \wedge \theta_2 \to \tilde{p}_{\mathbb{i}}(\bar{x})}{\left\{\begin{matrix} \tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \\ \theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x}), \theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x}) \end{matrix}\right\}} \qquad (21)$$

$$|\text{Consequent}| = 12 + 8 \cdot |\bar{x}| + |\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})| + |\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})| \leq 15 \cdot (1 + |\bar{x}|) + |\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})| + |\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})|$$

$\theta = \theta_1 \vee \theta_2$

Positive interpolation

$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to (\theta_1 \vee \theta_2)}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \to \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \to \tilde{p}_{\mathbb{i}_2}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2)} \quad (5) \qquad (22)$$

$$|\text{Consequent}| = 9 + 4 \cdot |\bar{x}| + |\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1| + |\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2| \leq 13 \cdot (1 + |\bar{x}|) + |\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1| + |\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2|$$

Positive interpolation

$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to (\theta_1 \vee \theta_2)}{\left\{\begin{matrix} \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}), \\ \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2 \end{matrix}\right\}} \qquad (23)$$

$$|\text{Consequent}| = 12 + 8 \cdot |\bar{x}| + |\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1| + |\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2| \leq 15 \cdot (1 + |\bar{x}|) + |\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1| + |\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2|$$

Negative interpolation

$$\frac{(\theta_1 \vee \theta_2) \to \tilde{p}_{\mathbb{i}}(\bar{x})}{(\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x}))} \quad (7) \qquad (24)$$

$$|\text{Consequent}| = 9 + 4 \cdot |\bar{x}| + |\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})| + |\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})| \leq 13 \cdot (1 + |\bar{x}|) + |\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})| + |\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})|$$

Negative interpolation

$$\frac{(\theta_1 \vee \theta_2) \to \tilde{p}_{\mathbb{i}}(\bar{x})}{\left\{\begin{matrix} \tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \\ \theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x}), \theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x}) \end{matrix}\right\}} \qquad (25)$$

$$|\text{Consequent}| = 12 + 8 \cdot |\bar{x}| + |\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})| + |\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})| \leq 15 \cdot (1 + |\bar{x}|) + |\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})| + |\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})|$$

Table 4: Binary interpolation rules for $\to$

Case		Laws	
$\theta = \theta_1 \to \theta_2, \theta_2 \neq \mathit{0}$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to (\theta_1 \to \theta_2)}{\begin{array}{l}(\tilde{p}_{\mathbb{i}}(\bar{x}) \to \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \tilde{p}_{\mathbb{i}_2}(\bar{x})) \wedge \\ (\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2)\end{array}}$	(9), (8)	(26)
$\lvert\text{Consequent}\rvert = 9 + 4 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2\rvert$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to (\theta_1 \to \theta_2)}{\left\{\begin{array}{l}\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}), \\ \theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2\end{array}\right\}}$		(27)
$\lvert\text{Consequent}\rvert = 12 + 8 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \theta_2\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \to \theta_2) \to \tilde{p}_{\mathbb{i}}(\bar{x})}{\begin{array}{l}((\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \tilde{p}_{\mathbb{i}_2}(\bar{x})) \to \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \to \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge \\ (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1) \wedge (\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x}))\end{array}}$	(11), (3), (1)	(28)
$\lvert\text{Consequent}\rvert = 13 + 6 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \to \theta_2) \to \tilde{p}_{\mathbb{i}}(\bar{x})}{\left\{\begin{array}{l}\tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \mathit{1} \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \\ \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1, \theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})\end{array}\right\}}$		(29)
$\lvert\text{Consequent}\rvert = 15 + 8 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert$			

Table 5: Unary interpolation rules for $\to$

Case		Laws	
$\theta = \theta_1 \to \mathit{0}$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to (\theta_1 \to \mathit{0})}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \to \mathit{0} \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \mathit{0}) \wedge (\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x}))}$	(9), (8)	(30)
$\lvert\text{Consequent}\rvert = 8 + 2 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to (\theta_1 \to \mathit{0})}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{0} \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \mathit{0}, \theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\}}$		(31)
$\lvert\text{Consequent}\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \to \mathit{0}) \to \tilde{p}_{\mathbb{i}}(\bar{x})}{((\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \mathit{0}) \to \mathit{0} \vee \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1)}$	(11)	(32)
$\lvert\text{Consequent}\rvert = 8 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \to \mathit{0}) \to \tilde{p}_{\mathbb{i}}(\bar{x})}{\{\mathit{0} \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\}}$		(33)
$\lvert\text{Consequent}\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert$			

Table 6: Unary interpolation rules for $\forall$ and $\exists$

Case	
$\theta = \forall x\, \theta_1$	
Positive interpolation $\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to \forall x\, \theta_1}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \to \forall x\, \tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1)}$	(34)
$\lvert\text{Consequent}\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert$	
Positive interpolation $\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to \forall x\, \theta_1}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \forall x\, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \forall x\, \tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\}}$	(35)
$\lvert\text{Consequent}\rvert = 10 + 4 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert$	
Negative interpolation $\dfrac{\forall x\, \theta_1 \to \tilde{p}_{\mathbb{i}}(\bar{x})}{(\forall x\, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x}))}$	(36)
$\lvert\text{Consequent}\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$	
Negative interpolation $\dfrac{\forall x\, \theta_1 \to \tilde{p}_{\mathbb{i}}(\bar{x})}{\{\forall x\, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \forall x\, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\}}$	(37)
$\lvert\text{Consequent}\rvert = 10 + 4 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$	
$\theta = \exists x\, \theta_1$	
Positive interpolation $\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to \exists x\, \theta_1}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \to \exists x\, \tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1)}$	(38)
$\lvert\text{Consequent}\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert$	
Positive interpolation $\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to \exists x\, \theta_1}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \exists x\, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \exists x\, \tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\}}$	(39)
$\lvert\text{Consequent}\rvert = 10 + 4 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert$	
Negative interpolation $\dfrac{\exists x\, \theta_1 \to \tilde{p}_{\mathbb{i}}(\bar{x})}{(\exists x\, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x}))}$	(40)
$\lvert\text{Consequent}\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$	
Negative interpolation $\dfrac{\exists x\, \theta_1 \to \tilde{p}_{\mathbb{i}}(\bar{x})}{\{\exists x\, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \exists x\, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\}}$	(41)
$\lvert\text{Consequent}\rvert = 10 + 4 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$	

Appendix

7.1. Substitutions 3.2

Let $X = \{x_i \,|\, 1 \leq i \leq n\} \subseteq \mathit{Var}_{\mathcal{L}}$. A substitution ϑ of $\mathcal{L}$ is a mapping $\vartheta : X \longrightarrow \mathit{Term}_{\mathcal{L}}$. ϑ may be written in the form $x_1/\vartheta(x_1), \dots, x_n/\vartheta(x_n)$. We denote $\mathit{dom}(\vartheta) = X \subseteq_{\mathcal{F}} \mathit{Var}_{\mathcal{L}}$ and $\mathit{range}(\vartheta) = \bigcup_{x \in X} \mathit{vars}(\vartheta(x)) \subseteq_{\mathcal{F}} \mathit{Var}_{\mathcal{L}}$. The set of all substitutions of $\mathcal{L}$ is designated as $\mathit{Subst}_{\mathcal{L}}$. Let $\vartheta, \vartheta' \in \mathit{Subst}_{\mathcal{L}}$. ϑ is a variable renaming of $\mathcal{L}$ iff $\vartheta : \mathit{dom}(\vartheta) \longrightarrow \mathit{Var}_{\mathcal{L}}$, for all $x, x' \in \mathit{dom}(\vartheta)$, $x \neq x'$, $\vartheta(x) \neq \vartheta(x')$. We define $\mathit{id}_{\mathcal{L}} : \mathit{Var}_{\mathcal{L}} \longrightarrow \mathit{Var}_{\mathcal{L}}$, $\mathit{id}_{\mathcal{L}}(x) = x$. Let $t \in \mathit{Term}_{\mathcal{L}}$. ϑ is applicable to t iff $\mathit{dom}(\vartheta) \supseteq \mathit{vars}(t) = \mathit{freevars}(t)$. Let ϑ be applicable to t. We define the application $t\vartheta \in \mathit{Term}_{\mathcal{L}}$ of ϑ to t by recursion on the structure of t:

$$t\vartheta = \begin{cases} \vartheta(t) & \mathit{if}\ t \in \mathit{Var}_{\mathcal{L}}, \\ f(t_1\vartheta, \dots, t_\tau\vartheta) & \mathit{if}\ t = f(t_1, \dots, t_\tau). \end{cases}$$

Let $\mathit{range}(\vartheta) \subseteq \mathit{dom}(\vartheta')$. We define the composition of ϑ and ϑ' as $\vartheta \circ \vartheta' : \mathit{dom}(\vartheta) \longrightarrow \mathit{Term}_{\mathcal{L}}$, $\vartheta \circ \vartheta'(x) = \vartheta(x)\vartheta'$, $\vartheta \circ \vartheta' \in \mathit{Subst}_{\mathcal{L}}$, $\mathit{dom}(\vartheta \circ \vartheta') = \mathit{dom}(\vartheta)$, $\mathit{range}(\vartheta \circ \vartheta') = \mathit{range}(\vartheta'|_{\mathit{range}(\vartheta)})$. Note that composition of substitutions is associative. ϑ' is a regular extension of ϑ iff $\mathit{dom}(\vartheta') \supseteq \mathit{dom}(\vartheta)$, $\vartheta'|_{\mathit{dom}(\vartheta)} = \vartheta$, $\vartheta'|_{\mathit{dom}(\vartheta') - \mathit{dom}(\vartheta)}$ is a variable renaming such that

$$\mathit{range}(\vartheta'|_{\mathit{dom}(\vartheta') - \mathit{dom}(\vartheta)}) \cap \mathit{range}(\vartheta) = \emptyset.$$

Let $a \in \mathit{Atom}_{\mathcal{L}}$. ϑ is applicable to a iff $\mathit{dom}(\vartheta) \supseteq \mathit{vars}(a) = \mathit{freevars}(a)$. Let ϑ be applicable to a and $a = p(t_1, \dots, t_\tau)$. We define the application of ϑ to a as $a\vartheta = p(t_1\vartheta, \dots, t_\tau\vartheta) \in \mathit{Atom}_{\mathcal{L}}$. Let $Qx\,a \in \mathit{QAtom}_{\mathcal{L}}$. ϑ is applicable to $Qx\,a$ iff $\mathit{dom}(\vartheta) \supseteq \mathit{freevars}(Qx\,a)$ and $x \not\in \mathit{range}(\vartheta|_{\mathit{freevars}(Qx\,a)})$. Let ϑ be applicable to $Qx\,a$. We define the application of ϑ to $Qx\,a$ as $(Qx\,a)\vartheta = Qx\,a(\vartheta|_{\mathit{freevars}(Qx\,a)} \cup x/x) \in \mathit{QAtom}_{\mathcal{L}}$. Let $\varepsilon_1 \diamond \varepsilon_2 \in \mathit{OrdLit}_{\mathcal{L}}$. ϑ is applicable to $\varepsilon_1 \diamond \varepsilon_2$ iff, for both i, ϑ is applicable to ε_i. Let ϑ be applicable to $\varepsilon_1 \diamond \varepsilon_2$. Then, for both i, ϑ is applicable to ε_i, $\mathit{dom}(\vartheta) \supseteq \mathit{freevars}(\varepsilon_i)$, $\mathit{dom}(\vartheta) \supseteq \mathit{freevars}(\varepsilon_1) \cup \mathit{freevars}(\varepsilon_2) = \mathit{freevars}(\varepsilon_1 \diamond \varepsilon_2)$. We define the application of ϑ to $\varepsilon_1 \diamond \varepsilon_2$ as $(\varepsilon_1 \diamond \varepsilon_2)\vartheta = \varepsilon_1\vartheta \diamond \varepsilon_2\vartheta \in \mathit{OrdLit}_{\mathcal{L}}$. Let $E \subseteq \mathsf{A}$, $\mathsf{A} = \mathit{Term}_{\mathcal{L}} \mid \mathsf{A} = \mathit{Atom}_{\mathcal{L}} \mid \mathsf{A} = \mathit{QAtom}_{\mathcal{L}} \mid \mathsf{A} = \mathit{OrdLit}_{\mathcal{L}}$. ϑ is applicable to E iff, for all $\varepsilon \in E$, ϑ is applicable to ε. Let ϑ be applicable to E. Then, for all $\varepsilon \in E$, ϑ is applicable to ε, $\mathit{dom}(\vartheta) \supseteq \mathit{freevars}(\varepsilon)$, $\mathit{dom}(\vartheta) \supseteq \bigcup_{\varepsilon \in E} \mathit{freevars}(\varepsilon) = \mathit{freevars}(E)$. We define the application of ϑ to E as $E\vartheta = \{\varepsilon\vartheta \,|\, \varepsilon \in E\} \subseteq \mathsf{A}$. Let $\varepsilon, \varepsilon' \in \mathsf{A} \mid \varepsilon, \varepsilon' \in \mathit{OrdCl}_{\mathcal{L}}$. ε' is an instance of ε of $\mathcal{L}$ iff there exists $\vartheta^* \in \mathit{Subst}_{\mathcal{L}}$ such that $\varepsilon' = \varepsilon\vartheta^*$. ε' is a variant of ε of $\mathcal{L}$ iff there exists a variable renaming $\rho^* \in \mathit{Subst}_{\mathcal{L}}$ such that $\varepsilon' = \varepsilon\rho^*$. Let $C \in \mathit{OrdCl}_{\mathcal{L}}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L}}$. C is an instance $\mid$ a variant of S of $\mathcal{L}$ iff there exists $C^* \in S$ such that C is an instance $\mid$ a variant of C^* of $\mathcal{L}$. We denote $\mathit{Inst}_{\mathcal{L}}(S) = \{C \,|\, C\ \mathit{is\ an\ instance\ of}\ S\ \mathit{of}\ \mathcal{L}\} \subseteq \mathit{OrdCl}_{\mathcal{L}}$ and $\mathit{Vrnt}_{\mathcal{L}}(S) = \{C \,|\, C\ \mathit{is\ a\ variant\ of}\ S\ \mathit{of}\ \mathcal{L}\} \subseteq \mathit{OrdCl}_{\mathcal{L}}$.

ϑ is a unifier of $\mathcal{L}$ for E iff $E\vartheta$ is a singleton set. Note that there does not exist a unifier for $\emptyset$. Let $\theta \in \mathit{Subst}_{\mathcal{L}}$. θ is a most general unifier of $\mathcal{L}$ for E iff θ is a unifier of $\mathcal{L}$ for E, and for every unifier ϑ of $\mathcal{L}$ for E, there exists $\gamma^* \in \mathit{Subst}_{\mathcal{L}}$ such that $\vartheta|_{\mathit{freevars}(E)} = \theta|_{\mathit{freevars}(E)} \circ \gamma^*$. By $\mathit{mgu}_{\mathcal{L}}(E) \subseteq \mathit{Subst}_{\mathcal{L}}$ we denote the set of all most general unifiers of $\mathcal{L}$ for E. Let $\overline{E} = E_0, \dots, E_n$, $E_i \subseteq \mathsf{A}_i$, either $\mathsf{A}_i = \mathit{Term}_{\mathcal{L}}$ or $\mathsf{A}_i = \mathit{Atom}_{\mathcal{L}}$ or $\mathsf{A}_i = \mathit{QAtom}_{\mathcal{L}}$ or $\mathsf{A}_i = \mathit{OrdLit}_{\mathcal{L}}$. ϑ is applicable to $\overline{E}$ iff, for all $i \leq n$, ϑ is applicable to E_i. Let ϑ be applicable to $\overline{E}$. Then, for all $i \leq n$, ϑ is applicable to E_i, $\mathit{dom}(\vartheta) \supseteq \mathit{freevars}(E_i)$, $\mathit{dom}(\vartheta) \supseteq \bigcup_{i \leq n} \mathit{freevars}(E_i) = \mathit{freevars}(\overline{E})$. We define the application of ϑ to $\overline{E}$ as $\overline{E}\vartheta = E_0\vartheta, \dots, E_n\vartheta$, $E_i\vartheta \subseteq \mathsf{A}_i$. ϑ is a unifier of $\mathcal{L}$ for $\overline{E}$ iff, for all $i \leq n$, ϑ is a unifier of $\mathcal{L}$ for E_i. Note that if there exists $i^* \leq n$ and $E_{i^*} = \emptyset$, then there does not exist a unifier for $\overline{E}$. θ is a most general unifier of $\mathcal{L}$ for $\overline{E}$ iff θ is a unifier of $\mathcal{L}$ for $\overline{E}$, and for every unifier ϑ of $\mathcal{L}$ for $\overline{E}$, there exists $\gamma^* \in \mathit{Subst}_{\mathcal{L}}$ such that $\vartheta|_{\mathit{freevars}(\overline{E})} = \theta|_{\mathit{freevars}(\overline{E})} \circ \gamma^*$. By $\mathit{mgu}_{\mathcal{L}}(\overline{E}) \subseteq \mathit{Subst}_{\mathcal{L}}$ we denote the set of all most general unifiers of $\mathcal{L}$ for $\overline{E}$.

Let $\overline{E_i} = t_1^i, \dots, t_m^i$, $t_j^i \in \mathit{Term}_{\mathcal{L}}$, $i \leq n$. We define the union of $\overline{E_i}$, $i \leq n$, as

$$\bigcup\{\overline{E_i} \,|\, i \leq n\} = \{t_1^i \,|\, i \leq n\}, \dots, \{t_m^i \,|\, i \leq n\}, \{t_j^i \,|\, i \leq n\} \subseteq \mathit{Term}_{\mathcal{L}}.$$

Note that if $m = 0$, then $\bigcup\{\overline{E_i} \,|\, i \leq n\} = \ell$.

7.2. Full proof of Theorem 2

Proof. Let $\vartheta \in \mathit{Subst}_{\mathcal{L}}$ be a unifier for $\overline{E}$. Then ϑ is applicable to $\overline{E}$, $\mathit{dom}(\vartheta) \supseteq \mathit{freevars}(\overline{E})$; there exists a variable renaming $\rho^* \in \mathit{Subst}_{\mathcal{L}}$, $\mathit{dom}(\rho^*) = \mathit{freevars}(\overline{E})$, and $\mathit{range}(\rho^*) \cap \mathit{boundvars}(\overline{E}) = \mathit{range}(\rho^*) \cap V = \emptyset$; for all $Qx\,a \in \mathit{qatoms}(\overline{E})$, $x \in \mathit{boundvars}(\overline{E})$, $\mathit{freevars}(Qx\,a) \subseteq \mathit{freevars}(\overline{E}) = \mathit{dom}(\rho^*)$, $x \not\in \mathit{range}(\rho^*) \supseteq \mathit{range}(\rho^*|_{\mathit{freevars}(Qx\,a)})$, ρ^* is applicable to $Qx\,a$; ρ^* is applicable to $\overline{E}$; $\mathit{range}(\rho^*) = \mathit{freevars}(\overline{E}\rho^*)$, $\mathit{freevars}(\overline{E}\rho^*) \cap \mathit{boundvars}(\overline{E}) \subseteq \mathit{freevars}(\overline{E}\rho^*) \cap V = \mathit{range}(\rho^*) \cap V = \emptyset$; $(\rho^*)^{-1} \in \mathit{Subst}_{\mathcal{L}}$, $\mathit{dom}((\rho^*)^{-1}) = \mathit{range}(\rho^*) = \mathit{freevars}(\overline{E}\rho^*)$, $\mathit{range}((\rho^*)^{-1}) = \mathit{dom}(\rho^*) = \mathit{freevars}(\overline{E})$, is a variable renaming, $\rho^* \circ (\rho^*)^{-1} = \mathit{id}_{\mathcal{L}}|_{\mathit{dom}(\rho^*)} = \mathit{id}_{\mathcal{L}}|_{\mathit{freevars}(\overline{E})} \in \mathit{Subst}_{\mathcal{L}}$; for all $Qx\,a \in \mathit{qatoms}(\overline{E})$, $\mathit{freevars}(Qx\,a) \subseteq \mathit{freevars}(\overline{E})$, $x \not\in \mathit{freevars}(Qx\,a) = \mathit{range}(\mathit{id}_{\mathcal{L}}|_{\mathit{freevars}(Qx\,a)})$, $\mathit{id}_{\mathcal{L}}|_{\mathit{freevars}(\overline{E})}$ is applicable to $Qx\,a$; $\mathit{id}_{\mathcal{L}}|_{\mathit{freevars}(\overline{E})}$ is applicable to $\overline{E}$; $\overline{E}(\mathit{id}_{\mathcal{L}}|_{\mathit{freevars}(\overline{E})}) = \overline{E}(\rho^* \circ (\rho^*)^{-1}) = (\overline{E}\rho^*)(\rho^*)^{-1}$, $(\rho^*)^{-1}$ is applicable to $\overline{E}\rho^*$; $\mathit{dom}(\vartheta) \supseteq \mathit{freevars}(\overline{E}) = \mathit{range}((\rho^*)^{-1})$, $(\rho^*)^{-1} \circ \vartheta \in \mathit{Subst}_{\mathcal{L}}$, $\mathit{dom}((\rho^*)^{-1} \circ \vartheta) = \mathit{dom}((\rho^*)^{-1}) = \mathit{freevars}(\overline{E}\rho^*)$, $\overline{E}\vartheta|_{\mathit{freevars}(\overline{E})} = \overline{E}(\mathit{id}_{\mathcal{L}}|_{\mathit{freevars}(\overline{E})} \circ \vartheta) = \overline{E}((\rho^* \circ (\rho^*)^{-1}) \circ \vartheta) = \overline{E}(\rho^* \circ ((\rho^*)^{-1} \circ \vartheta)) = (\overline{E}\rho^*)((\rho^*)^{-1} \circ \vartheta)$, $(\rho^*)^{-1} \circ \vartheta$ is applicable to $\overline{E}\rho^*$ and a unifier for $\overline{E}\rho^*$. We denote

$$F_i = \begin{cases} E_i\rho^* & \textit{if either } E_i\rho^* \subseteq_{\mathcal{F}} \mathit{Term}_{\mathcal{L}} \textit{ or } E_i\rho^* \subseteq_{\mathcal{F}} \mathit{Atom}_{\mathcal{L}}; \\[2ex] \bigcup\{\mathit{freetermseq}(a) \mid a \in E_i\rho^*\} & \begin{array}{l} \textit{if } E_i\rho^* \subseteq_{\mathcal{F}} \mathit{QAtom}_{\mathcal{L}}, \\ \quad \textit{there exist } Q^* \in \{\forall, \exists\}, x^* \in \mathit{Var}_{\mathcal{L}}, \\ \quad\quad p^* \in \mathit{Pred}_{\mathcal{L}}, I^* \subseteq_{\mathcal{F}} \mathbb{N}, \textit{ and} \\ \quad \textit{for all } a \in E_i\rho^*, a = Q^*x^*\, b, \mathit{preds}(b) = \{p^*\}, \\ \quad\quad \mathit{boundindset}(a) = I^*, \end{array} \\ \emptyset & \textit{else}; \\[2ex] \begin{array}{l} \{\varepsilon_1 \mid \varepsilon_1 \diamond^* \varepsilon_2 \in E_i\rho^*\}, \\ \{\varepsilon_2 \mid \varepsilon_1 \diamond^* \varepsilon_2 \in E_i\rho^*\} \end{array} & \begin{array}{l} \textit{if } E_i\rho^* \subseteq_{\mathcal{F}} \mathit{OrdLit}_{\mathcal{L}}, \\ \quad \textit{there exists } \diamond^* \in \{=, \prec\}, \\ \quad \textit{for all } l \in E_i\rho^*, l = \varepsilon_1 \diamond^* \varepsilon_2, \varepsilon_j \in \mathit{Atom}_{\mathcal{L}}, \end{array} \\ \begin{array}{l} \{\varepsilon_1 \mid \varepsilon_1 \diamond^* \varepsilon_2 \in E_i\rho^*\}, \\ \bigcup\{\mathit{freetermseq}(\varepsilon_2) \mid \varepsilon_1 \diamond^* \varepsilon_2 \in E_i\rho^*\} \end{array} & \begin{array}{l} \quad \textit{there exist } \diamond^* \in \{=, \prec\}, Q^* \in \{\forall, \exists\}, \\ \quad\quad x^* \in \mathit{Var}_{\mathcal{L}}, p^* \in \mathit{Pred}_{\mathcal{L}}, I^* \subseteq_{\mathcal{F}} \mathbb{N}, \\ \quad \textit{for all } l \in E_i\rho^*, l = \varepsilon_1 \diamond^* \varepsilon_2, \varepsilon_1 \in \mathit{Atom}_{\mathcal{L}}, \\ \quad\quad \varepsilon_2 \in \mathit{QAtom}_{\mathcal{L}}, \varepsilon_2 = Q^*x^*\, b, \mathit{preds}(b) = \{p^*\}, \\ \quad\quad \mathit{boundindset}(\varepsilon_2) = I^*, \end{array} \\ \begin{array}{l} \bigcup\{\mathit{freetermseq}(\varepsilon_1) \mid \varepsilon_1 \diamond^* \varepsilon_2 \in E_i\rho^*\}, \\ \{\varepsilon_2 \mid \varepsilon_1 \diamond^* \varepsilon_2 \in E_i\rho^*\} \end{array} & \begin{array}{l} \quad \textit{there exist } \diamond^* \in \{=, \prec\}, Q^* \in \{\forall, \exists\}, \\ \quad\quad x^* \in \mathit{Var}_{\mathcal{L}}, p^* \in \mathit{Pred}_{\mathcal{L}}, I^* \subseteq_{\mathcal{F}} \mathbb{N}, \\ \quad \textit{for all } l \in E_i\rho^*, l = \varepsilon_1 \diamond^* \varepsilon_2, \varepsilon_1 \in \mathit{QAtom}_{\mathcal{L}}, \\ \quad\quad \varepsilon_1 = Q^*x^*\, b, \mathit{preds}(b) = \{p^*\}, \\ \quad\quad \mathit{boundindset}(\varepsilon_1) = I^*, \varepsilon_2 \in \mathit{Atom}_{\mathcal{L}}, \end{array} \\ \begin{array}{l} \bigcup\{\mathit{freetermseq}(\varepsilon_1) \mid \varepsilon_1 \diamond^* \varepsilon_2 \in E_i\rho^*\}, \\ \bigcup\{\mathit{freetermseq}(\varepsilon_2) \mid \varepsilon_1 \diamond^* \varepsilon_2 \in E_i\rho^*\} \end{array} & \begin{array}{l} \quad \textit{there exist } \diamond^* \in \{=, \prec\}, Q_j^* \in \{\forall, \exists\}, \\ \quad\quad x_j^* \in \mathit{Var}_{\mathcal{L}}, p_j^* \in \mathit{Pred}_{\mathcal{L}}, I_j^* \subseteq_{\mathcal{F}} \mathbb{N}, j = 1, 2, \\ \quad \textit{for all } l \in E_i\rho^*, l = \varepsilon_1 \diamond^* \varepsilon_2, \varepsilon_j \in \mathit{QAtom}_{\mathcal{L}}, \\ \quad\quad \varepsilon_j = Q_j^*x_j^*\, b_j, \mathit{preds}(b_j) = \{p_j^*\}, \\ \quad\quad \mathit{boundindset}(\varepsilon_j) = I_j^*, \end{array} \\ \emptyset & \textit{else}; \end{cases}$$

$F_i = f_1^i, \dots, f_{n_i}^i$, either $f_j^i \subseteq_{\mathcal{F}} \mathit{Term}_{\mathcal{L}}$ or $f_j^i \subseteq_{\mathcal{F}} \mathit{Atom}_{\mathcal{L}}$, $\overline{F} = F_0, \dots, F_n = G_0, \dots, G_r$, either $G_i \subseteq_{\mathcal{F}} \mathit{Term}_{\mathcal{L}}$ or $G_i \subseteq_{\mathcal{F}} \mathit{Atom}_{\mathcal{L}}$. Hence, for all $i \leq n$, $(\rho^*)^{-1} \circ \vartheta$ is a unifier for $E_i\rho^*$ and F_i, $\mathit{freevars}(E_i\rho^*) = \mathit{vars}(F_i)$; $\mathit{freevars}(\overline{E}\rho^*) = \mathit{vars}(\overline{F})$, $(\rho^*)^{-1} \circ \vartheta$ is a unifier for $\overline{F}$, $\mathit{dom}((\rho^*)^{-1} \circ \vartheta) = \mathit{freevars}(\overline{E}\rho^*) = \mathit{vars}(\overline{F})$; by Theorem 1 for $\overline{F}$, $(\rho^*)^{-1} \circ \vartheta$, there exists $\delta^* \in \mathit{mgu}_{\mathcal{L}}(\overline{F})$, $\mathit{dom}(\delta^*) = \mathit{vars}(\overline{F}) = \mathit{freevars}(\overline{E}\rho^*)$, and $\mathit{range}(\delta^*) \subseteq \mathit{vars}(\overline{F}) = \mathit{freevars}(\overline{E}\rho^*)$; there exists $\gamma^* \in \mathit{Subst}_{\mathcal{L}}$, $\mathit{dom}(\gamma^*) = \mathit{range}(\delta^*)$, and $(\rho^*)^{-1} \circ \vartheta = \delta^* \circ \gamma^*$; $\mathit{boundvars}(\overline{E}\rho^*) = \mathit{boundvars}(\overline{E})$, $\mathit{range}(\delta^*) \cap \mathit{boundvars}(\overline{E}\rho^*) = \mathit{range}(\delta^*) \cap \mathit{boundvars}(\overline{E}) \subseteq \mathit{range}(\delta^*) \cap V \subseteq \mathit{freevars}(\overline{E}\rho^*) \cap V = \emptyset$; for all $Qx\,a \in \mathit{qatoms}(\overline{E}\rho^*)$, $x \in \mathit{boundvars}(\overline{E}\rho^*)$, $x \notin \mathit{range}(\delta^*) \supseteq \mathit{range}(\delta^*|_{\mathit{freevars}(Qx\,a)})$, δ^* is applicable to $Qx\,a$; δ^* is applicable to $\overline{E}\rho^*$; for all $i \leq n$, δ^* is a unifier for F_i and $E_i\rho^*$; δ^* is a unifier for $\overline{E}\rho^*$; $\mathit{range}(\rho^*) = \mathit{freevars}(\overline{E}\rho^*) = \mathit{dom}(\delta^*)$, $\rho^* \circ \delta^* \in \mathit{Subst}_{\mathcal{L}}$, $\mathit{dom}(\rho^* \circ \delta^*) = \mathit{dom}(\rho^*) = \mathit{freevars}(\overline{E})$; $(\overline{E}\rho^*)\delta^* = \overline{E}(\rho^* \circ \delta^*)$, $\rho^* \circ \delta^*$ is applicable to $\overline{E}$ and a unifier for $\overline{E}$; $\vartheta|_{\mathit{freevars}(\overline{E})} = \mathit{id}_{\mathcal{L}}|_{\mathit{freevars}(\overline{E})} \circ \vartheta = (\rho^* \circ (\rho^*)^{-1}) \circ \vartheta = \rho^* \circ ((\rho^*)^{-1} \circ \vartheta) = \rho^* \circ (\delta^* \circ \gamma^*) = (\rho^* \circ \delta^*) \circ \gamma^*$; $\rho^* \circ \delta^* \in \mathit{mgu}_{\mathcal{L}}(\overline{E})$. We put $\theta^* = \rho^* \circ \delta^* \in \mathit{mgu}_{\mathcal{L}}(\overline{E})$, $\mathit{dom}(\theta^*) = \mathit{freevars}(\overline{E})$. Then $\mathit{range}(\theta^*|_{\mathit{freevars}(\overline{E})}) \cap V = \mathit{range}(\delta^*) \cap V = \emptyset$. The theorem is proved. $\square$

8. Full proof of Lemma 3

Proof. It is straightforward to prove the following statements:

Let $n_\theta \in \mathbb{N}$ and $\theta \in \mathit{Form}_\mathcal{L}$. There exists $\theta' \in \mathit{Form}_\mathcal{L}$ such that (69)

(a) $\theta' \equiv \theta$;

(b) $|\theta'| \leq 2 \cdot |\theta|$; θ' can be built up from θ via a postorder traversal of θ with $\#\mathcal{O}(\theta) \in O(|\theta|)$ and the time, space complexity in $O(|\theta| \cdot (\log(1+n_\theta) + \log|\theta|))$;

(c) θ' does not contain $\neg$;

(d) $\theta' \in \mathit{Tcons}_\mathcal{L}$; or $\mathit{1}$ is not a subformula of θ'; for every subformula of θ' of the form $\varepsilon_1 \diamond \varepsilon_2$, $\diamond \in \{\wedge, \vee\}$, $\varepsilon_i \neq \mathit{0}, \mathit{1}$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq \mathit{Tcons}_\mathcal{L}$; for every subformula of θ' of the form $\varepsilon_1 \rightarrow \varepsilon_2$, $\varepsilon_1 \neq \mathit{0}, \mathit{1}$, $\varepsilon_2 \neq \mathit{1}$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq \mathit{Tcons}_\mathcal{L}$; for every subformula of θ' of the form $Qx\,\varepsilon_1$, $Q \in \{\forall, \exists\}$, $\varepsilon_1 \notin \mathit{Tcons}_\mathcal{L}$;

(e) $\mathit{tcons}(\theta') \subseteq \mathit{tcons}(\theta)$.

The proof is by induction on the structure of θ using (2) and the obvious simplification identities on $\boldsymbol{G}$ with respect to 0, 1, $c_1, c_2 \in [0,1]$, $\vee$, $\wedge$, $\Rightarrow$ (e.g. $0 \vee a = a$, $1 \vee a = 1$, $0 \wedge a = 0$, $1 \wedge a = a$, $0 \Rightarrow a = 1$, $1 \Rightarrow a = a$, $c_1 \vee c_2 = c_1$ iff $c_1 \geq c_2$, $c_1 \wedge c_2 = c_1$ iff $c_1 \leq c_2$, $c_1 \Rightarrow c_2 = 1$ iff $c_1 \leq c_2$, $c_1 \Rightarrow c_2 = c_2$ iff $c_1 > c_2$, etc.); the postorder traversal of θ uses the input θ and the output θ', $|\theta'| \underset{(69b)}{\leq} 2 \cdot |\theta| \in O(|\theta|)$, $\#\mathcal{O}(\theta) \in O(|\theta|)$; by (13) for n_θ, θ, $\emptyset$, θ', $q = 2$, $r = 1$, the time complexity of the postorder traversal of θ, is in $O(\#\mathcal{O}(\theta) \cdot (\log(1+n_\theta) + \log(\#\mathcal{O}(\theta) + |\theta|))) \subseteq O(|\theta| \cdot (\log(1+n_\theta) + \log|\theta|))$; by (14) for n_θ, θ, $\emptyset$, θ', $q = 2$, $r = 1$, the space complexity of the postorder traversal of θ, is in $O((\#\mathcal{O}(\theta) + |\theta|) \cdot (\log(1+n_\theta) + \log|\theta|)) \subseteq O(|\theta| \cdot (\log(1+n_\theta) + \log|\theta|))$. So, (69) holds.

Let $l \in \mathit{Lit}_\mathcal{L}$. There exists $C \in \mathit{SimOrdCl}_\mathcal{L}$ such that (70)

(a) for every interpretation $\mathfrak{A}$ for $\mathcal{L}$, for all $e \in \mathcal{S}_\mathfrak{A}$, $\mathfrak{A} \models_e l$ if and only if $\mathfrak{A} \models_e C$;

(b) $|C| \leq 3 \cdot |l|$, C can be built up from l with $\#\mathcal{O}(l) \in O(|l|)$.

In Table 2, for every form of literal, an order clause is assigned. We have $a, b \in \mathit{Atom}_\mathcal{L} - \mathit{Tcons}_\mathcal{L}$, $\bar{c} \in \overline{C}_\mathcal{L}$, $d \in \mathit{QAtom}_\mathcal{L}$. Then, concerning Table 2, for every form of l, $C \in \mathit{SimOrdCl}_\mathcal{L}$. For every interpretation $\mathfrak{A}$ for $\mathcal{L}$, for all $e \in \mathcal{S}_\mathfrak{A}$, concerning Table 2, for every form of l, $\mathfrak{A} \models_e l$ if and only if $\mathfrak{A} \models_e C$; (a) holds. Concerning Table 2, for every form of l, $|C| \leq 3 \cdot |l|$ and C can be built up from l with $\#\mathcal{O}(l) \in O(|l|)$; (b) holds. So, (70) holds.

Let $n_\theta \in \mathbb{N}$, $\theta \in \mathit{Form}_\mathcal{L} - \{\mathit{0}, \mathit{1}\}$, (69c,d) hold for θ; $\bar{x}$ be a sequence of variables, $\mathit{vars}(\theta) \subseteq \mathit{vars}(\bar{x}) \subseteq \mathit{Var}_\mathcal{L}$; $\mathbb{i} = (n_\theta, j_\mathbb{i}) \in \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$, $\tilde{p}_\mathbb{i} \in \tilde{\mathbb{P}}$, $\mathit{ar}(\tilde{p}_\mathbb{i}) = |\bar{x}|$. There exist $J = \{(n_\theta, j) \,|\, j_\mathbb{i} + 1 \leq j \leq n_J\} \subseteq \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$, $j_\mathbb{i} \leq n_J$, $\mathbb{i} \notin J$, a *CNF* $\psi^s \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}}$, $S^s \subseteq_\mathcal{F} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}}$, $s = +, -$, such that for both s, (71)

(a) $\|J\| \leq |\theta| - 1$;

(b) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}$ and $\mathfrak{A} \models \tilde{p}_\mathbb{i}(\bar{x}) \rightarrow \theta \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models \psi^+$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$;

(c) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}$ and $\mathfrak{A} \models \theta \rightarrow \tilde{p}_\mathbb{i}(\bar{x}) \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models \psi^-$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$;

(d) for every interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}$, $\mathfrak{A} \models \psi^s$ if and only if $\mathfrak{A} \models S^s$;

(e) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}$ and $\mathfrak{A} \models \tilde{p}_\mathbb{i}(\bar{x}) \rightarrow \theta \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models S^+$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$;

(f) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}$ and $\mathfrak{A} \models \theta \rightarrow \tilde{p}_\mathbb{i}(\bar{x}) \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\} \cup \{\tilde{p}_\mathbb{j} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models S^-$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_\mathbb{i}\}}$;

(g) $|\psi^s| \leq 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$, ψ^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ with $\#\mathcal{O}(\theta, \tilde{f}_0(\bar{x})) \in O(|\theta| \cdot (1 + |\bar{x}|))$;

(h) $|S^s| \leq 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$, S^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ with $\#\mathcal{O}(\theta, \tilde{f}_0(\bar{x})) \in O(|\theta| \cdot (1 + |\bar{x}|))$;

(i) $\psi^s = \bigwedge_{i \leq n_{\psi^s}} D^s_i$, $D^s_i \neq \tilde{p}_{\mathbb{i}}(\bar{x})$ is a factor, for all $i \leq n_{\psi^s}$, $\emptyset \neq \mathit{preds}(D^s_i) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$, for all $i < i' \leq n_{\psi^s}$, $\mathit{lits}(D^s_i) \neq \mathit{lits}(D^s_{i'})$;

(j) for all $C \in S^s$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$, $\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1} \notin S^s$;

(k) for all $a \in \mathit{qatoms}(\psi^s)$, there exists $\mathbb{j}^* \in J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;

(l) for all $\mathbb{j} \in \{\mathbb{i}\} \cup J$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\psi^s)$ satisfying, for all $a \in \mathit{atoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\mathit{qatoms}(\psi^s))$, for all $\mathbb{j} \in J$, if there exists $a^* \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(\psi^s)$ satisfying, for all $a \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$;

(m) for all $a \in \mathit{qatoms}(S^s)$, there exists $\mathbb{j}^* \in J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;

(n) for all $\mathbb{j} \in \{\mathbb{i}\} \cup J$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(S^s)$ satisfying, for all $a \in \mathit{atoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\mathit{qatoms}(S^s))$, for all $\mathbb{j} \in J$, if there exists $a^* \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S^s)$ satisfying, for all $a \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$;

(o) $\mathit{tcons}(\psi^s) = \mathit{tcons}(S^s) = \mathit{tcons}(\theta)$.

We proceed by induction on the structure of θ.

Case 1 (the base case): $\theta \in \mathit{Atom}_{\mathcal{L}}$. We put $n_J = j_{\mathbb{i}}$ and $J = \emptyset \subseteq \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$. Then $j_{\mathbb{i}} \leq n_J$ and $\mathbb{i} \notin J$. We put $\psi^+ = \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$ and $\psi^- = \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$. We have $\theta \in \mathit{Form}_{\mathcal{L}} - \{\mathit{0}, \mathit{1}\}$. Hence, $\theta \in \mathit{Atom}_{\mathcal{L}} - \{\mathit{0}, \mathit{1}\}$, $\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{Atom}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} - \mathit{Tcons}_{\mathcal{L}}$, for both s, $\psi^s \in \mathit{Lit}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$ is a *CNF*. We put $S^+ = \{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \theta \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \theta\}$ and $S^- = \{\theta \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \theta \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x})\}$. Then, for both s, for all $C \in S^s$, $C \subseteq \mathit{SimOrdLit}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$; $S^s \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$.

$\|J\| = 0 \leq |\theta| - 1$; (a) holds.

We have, for both s, $\psi^s \in \mathit{Lit}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$. Then, for both s, for every interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$, by (70a) for ψ^s, S^s, $\mathfrak{A}$, for all $e \in \mathcal{S}_{\mathfrak{A}}$, $\mathfrak{A} \models_e \psi^s$ if and only if $\mathfrak{A} \models_e S^s$; $\mathfrak{A} \models \psi^s$ if and only if $\mathfrak{A} \models S^s$; (d) holds.

Hence, there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta = \psi^+$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A}' \models \psi^+ \mid \mathfrak{A}' \models S^+$, satisfying $\mathfrak{A} = \mathfrak{A}' = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$; (b) and (e) hold.

Hence, there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x}) = \psi^-$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A}' \models \psi^- \mid \mathfrak{A}' \models S^-$, satisfying $\mathfrak{A} = \mathfrak{A}' = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$; (c) and (f) hold.

For both s, $|\psi^s| = 2 + |\bar{x}| + |\theta| \leq 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$ and ψ^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a trivial preorder traversal of θ with $\#\mathcal{O}(\theta, \tilde{f}_0(\bar{x})) \in O(|\theta| + |\bar{x}|) \subseteq O(|\theta| \cdot (1 + |\bar{x}|))$; (g) holds.

For both s, $|S^s| = 4 + 2 \cdot |\bar{x}| + 2 \cdot |\theta| \leq 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$ and S^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a trivial preorder traversal of θ with $\#\mathcal{O}(\theta, \tilde{f}_0(\bar{x})) \in O(|\theta| + |\bar{x}|) \subseteq O(|\theta| \cdot (1 + |\bar{x}|))$; (h) holds.

We put $n_{\psi^s} = 0$ and $D^s_0 = \psi^s$, $s = +, -$. We have $\theta \in \mathit{Form}_{\mathcal{L}} - \{\mathit{0}, \mathit{1}\}$, for both s, $\psi^s \in \mathit{Lit}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$. Then, for both s, $D^s_0 \neq \tilde{p}_{\mathbb{i}}(\bar{x})$ is a factor, $\emptyset \neq \mathit{preds}(D^s_0) \cap \tilde{\mathbb{P}} = (\mathit{preds}(\theta) \cup \{\tilde{p}_{\mathbb{i}}\}) \cap \tilde{\mathbb{P}} = \{\tilde{p}_{\mathbb{i}}\}$, trivially, for all $i < i' \leq n_{\psi^s}$, $\mathit{lits}(D^s_i) \neq \mathit{lits}(D^s_{i'})$; (i) holds.

We have $\theta \in \mathit{Form}_{\mathcal{L}} - \{\mathit{0}, \mathit{1}\}$. For both s, for all $C \in S^s$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} = (\mathit{preds}(\theta) \cup \{\tilde{p}_{\mathbb{i}}\}) \cap \tilde{\mathbb{P}} = \{\tilde{p}_{\mathbb{i}}\}$, $\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1} \notin S^s$; (j) holds.

For both s, $\mathit{qatoms}(\psi^s) = \emptyset$; (k) holds trivially.

We have $\theta \in \mathit{Atom}_{\mathcal{L}}$. For both s, $\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{atoms}(\psi^s)$, for all $a \in \mathit{atoms}(\psi^s) = \{\theta, \tilde{p}_{\mathbb{i}}(\bar{x})\}$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}}\}$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; $\mathit{qatoms}(\psi^s) = \emptyset$, $\tilde{p}_{\mathbb{i}} \notin \emptyset = \mathit{preds}(\mathit{qatoms}(\psi^s))$, trivially, for all $\mathbb{j} \in \emptyset = J$, if there exists $a^* \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(\psi^s)$ satisfying, for all $a \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; (l) holds.

For both s, $\mathit{qatoms}(S^s) = \emptyset$; (m) holds trivially.

We have $\theta \in \mathit{Atom}_{\mathcal{L}}$. For both s, $\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{atoms}(S^s)$, for all $a \in \mathit{atoms}(S^s) = \{\theta, \tilde{p}_{\mathbb{i}}(\bar{x})\}$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}}\}$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; $\mathit{qatoms}(S^s) = \emptyset$, $\tilde{p}_{\mathbb{i}} \notin \emptyset = \mathit{preds}(\mathit{qatoms}(S^s))$, trivially, for all $\mathbb{j} \in \emptyset = J$, if there exists $a^* \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S^s)$ satisfying, for all $a \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; (n) holds.

We have $\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{Atom}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} - \mathit{Tcons}_{\mathcal{L}}$. Then, for both s, $\mathit{tcons}(\psi^s) = \mathit{tcons}(S^s) = \mathit{tcons}(\theta) \cup \mathit{tcons}(\tilde{p}_{\mathbb{i}}(\bar{x})) = \mathit{tcons}(\theta) \cup \{\mathit{0}, \mathit{1}\} = \mathit{tcons}(\theta)$; (o) holds.

Case 2 (the induction case): $\theta \in \mathit{Form}_{\mathcal{L}} - \mathit{Atom}_{\mathcal{L}}$. We have (69c,d) hold for θ. We distinguish two cases for θ.

Case 2.1 (the binary interpolation case): $\theta = \theta_1 \diamond \theta_2$, $\diamond \in \{\wedge, \vee, \rightarrow\}$, $\theta_i \in \mathit{Form}_{\mathcal{L}} - \{\mathit{0}, \mathit{1}\}$. We have (69c,d) hold for θ, $\mathit{vars}(\theta) \subseteq \mathit{vars}(\bar{x}) \subseteq \mathit{Var}_{\mathcal{L}}$. Then, for both i, (69c,d) hold for θ_i, $\mathit{vars}(\theta_i) \subseteq \mathit{vars}(\theta) \subseteq \mathit{vars}(\bar{x}) \subseteq \mathit{Var}_{\mathcal{L}}$. We put $j_{\mathbb{i}_1} = j_{\mathbb{i}} + 1$ and $\mathbb{i}_1 = (n_\theta, j_{\mathbb{i}_1}) \in \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$. $\tilde{p}_{\mathbb{i}_1} \in \tilde{\mathbb{P}}$. We put $\mathit{ar}(\tilde{p}_{\mathbb{i}_1}) = |\bar{x}|$. We get by the induction hypothesis for n_θ, θ_1, $\bar{x}$, $\mathbb{i}_1$, $\tilde{p}_{\mathbb{i}_1}$ that there exist $J_1 = \{(n_\theta, j) \,|\, j_{\mathbb{i}_1} + 1 \leq j \leq n_{J_1}\} \subseteq \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$, $j_{\mathbb{i}_1} \leq n_{J_1}$, $\mathbb{i}_1 \notin J_1$, a *CNF* $\psi^s_1 \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}}$, $S^s_1 \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}}$, $s = +, -$, and for both s, (a–o) hold for θ_1, $\bar{x}$, $\tilde{p}_{\mathbb{i}_1}$, J_1, ψ^s_1, S^s_1. We put $j_{\mathbb{i}_2} = n_{J_1} + 1$ and $\mathbb{i}_2 = (n_\theta, j_{\mathbb{i}_2}) \in \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$. $\tilde{p}_{\mathbb{i}_2} \in \tilde{\mathbb{P}}$. We put $\mathit{ar}(\tilde{p}_{\mathbb{i}_2}) = |\bar{x}|$. We get by the induction hypothesis for n_θ, θ_2, $\bar{x}$, $\mathbb{i}_2$, $\tilde{p}_{\mathbb{i}_2}$ that there exist $J_2 = \{(n_\theta, j) \,|\, j_{\mathbb{i}_2} + 1 \leq j \leq n_{J_2}\} \subseteq \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$, $j_{\mathbb{i}_2} \leq n_{J_2}$, $\mathbb{i}_2 \notin J_2$, a *CNF* $\psi^s_2 \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_2}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_2\}}$, $S^s_2 \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_2}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_2\}}$, $s = +, -$, and for both s, (a–o) hold for θ_2, $\bar{x}$, $\tilde{p}_{\mathbb{i}_2}$,

J_2, ψ_2^s, S_2^s. We put $n_J = n_{J_2}$ and $J = \{(n_\theta, j) \,|\, j_{\mathbb{i}} + 1 \leq j \leq n_J\} \subseteq \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$. Then $j_{\mathbb{i}} < j_{\mathbb{i}_1} \leq n_{J_1} < j_{\mathbb{i}_2} \leq n_J$, $\mathbb{i} \notin J$,

$$J = \{(n_\theta, j_{\mathbb{i}_1})\} \cup \{(n_\theta, j) \,|\, j_{\mathbb{i}_1} + 1 \leq j \leq n_{J_1}\} \cup \{(n_\theta, j_{\mathbb{i}_2})\} \cup \{(n_\theta, j) \,|\, j_{\mathbb{i}_2} + 1 \leq j \leq n_{J_2}\} = \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2, \tag{72}$$

$$\{\mathbb{i}\}, \{\mathbb{i}_1\}, J_1, \{\mathbb{i}_2\}, J_2 \text{ are pairwise disjoint.} \tag{73}$$

In Tables 3 and 4, for every form of θ, positive and negative binary interpolation rules of the respective forms

$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}}{\mathit{Prefix}^+ \wedge \gamma_1^{s_1^+} \wedge \gamma_2^{s_2^+} \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}}}, \quad s_i^+ \in \{+, -\},$$

$$\frac{\theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}}{\mathit{Prefix}^- \wedge \gamma_1^{s_1^-} \wedge \gamma_2^{s_2^-} \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}}}, \quad s_i^- \in \{+, -\},$$

$$\mathit{Prefix}^s \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}},$$

$$\gamma_i^s = \begin{cases} \tilde{p}_{\mathbb{i}_i}(\bar{x}) \rightarrow \theta_i & \mathit{if}\ s = +, \\ \theta_i \rightarrow \tilde{p}_{\mathbb{i}_i}(\bar{x}) & \mathit{if}\ s = -, \end{cases} \quad \gamma_i^s \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\}},$$

are assigned. We put

$$\psi^s = \mathit{Prefix}^s \wedge \psi_1^{s_1^s} \wedge \psi_2^{s_2^s} \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2 \overset{(72)}{=} J\}}, \quad s = +, -.$$

Then, concerning Tables 3 and 4, for every form of θ, for both s, $\mathit{Prefix}^s \neq \mathit{0}, \mathit{1}$, Prefix^s is a *CNF*, for every s and i, by the induction hypothesis (i) for θ_i, $\psi_i^{s_i^s} \neq \mathit{0}, \mathit{1}$; we have, for every s and i, $\psi_i^{s_i^s}$ is a *CNF*; for both s, ψ^s is a *CNF*. In Tables 3 and 4, for every form of θ, positive and negative binary interpolation rules of the respective forms

$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}}{\mathit{ClPrefix}^+ \cup \{\gamma_1^{s_1^+}\} \cup \{\gamma_2^{s_2^+}\} \subseteq_{\mathcal{F}} \mathit{OrdForm}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}}}, \quad s_i^+ \in \{+, -\},$$

$$\frac{\theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}}{\mathit{ClPrefix}^- \cup \{\gamma_1^{s_1^-}\} \cup \{\gamma_2^{s_2^-}\} \subseteq_{\mathcal{F}} \mathit{OrdForm}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}}}, \quad s_i^- \in \{+, -\},$$

$$\mathit{ClPrefix}^s \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}},$$

are assigned. We put

$$S^s = \mathit{ClPrefix}^s \cup S_1^{s_1^s} \cup S_2^{s_2^s} \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2 \overset{(72)}{=} J\}}, \quad s = +, -.$$

Concerning Tables 3 and 4, for every form of θ, for both s, for all $C \in \mathit{ClPrefix}^s$, $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(C)$; we have, for every s and i, by the induction hypothesis (j) for θ_i, for all $C \in S_i^{s_i^s}$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}$; for both s, for all $C_0 \in \mathit{ClPrefix}^s$, $C_1 \in S_1^{s_1^s}$, and $C_2 \in S_2^{s_2^s}$, $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(C_0)$, $\tilde{p}_{\mathbb{i}} \in \tilde{\mathbb{P}}$, for both i, $\tilde{p}_{\mathbb{i}} \underset{(73)}{\notin} \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(C_i) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(C_i)$, $\mathit{preds}(C_0) \neq \mathit{preds}(C_i)$, $C_0 \neq C_i$; $(\mathit{preds}(C_1) \cap \tilde{\mathbb{P}}) \cap (\mathit{preds}(C_2) \cap \tilde{\mathbb{P}}) \subseteq (\{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}) \cap (\{\tilde{p}_{\mathbb{i}_2}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_2\}) \overset{(73)}{=} \emptyset$, for both i, $\emptyset \neq \mathit{preds}(C_i) \cap \tilde{\mathbb{P}}$, $\mathit{preds}(C_1) \cap \tilde{\mathbb{P}} \neq \mathit{preds}(C_2) \cap \tilde{\mathbb{P}}$, $\mathit{preds}(C_1) \neq \mathit{preds}(C_2)$, $C_1 \neq C_2$;

$$\text{for both } s, \mathit{ClPrefix}^s, S_1^{s_1^s}, S_2^{s_2^s} \text{ are pairwise disjoint.} \tag{74}$$

$|\theta| = 1 + |\theta_1| + |\theta_2|$, for both i, by the induction hypothesis (a) for θ_i, $\|J\| \underset{(73)}{\overset{(72)}{=}} \|\{\mathbb{i}_1\}\| + \|J_1\| + \|\{\mathbb{i}_2\}\| + \|J_2\| = 2 + \|J_1\| + \|J_2\| \leq 2 + |\theta_1| - 1 + |\theta_2| - 1 = |\theta| - 1$; (a) holds.

Concerning Tables 3 and 4, for every form of θ, for both s,

$$\mathit{Prefix}^s = \bigwedge_{i \le n^s} \bigvee_{j \le m_i^s} l_{j,i}^s, \quad l_{j,i}^s \in \mathit{Lit}_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\} \cup \{\tilde p_{\mathbb{i}_1}\} \cup \{\tilde p_{\mathbb{i}_2}\}}, \tag{75}$$

$$\mathit{ClPrefix}^s = \Big\{ \bigvee_{j \le m_i^s} C_{j,i}^s \,|\, i \le n^s \Big\} \tag{76}$$

so that for all $i \le n^s$ and $j \le m_i^s$, by (70) for $l_{j,i}^s$, $C_{j,i}^s \in \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\} \cup \{\tilde p_{\mathbb{i}_1}\} \cup \{\tilde p_{\mathbb{i}_2}\}}$ satisfying, for every interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\} \cup \{\tilde p_{\mathbb{i}_1}\} \cup \{\tilde p_{\mathbb{i}_2}\}$, for all $e \in \mathcal{S}_{\mathfrak{A}}$, $\mathfrak{A} \models_e l_{j,i}^s$ if and only if $\mathfrak{A} \models_e C_{j,i}^s$; $\mathfrak{A} \models_e \bigvee_{j \le m_i^s} l_{j,i}^s$ if and only if $\mathfrak{A} \models_e \bigvee_{j \le m_i^s} C_{j,i}^s$; $\mathfrak{A} \models \bigvee_{j \le m_i^s} l_{j,i}^s$ if and only if $\mathfrak{A} \models \bigvee_{j \le m_i^s} C_{j,i}^s$;

for every form of θ, for both s, for every interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\} \cup \{\tilde p_{\mathbb{i}_1}\} \cup \{\tilde p_{\mathbb{i}_2}\}$, $\mathfrak{A} \models \mathit{Prefix}^s$ if and only if $\mathfrak{A} \models \mathit{ClPrefix}^s$. (77)

Let $\mathfrak{A}$ be an interpretation for $\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\}$. We define an expansion $\mathfrak{A}^\#$ of $\mathfrak{A}$ to $\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\} \cup \{\tilde p_{\mathbb{i}_1}\} \cup \{\tilde p_{\mathbb{i}_2}\}$ as follows:

$$\tilde p_{\mathbb{i}_i}^{\mathfrak{A}^\#}(u_1, \dots, u_{|\bar x|}) = \begin{cases} \|\theta_i\|_e^{\mathfrak{A}} & \text{if there exists } e \in \mathcal{S}_{\mathfrak{A}} \text{ such that } \|\bar x\|_e^{\mathfrak{A}} = u_1, \dots, u_{|\bar x|}, \\ 0 & \text{else}, \end{cases} \qquad i = 1, 2.$$

Then, for both i, for all $e \in \mathcal{S}_{\mathfrak{A}^\#}$, $\|\tilde p_{\mathbb{i}_i}(\bar x)\|_e^{\mathfrak{A}^\#} = \tilde p_{\mathbb{i}_i}^{\mathfrak{A}^\#}(\|\bar x\|_e^{\mathfrak{A}^\#}) = \tilde p_{\mathbb{i}_i}^{\mathfrak{A}^\#}(\|\bar x\|_e^{\mathfrak{A}}) = \|\theta_i\|_e^{\mathfrak{A}} = \|\theta_i\|_e^{\mathfrak{A}^\#}$,

$$\|\gamma_i^+\|_e^{\mathfrak{A}^\#} \mid \|\gamma_i^-\|_e^{\mathfrak{A}^\#} = \|\tilde p_{\mathbb{i}_i}(\bar x)\|_e^{\mathfrak{A}^\#} \Rightarrow \|\theta_i\|_e^{\mathfrak{A}^\#} \mid \|\theta_i\|_e^{\mathfrak{A}^\#} \Rightarrow \|\tilde p_{\mathbb{i}_i}(\bar x)\|_e^{\mathfrak{A}^\#} = \|\theta_i\|_e^{\mathfrak{A}^\#} \Rightarrow \|\theta_i\|_e^{\mathfrak{A}^\#} = 1;$$

for every s and i, $\mathfrak{A}^\# \models \gamma_i^s$.

Let $s = +$, $\mathfrak{A} \models \tilde p_{\mathbb{i}}(\bar x) \rightarrow \theta \mid s = -$, $\mathfrak{A} \models \theta \rightarrow \tilde p_{\mathbb{i}}(\bar x)$. Then $\mathfrak{A}^\# \models \tilde p_{\mathbb{i}}(\bar x) \rightarrow \theta \mid \mathfrak{A}^\# \models \theta \rightarrow \tilde p_{\mathbb{i}}(\bar x)$, for all $e \in \mathcal{S}_{\mathfrak{A}^\#}$,

$$\begin{aligned}
&\|\tilde p_{\mathbb{i}}(\bar x) \rightarrow \theta\|_e^{\mathfrak{A}^\#} = 1 \mid \|\theta \rightarrow \tilde p_{\mathbb{i}}(\bar x)\|_e^{\mathfrak{A}^\#} = 1, \\
&\|\tilde p_{\mathbb{i}}(\bar x) \rightarrow (\theta_1 \diamond \theta_2)\|_e^{\mathfrak{A}^\#} = 1 \mid \|(\theta_1 \diamond \theta_2) \rightarrow \tilde p_{\mathbb{i}}(\bar x)\|_e^{\mathfrak{A}^\#} = 1, \\
&\|\tilde p_{\mathbb{i}}(\bar x)\|_e^{\mathfrak{A}^\#} \Rightarrow (\|\theta_1\|_e^{\mathfrak{A}^\#} \diamond \|\theta_2\|_e^{\mathfrak{A}^\#}) = 1 \mid (\|\theta_1\|_e^{\mathfrak{A}^\#} \diamond \|\theta_2\|_e^{\mathfrak{A}^\#}) \Rightarrow \|\tilde p_{\mathbb{i}}(\bar x)\|_e^{\mathfrak{A}^\#} = 1, \\
&\|\tilde p_{\mathbb{i}}(\bar x)\|_e^{\mathfrak{A}^\#} \Rightarrow (\|\tilde p_{\mathbb{i}_1}(\bar x)\|_e^{\mathfrak{A}^\#} \diamond \|\tilde p_{\mathbb{i}_2}(\bar x)\|_e^{\mathfrak{A}^\#}) = 1 \mid (\|\tilde p_{\mathbb{i}_1}(\bar x)\|_e^{\mathfrak{A}^\#} \diamond \|\tilde p_{\mathbb{i}_2}(\bar x)\|_e^{\mathfrak{A}^\#}) \Rightarrow \|\tilde p_{\mathbb{i}}(\bar x)\|_e^{\mathfrak{A}^\#} = 1,
\end{aligned}$$

concerning Tables 3 and 4, for every form of θ, by the corresponding laws in the column Laws, $\|\mathit{Prefix}^s\|_e^{\mathfrak{A}^\#} = 1$; $\mathfrak{A}^\# \models \mathit{Prefix}^s$, by (77) for θ, s, $\mathfrak{A}^\#$, $\mathfrak{A}^\# \models \mathit{ClPrefix}^s$; we have, for every s and i, $\mathfrak{A}^\# \models \gamma_i^s$; for both i, $\mathfrak{A}^\#|_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}_i}\}} \models \gamma_i^{s_i^s}$, by the induction hypothesis (b) | (c) $\big|$ (e) | (f) for θ_i, $\mathfrak{A}^\#|_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}_i}\}}$, there exists an interpretation $\mathfrak{A}_i^s$ for $\mathcal{L} \cup \{\tilde p_{\mathbb{i}_i}\} \cup \{\tilde p_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}$ and $\mathfrak{A}_i^s \models \psi_i^{s_i^s} \;\big|\; S_i^{s_i^s}$, $\mathfrak{A}_i^s|_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}_i}\}} = \mathfrak{A}^\#|_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}_i}\}}$. By (73), $\{\tilde p_{\mathbb{i}}\}$, $\{\tilde p_{\mathbb{i}_1}\}$, $\{\tilde p_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}$, $\{\tilde p_{\mathbb{i}_2}\}$, $\{\tilde p_{\mathbb{j}} \,|\, \mathbb{j} \in J_2\}$ are pairwise disjoint. We define an expansion $\mathfrak{A}^s$ of $\mathfrak{A}$ to $\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\} \cup \{\tilde p_{\mathbb{j}} \,|\, \mathbb{j} \in J \overset{(72)}{=\!=} \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2\}$ as follows:

$$\tilde p_{\mathbb{j}}^{\mathfrak{A}^s} = \tilde p_{\mathbb{j}}^{\mathfrak{A}_i^s}, \quad \mathbb{j} \in \{\mathbb{i}_i\} \cup J_i, i = 1, 2.$$

We get

$$\begin{gathered}
\mathfrak{A}^s|_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\} \cup \{\tilde p_{\mathbb{i}_1}\} \cup \{\tilde p_{\mathbb{i}_2}\}} = \mathfrak{A}^\# \models \mathit{Prefix}^s, \mathit{ClPrefix}^s, \\
\text{for both } i, \mathfrak{A}^s|_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}_i}\} \cup \{\tilde p_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}} = \mathfrak{A}_i^s \models \psi_i^{s_i^s} \;\big|\; S_i^{s_i^s}.
\end{gathered}$$

We put $\mathfrak{A}' = \mathfrak{A}^s$, an interpretation for $\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\} \cup \{\tilde p_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$. Then $\mathfrak{A}' \models \psi^s \;\big|\; S^s$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\}} = \mathfrak{A}$.

Let $s = + \mid s = -$ and $\mathfrak{A}'$ be an interpretation for $\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\} \cup \{\tilde p_{\mathbb{j}} \,|\, \mathbb{j} \in J \overset{(72)}{=\!=} \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2\}$ such that $\mathfrak{A}' \models \psi^s \;\big|\; S^s$. We denote $\mathfrak{A}^\# = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}}\} \cup \{\tilde p_{\mathbb{i}_1}\} \cup \{\tilde p_{\mathbb{i}_2}\}}$. Then $\mathfrak{A}^\# \models \mathit{Prefix}^s \;\big|\; \mathfrak{A}^\# \models \mathit{ClPrefix}^s$, by (77) for θ, s, $\mathfrak{A}^\#$, $\mathfrak{A}^\# \models \mathit{Prefix}^s$; $\mathfrak{A}^\# \models \mathit{Prefix}^s$, for both i, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}_i}\} \cup \{\tilde p_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}} \models \psi_i^{s_i^s} \;\big|\; S_i^{s_i^s}$, by the induction hypothesis (b) | (c) $\big|$ (e) | (f) for θ_i, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}_i}\} \cup \{\tilde p_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}}$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde p_{\mathbb{i}_i}\}} \models \gamma_i^{s_i^s}$, $\mathfrak{A}^\# \models \gamma_i^{s_i^s}$, for all $e \in \mathcal{S}_{\mathfrak{A}^\#}$, $\|\mathit{Prefix}^s\|_e^{\mathfrak{A}^\#} = 1$, $\|\gamma_i^{s_i^s}\|_e^{\mathfrak{A}^\#} = 1$; if $s_i^s = +$, then $1 = \|\gamma_i^{s_i^s}\|_e^{\mathfrak{A}^\#} = \|\gamma_i^+\|_e^{\mathfrak{A}^\#} = \|\tilde p_{\mathbb{i}_i}(\bar x)\|_e^{\mathfrak{A}^\#} \Rightarrow \|\theta_i\|_e^{\mathfrak{A}^\#}$, $\|\tilde p_{\mathbb{i}_i}(\bar x)\|_e^{\mathfrak{A}^\#} \le \|\theta_i\|_e^{\mathfrak{A}^\#}$; if $s_i^s = -$, then $1 = \|\gamma_i^{s_i^s}\|_e^{\mathfrak{A}^\#} =$

$\|\gamma_i^-\|_e^{\mathfrak{A}^\#} = \|\theta_i\|_e^{\mathfrak{A}^\#} \Rightarrow \|\tilde{p}_{\mathbb{i}_i}(\bar{x})\|_e^{\mathfrak{A}^\#}$, $\|\theta_i\|_e^{\mathfrak{A}^\#} \leq \|\tilde{p}_{\mathbb{i}_i}(\bar{x})\|_e^{\mathfrak{A}^\#}$; concerning Tables 3 and 4, for every form of θ, for both s, by the corresponding laws in the column Laws, $\|\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow (\theta_1 \diamond \theta_2)\|_e^{\mathfrak{A}^\#} = 1 \mid \|(\theta_1 \diamond \theta_2) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^\#} = 1$, $\|\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta\|_e^{\mathfrak{A}^\#} = 1 \mid \|\theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^\#} = 1$; $\mathfrak{A}^\# \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \mid \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} = \mathfrak{A}^\#|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \mid \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})$. We put $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$, an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$. Then $\mathfrak{A} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \mid \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})$, $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$; (b), (c), (e), (f) hold.

Let $s = + \mid s = -$ and $\mathfrak{A}$ be an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J \overset{(72)}{=\!=} \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2\}$ such that $\mathfrak{A} \models \psi^s$. We denote $\mathfrak{A}^\# = \mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}}$. Then $\mathfrak{A}^\# \models \mathit{Prefix}^s$, by (77) for θ, s, $\mathfrak{A}^\#$, $\mathfrak{A}^\# \models \mathit{ClPrefix}^s$, for both i, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}} \models \psi_i^{s_i^s}$, by the induction hypothesis (d) for θ_i, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}}$, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}} \models S_i^{s_i^s}$, $\mathfrak{A} \models \mathit{ClPrefix}^s, S_i^{s_i^s}$; $\mathfrak{A} \models S^s$.

Let $s = + \mid s = -$ and $\mathfrak{A}$ be an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J \overset{(72)}{=\!=} \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2\}$ such that $\mathfrak{A} \models S^s$. We denote $\mathfrak{A}^\# = \mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}}$. Then $\mathfrak{A}^\# \models \mathit{ClPrefix}^s$, by (77) for θ, s, $\mathfrak{A}^\#$, $\mathfrak{A}^\# \models \mathit{Prefix}^s$, for both i, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}} \models S_i^{s_i^s}$, by the induction hypothesis (d) for θ_i, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}}$, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}} \models \psi_i^{s_i^s}$, $\mathfrak{A} \models \mathit{Prefix}^s, \psi_i^{s_i^s}$; $\mathfrak{A} \models \psi^s$; (d) holds.

Concerning Tables 3 and 4, for every form of θ, for both s, $|\mathit{Prefix}^s| \leq 13 \cdot (1 + |\bar{x}|)$, Prefix^s can be built up from $\tilde{f}_0(\bar{x})$ with $\#\mathcal{O}(\tilde{f}_0(\bar{x})) \in O(|\mathit{Prefix}^s|) = O(1 + |\bar{x}|)$; we have, for every s and i, by the induction hypothesis (g) for θ_i, $|\psi_i^{s_i^s}| \leq 15 \cdot |\theta_i| \cdot (1 + |\bar{x}|)$, $\psi_i^{s_i^s}$ can be built up from θ_i and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ_i with $\#\mathcal{O}(\theta_i, \tilde{f}_0(\bar{x})) \in O(|\theta_i| \cdot (1 + |\bar{x}|))$; $|\theta| = 1 + |\theta_1| + |\theta_2|$, for both s, $|\psi^s| = 2 + |\mathit{Prefix}^s| + |\psi_1^{s_1^s}| + |\psi_2^{s_2^s}| \leq 2 + 13 \cdot (1 + |\bar{x}|) + 15 \cdot |\theta_1| \cdot (1 + |\bar{x}|) + 15 \cdot |\theta_2| \cdot (1 + |\bar{x}|) \leq 15 \cdot (1 + |\theta_1| + |\theta_2|) \cdot (1 + |\bar{x}|) = 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$, ψ^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ with $\#\mathcal{O}(\theta, \tilde{f}_0(\bar{x})) \in O(1 + |\bar{x}| + |\theta_1| \cdot (1 + |\bar{x}|) + |\theta_2| \cdot (1 + |\bar{x}|)) = O((1 + |\theta_1| + |\theta_2|) \cdot (1 + |\bar{x}|)) = O(|\theta| \cdot (1 + |\bar{x}|))$; (g) holds.

Concerning Tables 3 and 4, for every form of θ, for both s, $|\mathit{ClPrefix}^s| \leq 15 \cdot (1 + |\bar{x}|)$, $\mathit{ClPrefix}^s$ can be built up from $\tilde{f}_0(\bar{x})$ with $\#\mathcal{O}(\tilde{f}_0(\bar{x})) \in O(|\mathit{ClPrefix}^s|) = O(1 + |\bar{x}|)$; we have, for every s and i, by the induction hypothesis (h) for θ_i, $|S_i^{s_i^s}| \leq 15 \cdot |\theta_i| \cdot (1 + |\bar{x}|)$, $S_i^{s_i^s}$ can be built up from θ_i and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ_i with $\#\mathcal{O}(\theta_i, \tilde{f}_0(\bar{x})) \in O(|\theta_i| \cdot (1 + |\bar{x}|))$; $|\theta| = 1 + |\theta_1| + |\theta_2|$, for both s, $|S^s| \overset{(74)}{=\!=} |\mathit{ClPrefix}^s| + |S_1^{s_1^s}| + |S_2^{s_2^s}| \leq 15 \cdot (1 + |\bar{x}|) + 15 \cdot |\theta_1| \cdot (1 + |\bar{x}|) + 15 \cdot |\theta_2| \cdot (1 + |\bar{x}|) = 15 \cdot (1 + |\theta_1| + |\theta_2|) \cdot (1 + |\bar{x}|) = 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$, S^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ with $\#\mathcal{O}(\theta, \tilde{f}_0(\bar{x})) \in O(1 + |\bar{x}| + |\theta_1| \cdot (1 + |\bar{x}|) + |\theta_2| \cdot (1 + |\bar{x}|)) = O((1 + |\theta_1| + |\theta_2|) \cdot (1 + |\bar{x}|)) = O(|\theta| \cdot (1 + |\bar{x}|))$; (h) holds.

Concerning Tables 3 and 4, for every form of θ, for both s, $\mathit{Prefix}^s \overset{(75)}{=\!=} \bigwedge_{i \leq n^s} \bigvee_{j \leq m_i^s} l_{j,i}^s$, $l_{j,i}^s \in \mathit{Lit}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}}$, for all $i \leq n^s$, $\bigvee_{j \leq m_i^s} l_{j,i}^s \neq \tilde{p}_{\mathbb{i}}(\bar{x})$ is a factor,

$$\tilde{p}_{\mathbb{i}} \in \mathit{preds}(\bigvee_{j \leq m_i^s} l_{j,i}^s),$$

$$\emptyset \neq \mathit{preds}(\bigvee_{j \leq m_i^s} l_{j,i}^s) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2 \overset{(72)}{=\!=} J\},$$

for all $i < i' \leq n^s$, $\mathit{lits}(\bigvee_{j \leq m_i^s} l_{j,i}^s) \neq \mathit{lits}(\bigvee_{j \leq m_{i'}^s} l_{j,i'}^s)$; for every s and i, by the induction hypothesis (i) for θ_i, $\psi_i^{s_i^s} = \bigwedge_{k \leq n_{\psi_i^{s_i^s}}} D_{k,i}^s$, for all $k \leq n_{\psi_i^{s_i^s}}$, $D_{k,i}^s \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}}$ is a factor, $\emptyset \neq \mathit{preds}(D_{k,i}^s) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2 \overset{(72)}{=\!=} J\}$, for all $k < k' \leq n_{\psi_i^{s_i^s}}$, $\mathit{lits}(D_{k,i}^s) \neq \mathit{lits}(D_{k',i}^s)$; $\tilde{p}_{\mathbb{i}} \underset{(73)}{\notin} \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}$, for all $k \leq n_{\psi_i^{s_i^s}}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(D_{k,i}^s) \subseteq \mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}}$, $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(\tilde{p}_{\mathbb{i}}(\bar{x}))$, $\mathit{preds}(D_{k,i}^s) \neq \mathit{preds}(\tilde{p}_{\mathbb{i}}(\bar{x}))$, $D_{k,i}^s \neq \tilde{p}_{\mathbb{i}}(\bar{x})$. We put $n_{\psi^s} = n^s + n_{\psi_1^{s_1^s}} + n_{\psi_2^{s_2^s}} + 2$ and

$$D_i^s = \begin{cases} \bigvee_{j \leq m_i^s} l_{j,i}^s & \mathit{if}\ i \leq n^s, \\ D_{i-(n^s+1),1}^s & \mathit{if}\ n^s + 1 \leq i \leq n^s + n_{\psi_1^{s_1^s}} + 1, \\ D_{i-(n^s+n_{\psi_1^{s_1^s}}+2),2}^s & \mathit{if}\ n^s + n_{\psi_1^{s_1^s}} + 2 \leq i \leq n^s + n_{\psi_1^{s_1^s}} + n_{\psi_2^{s_2^s}} + 2, \end{cases} \qquad s = +, -.$$

Then, for both s, $\psi^s \overset{(75)}{=\!=} \bigwedge_{i \leq n^s} (\bigvee_{j \leq m_i^s} l_{j,i}^s) \wedge \bigwedge_{k \leq n_{\psi_1^{s_1^s}}} D_{k,1}^s \wedge \bigwedge_{k \leq n_{\psi_2^{s_2^s}}} D_{k,2}^s = \bigwedge_{i \leq n_{\psi^s}} D_i^s$, for all $i \leq n_{\psi^s}$, $D_i^s \neq \tilde{p}_{\mathbb{i}}(\bar{x})$ is a factor, $\emptyset \neq \mathit{preds}(D_i^s) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$; we have, for all $i \leq n^s$, $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(\bigvee_{j \leq m_i^s} l_{j,i}^s)$, for both i, for all $k \leq n_{\psi_i^{s_i^s}}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(D_{k,i}^s)$, $\emptyset \neq \mathit{preds}(D_{k,i}^s) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}$; for all $k_0 \leq n^s$, $k_1 \leq n_{\psi_1^{s_1^s}}$, and $k_2 \leq n_{\psi_2^{s_2^s}}$, $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(\bigvee_{j \leq m_{k_0}^s} l_{j,k_0}^s)$, for both i, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(D_{k_i,i}^s)$, $\mathit{preds}(\bigvee_{j \leq m_{k_0}^s} l_{j,k_0}^s) \neq \mathit{preds}(D_{k_i,i}^s)$, $\mathit{lits}(\bigvee_{j \leq m_{k_0}^s} l_{j,k_0}^s) \neq \mathit{lits}(D_{k_i,i}^s)$; $(\mathit{preds}(D_{k_1,1}^s) \cap \tilde{\mathbb{P}}) \cap (\mathit{preds}(D_{k_2,2}^s) \cap \tilde{\mathbb{P}}) \subseteq (\{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}) \cap (\{\tilde{p}_{\mathbb{i}_2}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_2\}) \overset{(73)}{=\!=} \emptyset$, for both i, $\emptyset \neq \mathit{preds}(D_{k_i,i}^s) \cap \tilde{\mathbb{P}}$, $\mathit{preds}(D_{k_1,1}^s) \cap \tilde{\mathbb{P}} \neq \mathit{preds}(D_{k_2,2}^s) \cap \tilde{\mathbb{P}}$, $\mathit{preds}(D_{k_1,1}^s) \neq \mathit{preds}(D_{k_2,2}^s)$, $\mathit{lits}(D_{k_1,1}^s) \neq \mathit{lits}(D_{k_2,2}^s)$; for all $i < i' \leq n_{\psi^s}$, $\mathit{lits}(D_i^s) \neq \mathit{lits}(D_{i'}^s)$; (i) holds.

Concerning Tables 3 and 4, for every form of θ, for both s, for all $C \in \mathit{ClPrefix}^s$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2 \overset{(72)}{=\!=} J\}$, $\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1} \notin \mathit{ClPrefix}^s$; for every s and i, by the induction hypothesis (j) for θ_i, for all $C \in S_i^{s_i^s}$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2 \overset{(72)}{=\!=} J\}$; $\tilde{p}_{\mathbb{i}} \underset{(73)}{\notin} \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}$, $\tilde{p}_{\mathbb{i}} \in \tilde{\mathbb{P}}$, for all $C \in S_i^{s_i^s}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(C)$, for both $\diamond \in \{\eqcirc, \prec\}$, $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(\tilde{p}_{\mathbb{i}}(\bar{x}) \diamond \mathit{1})$, $\mathit{preds}(C) \neq \mathit{preds}(\tilde{p}_{\mathbb{i}}(\bar{x}) \diamond \mathit{1})$, $C \neq \tilde{p}_{\mathbb{i}}(\bar{x}) \diamond \mathit{1}$; $\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1} \notin S_i^{s_i^s}$; for all $C \in S^s$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$, $\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1} \notin S^s$; (j) holds.

Concerning Tables 3 and 4, for every form of θ, for both s, trivially, for all $a \in \emptyset = \mathit{qatoms}(\mathit{Prefix}^s)$, there exists $\mathbb{j}^* \in J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; for every s and i, by the induction hypothesis (k) for θ_i, for all $a \in \mathit{qatoms}(\psi_i^{s^s_i})$, there exists $\mathbb{j}^* \in J_i \underset{(72)}{\subseteq} J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; for both s, for all $a \in \mathit{qatoms}(\psi^s) = \mathit{qatoms}(\mathit{Prefix}^s) \cup \mathit{qatoms}(\psi_1^{s^s_1}) \cup \mathit{qatoms}(\psi_2^{s^s_2})$, there exists $\mathbb{j}^* \in J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; (k) holds.

Concerning Tables 3 and 4, for every form of θ, for both s, for all $\mathbb{j} \in \{\mathbb{i}\} \cup \{\mathbb{i}_1\} \cup \{\mathbb{i}_2\}$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\mathit{Prefix}^s)$ satisfying, for all $a \in \mathit{atoms}(\mathit{Prefix}^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\mathit{qatoms}(\mathit{Prefix}^s) = \emptyset$, $\tilde{p}_{\mathbb{i}} \notin \emptyset = \mathit{preds}(\mathit{qatoms}(\mathit{Prefix}^s))$; we have, for every s and i, by the induction hypothesis (l) for θ_i, for all $\mathbb{j} \in \{\mathbb{i}_i\} \cup J_i$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\psi_i^{s^s_i})$ satisfying, for all $a \in \mathit{atoms}(\psi_i^{s^s_i})$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}_i} \notin \mathit{preds}(\mathit{qatoms}(\psi_i^{s^s_i}))$, for all $\mathbb{j} \in J_i$, if there exists $a^* \in \mathit{qatoms}(\psi_i^{s^s_i})$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(\psi_i^{s^s_i})$ satisfying, for all $a \in \mathit{qatoms}(\psi_i^{s^s_i})$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; we have, for every s and i, $\psi_i^{s^s_i} \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}}$; for every s and i, $\tilde{p}_{\mathbb{i}} \underset{(73)}{\notin} \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\psi_i^{s^s_i}) \subseteq \mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}}$; we have, for both s, $\mathit{Prefix}^s \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}}$; for every s and i, $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\} \cap (\{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}) \overset{(73)}{=\!=} \emptyset$, $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\} \cap \mathit{preds}(\mathit{Prefix}^s) \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\} \cap \mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}} = \emptyset$, $(\mathit{preds}(\psi_1^{s^s_1}) \cap \tilde{\mathbb{P}}) \cap (\mathit{preds}(\psi_2^{s^s_2}) \cap \tilde{\mathbb{P}}) \subseteq (\mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}} \cap \tilde{\mathbb{P}}) \cap (\mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_2}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_2\}} \cap \tilde{\mathbb{P}}) = (\{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}) \cap (\{\tilde{p}_{\mathbb{i}_2}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_2\}) \overset{(73)}{=\!=} \emptyset$; $(q)\mathit{atoms}(\psi^s) = (q)\mathit{atoms}(\mathit{Prefix}^s) \cup (q)\mathit{atoms}(\psi_1^{s^s_1}) \cup (q)\mathit{atoms}(\psi_2^{s^s_2})$; $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\mathit{qatoms}(\psi_i^{s^s_i})) \subseteq \mathit{preds}(\psi_i^{s^s_i})$,

$$\begin{gathered}\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\mathit{qatoms}(\mathit{Prefix}^s)) \cup \mathit{preds}(\mathit{qatoms}(\psi_1^{s^s_1})) \cup \mathit{preds}(\mathit{qatoms}(\psi_2^{s^s_2})) = \\ \mathit{preds}(\mathit{qatoms}(\mathit{Prefix}^s) \cup \mathit{qatoms}(\psi_1^{s^s_1}) \cup \mathit{qatoms}(\psi_2^{s^s_2})) = \mathit{preds}(\mathit{qatoms}(\psi^s)).\end{gathered}$$

Let $\mathbb{j} \in \{\mathbb{i}\} \cup J \overset{(72)}{=\!=} \{\mathbb{i}\} \cup \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2$. We distinguish three cases for $\mathbb{j}$.

Case 2.1.1: $\mathbb{j} = \mathbb{i}$. Then, for both s, $\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{atoms}(\mathit{Prefix}^s) \subseteq \mathit{atoms}(\psi^s)$; we have, for every s and i, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\psi_i^{s^s_i})$; for both s, for all $a \in \mathit{atoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}}\}$, for both i, $a \notin \mathit{atoms}(\psi_i^{s^s_i})$, $a \in \mathit{atoms}(\mathit{Prefix}^s)$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; (l) holds.

Case 2.1.2: $\mathbb{j} = \mathbb{i}_i$, $i = 1, 2$. Then, for both s, $\tilde{p}_{\mathbb{i}_i}(\bar{x}) \in \mathit{atoms}(\mathit{Prefix}^s)$, $\mathit{atoms}(\psi_i^{s^s_i}) \subseteq \mathit{atoms}(\psi^s)$, $\tilde{p}_{\mathbb{i}_i} \in \mathit{preds}(\psi_i^{s^s_i})$, $\tilde{p}_{\mathbb{i}_i} \in \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{i}_i} \in \mathit{preds}(\psi_i^{s^s_i}) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{i}_i} \notin \mathit{preds}(\psi_{3-i}^{s^s_{3-i}}) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{i}_i} \notin \mathit{preds}(\psi_{3-i}^{s^s_{3-i}})$; for all $a \in \mathit{atoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_i}\}$, $a \notin \mathit{atoms}(\psi_{3-i}^{s^s_{3-i}})$, $a \in \mathit{atoms}(\mathit{Prefix}^s) \cup \mathit{atoms}(\psi_i^{s^s_i})$, for both the cases $a \in \mathit{atoms}(\mathit{Prefix}^s)$ and $a \in \mathit{atoms}(\psi_i^{s^s_i})$, $a = \tilde{p}_{\mathbb{i}_i}(\bar{x})$; $a = \tilde{p}_{\mathbb{i}_i}(\bar{x})$; we have, for every s and i, $\tilde{p}_{\mathbb{i}_i} \notin \mathit{preds}(\mathit{qatoms}(\psi_i^{s^s_i}))$; for both s, for all $a \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_i}\}$, $a \notin \emptyset = \mathit{qatoms}(\mathit{Prefix}^s)$, $a \notin \mathit{qatoms}(\psi_i^{s^s_i})$, $a \notin \mathit{qatoms}(\psi_{3-i}^{s^s_{3-i}})$, $a \notin \mathit{qatoms}(\psi^s)$; trivially, if there exists $a^* \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{i}_i}\}$, there exists $Qx\, \tilde{p}_{\mathbb{i}_i}(\bar{x}) \in \mathit{qatoms}(\psi^s)$ satisfying, for all $a \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_i}\}$, $a = Qx\, \tilde{p}_{\mathbb{i}_i}(\bar{x})$; (l) holds.

Case 2.1.3: $\mathbb{j} \in J_i$, $i = 1, 2$. Then, for both s, $\tilde{p}_{\mathbb{j}} \notin \mathit{preds}(\mathit{Prefix}^s)$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\psi_i^{s^s_i}) \subseteq \mathit{atoms}(\psi^s)$, $\tilde{p}_{\mathbb{j}} \in \mathit{preds}(\psi_i^{s^s_i})$, $\tilde{p}_{\mathbb{j}} \in \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \in \mathit{preds}(\psi_i^{s^s_i}) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \notin \mathit{preds}(\psi_{3-i}^{s^s_{3-i}}) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \notin \mathit{preds}(\psi_{3-i}^{s^s_{3-i}})$; for all $a \in \mathit{atoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \notin \mathit{atoms}(\mathit{Prefix}^s)$, $a \notin \mathit{atoms}(\psi_{3-i}^{s^s_{3-i}})$, $a \in \mathit{atoms}(\psi_i^{s^s_i})$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; if there exists $a^* \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, $a^* \notin \emptyset = \mathit{qatoms}(\mathit{Prefix}^s)$, $a^* \notin \mathit{qatoms}(\psi_{3-i}^{s^s_{3-i}})$, $a^* \in \mathit{qatoms}(\psi_i^{s^s_i})$, there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(\psi_i^{s^s_i}) \subseteq \mathit{qatoms}(\psi^s)$, for all $a \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \notin \emptyset = \mathit{qatoms}(\mathit{Prefix}^s)$, $a \notin \mathit{qatoms}(\psi_{3-i}^{s^s_{3-i}})$, $a \in \mathit{qatoms}(\psi_i^{s^s_i})$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; (l) holds.

So, in all Cases 2.1.1–2.1.3, (l) holds; (l) holds.

Concerning Tables 3 and 4, for every form of θ, for both s, trivially, for all $a \in \emptyset = \mathit{qatoms}(\mathit{ClPrefix}^s)$, there exists $\mathbb{j}^* \in J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; for every s and i, by the induction hypothesis (m) for θ_i, for all $a \in \mathit{qatoms}(S_i^{s^s_i})$, there exists $\mathbb{j}^* \in J_i \underset{(72)}{\subseteq} J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; for both s, for all $a \in \mathit{qatoms}(S^s) = \mathit{qatoms}(\mathit{ClPrefix}^s) \cup \mathit{qatoms}(S_1^{s^s_1}) \cup \mathit{qatoms}(S_2^{s^s_2})$, there exists $\mathbb{j}^* \in J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; (m) holds.

Concerning Tables 3 and 4, for every form of θ, for both s, for all $\mathbb{j} \in \{\mathbb{i}\} \cup \{\mathbb{i}_1\} \cup \{\mathbb{i}_2\}$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\mathit{ClPrefix}^s)$ satisfying, for all $a \in \mathit{atoms}(\mathit{ClPrefix}^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\mathit{qatoms}(\mathit{ClPrefix}^s) = \emptyset$, $\tilde{p}_{\mathbb{i}} \notin \emptyset = \mathit{preds}(\mathit{qatoms}(\mathit{ClPrefix}^s))$; we have, for every s and i, by the induction hypothesis (n) for θ_i, for all $\mathbb{j} \in \{\mathbb{i}_i\} \cup J_i$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(S_i^{s^s_i})$ satisfying, for all $a \in \mathit{atoms}(S_i^{s^s_i})$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}_i} \notin \mathit{preds}(\mathit{qatoms}(S_i^{s^s_i}))$, for all $\mathbb{j} \in J_i$, if there exists $a^* \in \mathit{qatoms}(S_i^{s^s_i})$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S_i^{s^s_i})$ satisfying, for all $a \in \mathit{qatoms}(S_i^{s^s_i})$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; we have, for every s and i, $S_i^{s^s_i} \subseteq \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}}$; for every s and i, $\tilde{p}_{\mathbb{i}} \underset{(73)}{\notin} \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(S_i^{s^s_i}) \subseteq \mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_i}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\}}$; we have, for both s, $\mathit{ClPrefix}^s \subseteq \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}}$; for every s and i, $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\} \cap (\{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}) \overset{(73)}{=\!=} \emptyset$, $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\} \cap \mathit{preds}(\mathit{ClPrefix}^s) \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_i\} \cap \mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}} = \emptyset$, $(\mathit{preds}(S_1^{s^s_1}) \cap \tilde{\mathbb{P}}) \cap (\mathit{preds}(S_2^{s^s_2}) \cap \tilde{\mathbb{P}}) \subseteq (\mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}} \cap \tilde{\mathbb{P}}) \cap (\mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_2}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_2\}} \cap \tilde{\mathbb{P}}) = (\{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}) \cap (\{\tilde{p}_{\mathbb{i}_2}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_2\}) \overset{(73)}{=\!=} \emptyset$; $(q)\mathit{atoms}(S^s) = (q)\mathit{atoms}(\mathit{ClPrefix}^s) \cup (q)\mathit{atoms}(S_1^{s^s_1}) \cup (q)\mathit{atoms}(S_2^{s^s_2})$; $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\mathit{qatoms}(S_i^{s^s_i})) \subseteq \mathit{preds}(S_i^{s^s_i})$,

$$\begin{gathered}\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\mathit{qatoms}(\mathit{ClPrefix}^s)) \cup \mathit{preds}(\mathit{qatoms}(S_1^{s^s_1})) \cup \mathit{preds}(\mathit{qatoms}(S_2^{s^s_2})) = \\ \mathit{preds}(\mathit{qatoms}(\mathit{ClPrefix}^s) \cup \mathit{qatoms}(S_1^{s^s_1}) \cup \mathit{qatoms}(S_2^{s^s_2})) = \mathit{preds}(\mathit{qatoms}(S^s)).\end{gathered}$$

Let $\mathbb{j} \in \{\mathbb{i}\} \cup J \overset{(72)}{=\!=} \{\mathbb{i}\} \cup \{\mathbb{i}_1\} \cup J_1 \cup \{\mathbb{i}_2\} \cup J_2$. We distinguish three cases for $\mathbb{j}$.

Case 2.1.4: $\mathbb{j} = \mathbb{i}$. Then, for both s, $\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{atoms}(\mathit{ClPrefix}^s) \subseteq \mathit{atoms}(S^s)$; we have, for every s and i, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(S_i^{s_i^s})$; for both s, for all $a \in \mathit{atoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}}\}$, for both i, $a \notin \mathit{atoms}(S_i^{s_i^s})$, $a \in \mathit{atoms}(\mathit{ClPrefix}^s)$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; (n) holds.

Case 2.1.5: $\mathbb{j} = \mathbb{i}_i$, $i = 1, 2$. Then, for both s, $\tilde{p}_{\mathbb{i}_i}(\bar{x}) \in \mathit{atoms}(\mathit{ClPrefix}^s), \mathit{atoms}(S_i^{s_i^s}) \subseteq \mathit{atoms}(S^s)$, $\tilde{p}_{\mathbb{i}_i} \in \mathit{preds}(S_i^{s_i^s})$, $\tilde{p}_{\mathbb{i}_i} \in \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{i}_i} \in \mathit{preds}(S_i^{s_i^s}) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{i}_i} \notin \mathit{preds}(S_{3-i}^{s_{3-i}^s}) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{i}_i} \notin \mathit{preds}(S_{3-i}^{s_{3-i}^s})$; for all $a \in \mathit{atoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_i}\}$, $a \notin \mathit{atoms}(S_{3-i}^{s_{3-i}^s})$, $a \in \mathit{atoms}(\mathit{ClPrefix}^s) \cup \mathit{atoms}(S_i^{s_i^s})$, for both the cases $a \in \mathit{atoms}(\mathit{ClPrefix}^s)$ and $a \in \mathit{atoms}(S_i^{s_i^s})$, $a = \tilde{p}_{\mathbb{i}_i}(\bar{x})$; $a = \tilde{p}_{\mathbb{i}_i}(\bar{x})$; we have, for every s and i, $\tilde{p}_{\mathbb{i}_i} \notin \mathit{preds}(\mathit{qatoms}(S_i^{s_i^s}))$; for both s, for all $a \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_i}\}$, $a \notin \emptyset = \mathit{qatoms}(\mathit{ClPrefix}^s)$, $a \notin \mathit{qatoms}(S_i^{s_i^s})$, $a \notin \mathit{qatoms}(S_{3-i}^{s_{3-i}^s})$, $a \notin \mathit{qatoms}(S^s)$; trivially, if there exists $a^* \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{i}_i}\}$, there exists $Qx\, \tilde{p}_{\mathbb{i}_i}(\bar{x}) \in \mathit{qatoms}(S^s)$ satisfying, for all $a \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_i}\}$, $a = Qx\, \tilde{p}_{\mathbb{i}_i}(\bar{x})$; (n) holds.

Case 2.1.6: $\mathbb{j} \in J_i$, $i = 1, 2$. Then, for both s, $\tilde{p}_{\mathbb{j}} \notin \mathit{preds}(\mathit{ClPrefix}^s)$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(S_i^{s_i^s}) \subseteq \mathit{atoms}(S^s)$, $\tilde{p}_{\mathbb{j}} \in \mathit{preds}(S_i^{s_i^s})$, $\tilde{p}_{\mathbb{j}} \in \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \in \mathit{preds}(S_i^{s_i^s}) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \notin \mathit{preds}(S_{3-i}^{s_{3-i}^s}) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \notin \mathit{preds}(S_{3-i}^{s_{3-i}^s})$; for all $a \in \mathit{atoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \notin \mathit{atoms}(\mathit{ClPrefix}^s)$, $a \notin \mathit{atoms}(S_{3-i}^{s_{3-i}^s})$, $a \in \mathit{atoms}(S_i^{s_i^s})$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; if there exists $a^* \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, $a^* \notin \emptyset = \mathit{qatoms}(\mathit{ClPrefix}^s)$, $a^* \notin \mathit{qatoms}(S_{3-i}^{s_{3-i}^s})$, $a^* \in \mathit{qatoms}(S_i^{s_i^s})$, there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S_i^{s_i^s}) \subseteq \mathit{qatoms}(S^s)$, for all $a \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \notin \emptyset = \mathit{qatoms}(\mathit{ClPrefix}^s)$, $a \notin \mathit{qatoms}(S_{3-i}^{s_{3-i}^s})$, $a \in \mathit{qatoms}(S_i^{s_i^s})$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; (n) holds.

So, in all Cases 2.1.4–2.1.6, (n) holds; (n) holds.

$\tilde{p}_{\mathbb{i}}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_2}(\bar{x}) \in \mathit{Atom}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{i}_2}\}} - \mathit{Tcons}_{\mathcal{L}}$, concerning Tables 3 and 4, for every form of θ, for both s, $\mathit{tcons}(\mathit{Prefix}^s) = \mathit{tcons}(\mathit{ClPrefix}^s) = \{\mathit{0}, \mathit{1}\}$; we have, for every s and i, by the induction hypothesis (o) for θ_i, $\mathit{tcons}(\psi_i^{s_i^s}) = \mathit{tcons}(S_i^{s_i^s}) = \mathit{tcons}(\theta_i)$; for both s, $\mathit{tcons}(\psi^s) = \mathit{tcons}(\mathit{Prefix}^s) \cup \mathit{tcons}(\psi_1^{s_1^s}) \cup \mathit{tcons}(\psi_2^{s_2^s}) = \{\mathit{0}, \mathit{1}\} \cup \mathit{tcons}(\theta_1) \cup \mathit{tcons}(\theta_2) = \mathit{tcons}(\theta)$, $\mathit{tcons}(S^s) = \mathit{tcons}(\mathit{ClPrefix}^s) \cup \mathit{tcons}(S_1^{s_1^s}) \cup \mathit{tcons}(S_2^{s_2^s}) = \{\mathit{0}, \mathit{1}\} \cup \mathit{tcons}(\theta_1) \cup \mathit{tcons}(\theta_2) = \mathit{tcons}(\theta)$; (o) holds.

Case 2.2 (the unary interpolation case): Either $\theta = \theta_1 \rightarrow \mathit{0}$, $\theta_1 \in \mathit{Form}_{\mathcal{L}} - \{\mathit{0}, \mathit{1}\}$, or $\theta = Qx\, \theta_1$, $\theta_1 \in \mathit{Form}_{\mathcal{L}} - \mathit{Tcons}_{\mathcal{L}} \subseteq \mathit{Form}_{\mathcal{L}} - \{\mathit{0}, \mathit{1}\}$. We have (69c,d) hold for θ, $\mathit{vars}(\theta) \subseteq \mathit{vars}(\bar{x}) \subseteq \mathit{Var}_{\mathcal{L}}$. Then, for both the cases $\theta = \theta_1 \rightarrow \mathit{0}$ and $\theta = Qx\, \theta_1$, $\theta_1 \in \mathit{Form}_{\mathcal{L}} - \{\mathit{0}, \mathit{1}\}$, (69c,d) hold for θ_1; $\theta_1 \in \mathit{Form}_{\mathcal{L}} - \{\mathit{0}, \mathit{1}\}$, (69c,d) hold for θ_1, $\mathit{vars}(\theta_1) \subseteq \mathit{vars}(\theta) \subseteq \mathit{vars}(\bar{x}) \subseteq \mathit{Var}_{\mathcal{L}}$, $x \in \mathit{vars}(\theta) \subseteq \mathit{vars}(\bar{x}) \subseteq \mathit{Var}_{\mathcal{L}}$. We put $j_{\mathbb{i}_1} = j_{\mathbb{i}} + 1$ and $\mathbb{i}_1 = (n_\theta, j_{\mathbb{i}_1}) \in \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$. $\tilde{p}_{\mathbb{i}_1} \in \tilde{\mathbb{P}}$. We put $\mathit{ar}(\tilde{p}_{\mathbb{i}_1}) = |\bar{x}|$. We get by the induction hypothesis for n_θ, θ_1, $\bar{x}$, $\mathbb{i}_1$, $\tilde{p}_{\mathbb{i}_1}$ that there exist $J_1 = \{(n_\theta, j) \,|\, j_{\mathbb{i}_1} + 1 \leq j \leq n_{J_1}\} \subseteq \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$, $j_{\mathbb{i}_1} \leq n_{J_1}$, $\mathbb{i}_1 \notin J_1$, a *CNF* $\psi_1^s \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}}$, $S_1^s \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}}$, $s = +, -$, and for both s, (a–o) hold for θ_1, $\bar{x}$, $\tilde{p}_{\mathbb{i}_1}$, J_1, ψ_1^s, S_1^s. We put $n_J = n_{J_1}$ and $J = \{(n_\theta, j) \,|\, j_{\mathbb{i}} + 1 \leq j \leq n_J\} \subseteq \{(n_\theta, j) \,|\, j \in \mathbb{N}\}$. Then $j_{\mathbb{i}} < j_{\mathbb{i}_1} \leq n_J$, $\mathbb{i} \notin J$,

$$J = \{(n_\theta, j_{\mathbb{i}_1})\} \cup \{(n_\theta, j) \,|\, j_{\mathbb{i}_1} + 1 \leq j \leq n_{J_1}\} = \{\mathbb{i}_1\} \cup J_1, \tag{78}$$

$$\{\mathbb{i}\}, \{\mathbb{i}_1\}, J_1 \text{ are pairwise disjoint.} \tag{79}$$

In Tables 5 and 6, for every form of θ, positive and negative unary interpolation rules of the respective forms

$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}}{\mathit{Prefix}^+ \wedge \gamma_1^{s_1^+} \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}}}, \quad s_1^+ \in \{+, -\},$$

$$\frac{\theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}}{\mathit{Prefix}^- \wedge \gamma_1^{s_1^-} \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}}}, \quad s_1^- \in \{+, -\},$$

$$\mathit{Prefix}^s \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}},$$

$$\gamma_1^s = \begin{cases} \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1 & \mathit{if}\ s = +, \\ \theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x}) & \mathit{if}\ s = -, \end{cases} \quad \gamma_1^s \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\}},$$

are assigned. We put

$$\psi^s = \mathit{Prefix}^s \wedge \psi_1^{s_1^s} \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1 \overset{(78)}{=\!=} J\}}, \quad s = +, -.$$

Then, concerning Tables 5 and 6, for every form of θ, for both s, $\mathit{Prefix}^s \neq \mathit{0}, \mathit{1}$, Prefix^s is a *CNF*, for both s, by the induction hypothesis (i) for θ_1, $\psi_1^{s_1^s} \neq \mathit{0}, \mathit{1}$; we have, for both s, $\psi_1^{s_1^s}$ is a *CNF*;

for both s, ψ^s is a *CNF*. In Tables 5 and 6, for every form of θ, positive and negative unary interpolation rules of the respective forms

$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \in \mathit{Form}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}}}{\mathit{ClPrefix}^{+} \cup \{\gamma_1^{s_1^+}\} \subseteq_{\mathcal{F}} \mathit{OrdForm}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{i}_1}\}}}, \quad s_1^+ \in \{+,-\},$$

$$\frac{\theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{Form}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}}}{\mathit{ClPrefix}^{-} \cup \{\gamma_1^{s_1^-}\} \subseteq_{\mathcal{F}} \mathit{OrdForm}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{i}_1}\}}}, \quad s_1^- \in \{+,-\},$$

$$\mathit{ClPrefix}^s \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{i}_1}\}},$$

are assigned. We put

$$S^s = \mathit{ClPrefix}^s \cup S_1^{s_1^s} \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in\{\mathbb{i}_1\}\cup J_1 \overset{(78)}{=} J\}}, \quad s = +,-.$$

Concerning Tables 5 and 6, for every form of θ, for both s, for all $C \in \mathit{ClPrefix}^s$, $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(C)$; we have, for both s, by the induction hypothesis (j) for θ_1, for all $C \in S_1^{s_1^s}$, $\mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j} \in J_1\}$; for both s, for all $C_0 \in \mathit{ClPrefix}^s$ and $C_1 \in S_1^{s_1^s}$, $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(C_0)$, $\tilde{p}_{\mathbb{i}} \in \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{i}} \underset{(79)}{\notin} \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j} \in J_1\}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(C_1) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j} \in J_1\}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(C_1)$, $\mathit{preds}(C_0) \neq \mathit{preds}(C_1)$, $C_0 \neq C_1$;

$$\text{for both } s,\ \mathit{ClPrefix}^s \cap S_1^{s_1^s} = \emptyset. \tag{80}$$

For both the cases $\theta = \theta_1 \rightarrow 0$ and $\theta = Qx\,\theta_1$, $|\theta| = 2 + |\theta_1|$; $|\theta| = 2 + |\theta_1|$, by the induction hypothesis (a) for θ_1, $\|J\| \underset{(79)}{\overset{(78)}{=}} \|\{\mathbb{i}_1\}\| + \|J_1\| = 1 + \|J_1\| \leq 1 + |\theta_1| - 1 = |\theta| - 2 \leq |\theta| - 1$; (a) holds.

Concerning Tables 5 and 6, for every form of θ, for both s,

$$\mathit{Prefix}^s = \bigvee_{j \leq m_0^s} l_{j,0}^s, \quad l_{j,0}^s \in \mathit{Lit}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{i}_1}\}}, \tag{81}$$

$$\mathit{ClPrefix}^s = \bigvee_{j \leq m_0^s} C_{j,0}^s \tag{82}$$

so that for all $j \leq m_0^s$, by (70) for $l_{j,0}^s$, $C_{j,0}^s \in \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{i}_1}\}}$ satisfying, for every interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}$, for all $e \in \mathcal{S}_{\mathfrak{A}}$, $\mathfrak{A} \models_e l_{j,0}^s$ if and only if $\mathfrak{A} \models_e C_{j,0}^s$; $\mathfrak{A} \models_e \bigvee_{j \leq m_0^s} l_{j,0}^s$ if and only if $\mathfrak{A} \models_e \bigvee_{j \leq m_0^s} C_{j,0}^s$; $\mathfrak{A} \models \bigvee_{j \leq m_0^s} l_{j,0}^s$ if and only if $\mathfrak{A} \models \bigvee_{j \leq m_0^s} C_{j,0}^s$;

$$\text{for every form of } \theta,\text{ for both } s,\text{ for every interpretation } \mathfrak{A} \text{ for } \mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\},\ \mathfrak{A} \models \mathit{Prefix}^s \text{ if and only if } \mathfrak{A} \models \mathit{ClPrefix}^s. \tag{83}$$

Let $\mathfrak{A}$ be an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$. We define an expansion $\mathfrak{A}^{\#}$ of $\mathfrak{A}$ to $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}$ as follows:

$$\tilde{p}_{\mathbb{i}_1}^{\mathfrak{A}^{\#}}(u_1, \dots, u_{|\bar{x}|}) = \begin{cases} \|\theta_1\|_e^{\mathfrak{A}} & \text{if there exists } e \in \mathcal{S}_{\mathfrak{A}} \text{ such that } \|\bar{x}\|_e^{\mathfrak{A}} = u_1, \dots, u_{|\bar{x}|}, \\ 0 & \text{else.} \end{cases}$$

Then, for all $e \in \mathcal{S}_{\mathfrak{A}^{\#}}$, $\|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = \tilde{p}_{\mathbb{i}_1}^{\mathfrak{A}^{\#}}(\|\bar{x}\|_e^{\mathfrak{A}^{\#}}) = \tilde{p}_{\mathbb{i}_1}^{\mathfrak{A}^{\#}}(\|\bar{x}\|_e^{\mathfrak{A}}) = \|\theta_1\|_e^{\mathfrak{A}} = \|\theta_1\|_e^{\mathfrak{A}^{\#}}$,

$$\|\gamma_1^+\|_e^{\mathfrak{A}^{\#}} \mid \|\gamma_1^-\|_e^{\mathfrak{A}^{\#}} = \|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow \|\theta_1\|_e^{\mathfrak{A}^{\#}} \mid \|\theta_1\|_e^{\mathfrak{A}^{\#}} \Rightarrow \|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = \|\theta_1\|_e^{\mathfrak{A}^{\#}} \Rightarrow \|\theta_1\|_e^{\mathfrak{A}^{\#}} = 1;$$

for both s, $\mathfrak{A}^{\#} \models \gamma_1^s$.

Let $s = +$, $\mathfrak{A} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \bigm| s = -$, $\mathfrak{A} \models \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})$. Then $\mathfrak{A}^{\#} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \bigm| \mathfrak{A}^{\#} \models \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})$, for all $e \in \mathcal{S}_{\mathfrak{A}^{\#}}$,

$$\|\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta\|_e^{\mathfrak{A}^{\#}} = 1 \bigm| \|\theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1,$$

$$\|\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow (\theta_1 \rightarrow \mathit{0})\|_e^{\mathfrak{A}^{\#}} = 1 \mid \|\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow Qx\,\theta_1\|_e^{\mathfrak{A}^{\#}} = 1 \bigm|$$
$$\|(\theta_1 \rightarrow \mathit{0}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1 \mid \|Qx\,\theta_1 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1,$$

$$\|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow (\|\theta_1\|_e^{\mathfrak{A}^{\#}} \Rightarrow 0) = 1 \mid \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow \mathop{\mathbf{X}}_{u \in \mathcal{U}_{\mathfrak{A}^{\#}}} \|\theta_1\|_{e[x/u]}^{\mathfrak{A}^{\#}} = 1 \bigm|$$
$$(\|\theta_1\|_e^{\mathfrak{A}^{\#}} \Rightarrow 0) \Rightarrow \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1 \mid \mathop{\mathbf{X}}_{u \in \mathcal{U}_{\mathfrak{A}^{\#}}} \|\theta_1\|_{e[x/u]}^{\mathfrak{A}^{\#}} \Rightarrow \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1,$$

$$\|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow (\|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow 0) = 1 \mid \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow \mathop{\mathbf{X}}_{u \in \mathcal{U}_{\mathfrak{A}^{\#}}} \|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_{e[x/u]}^{\mathfrak{A}^{\#}} = 1 \bigm|$$
$$(\|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow 0) \Rightarrow \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1 \mid \mathop{\mathbf{X}}_{u \in \mathcal{U}_{\mathfrak{A}^{\#}}} \|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_{e[x/u]}^{\mathfrak{A}^{\#}} \Rightarrow \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1,$$

$$\|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow (\|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow 0) = 1 \mid \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow \|Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1 \bigm|$$
$$(\|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow 0) \Rightarrow \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1 \mid \|Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1, \quad \mathbf{X} \in \Big\{\bigwedge, \bigvee\Big\},$$

concerning Tables 5 and 6, for every form of θ, by the corresponding laws in the column Laws, $\|\mathit{Prefix}^s\|_e^{\mathfrak{A}^{\#}} = 1$; $\mathfrak{A}^{\#} \models \mathit{Prefix}^s$, by (83) for θ, s, $\mathfrak{A}^{\#}$, $\mathfrak{A}^{\#} \models \mathit{ClPrefix}^s$; we have, for both s, $\mathfrak{A}^{\#} \models \gamma_1^s$; $\mathfrak{A}^{\#}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\}} \models \gamma_1^{s_1^s}$, by the induction hypothesis (b) | (c) $\bigm|$ (e) | (f) for θ_1, $\mathfrak{A}^{\#}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\}}$, there exists an interpretation $\mathfrak{A}_1^s$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}$ and $\mathfrak{A}_1^s \models \psi_1^{s_1^s} \bigm| S_1^{s_1^s}$, $\mathfrak{A}_1^s|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\}} = \mathfrak{A}^{\#}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\}}$. By (79), $\{\tilde{p}_{\mathbb{i}}\}$, $\{\tilde{p}_{\mathbb{i}_1}\}$, $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}$ are pairwise disjoint. We define an expansion $\mathfrak{A}^s$ of $\mathfrak{A}$ to $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J \overset{(78)}{=\!=} \{\mathbb{i}_1\} \cup J_1\}$ as follows:

$$\tilde{p}_{\mathbb{j}}^{\mathfrak{A}^s} = \tilde{p}_{\mathbb{j}}^{\mathfrak{A}_1^s}, \quad \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1.$$

We get $\mathfrak{A}^s|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}} = \mathfrak{A}^{\#} \models \mathit{Prefix}^s, \mathit{ClPrefix}^s$ and $\mathfrak{A}^s|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}} = \mathfrak{A}_1^s \models \psi_1^{s_1^s} \bigm| S_1^{s_1^s}$. We put $\mathfrak{A}' = \mathfrak{A}^s$, an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$. Then $\mathfrak{A}' \models \psi^s \bigm| S^s$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} = \mathfrak{A}$.

Let $s = + \mid s = -$ and $\mathfrak{A}'$ be an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J \overset{(78)}{=\!=} \{\mathbb{i}_1\} \cup J_1\}$ such that $\mathfrak{A}' \models \psi^s \bigm| S^s$. We denote $\mathfrak{A}^{\#} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}}$. Then $\mathfrak{A}^{\#} \models \mathit{Prefix}^s \bigm| \mathfrak{A}^{\#} \models \mathit{ClPrefix}^s$, by (83) for θ, s, $\mathfrak{A}^{\#}$, $\mathfrak{A}^{\#} \models \mathit{Prefix}^s$; $\mathfrak{A}^{\#} \models \mathit{Prefix}^s$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}} \models \psi_1^{s_1^s} \bigm| S_1^{s_1^s}$, by the induction hypothesis (b) | (c) $\bigm|$ (e) | (f) for θ_1, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}}$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\}} \models \gamma_1^{s_1^s}$, $\mathfrak{A}^{\#} \models \gamma_1^{s_1^s}$, for all $e \in \mathcal{S}_{\mathfrak{A}^{\#}}$, $\|\mathit{Prefix}^s\|_e^{\mathfrak{A}^{\#}} = 1$, $\|\gamma_1^{s_1^s}\|_e^{\mathfrak{A}^{\#}} = 1$; if $s_1^s = +$, then $1 = \|\gamma_1^{s_1^s}\|_e^{\mathfrak{A}^{\#}} = \|\gamma_1^+\|_e^{\mathfrak{A}^{\#}} = \|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \Rightarrow \|\theta_1\|_e^{\mathfrak{A}^{\#}}$, $\|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \leq \|\theta_1\|_e^{\mathfrak{A}^{\#}}$; if $s_1^s = -$, then $1 = \|\gamma_1^{s_1^s}\|_e^{\mathfrak{A}^{\#}} = \|\gamma_1^-\|_e^{\mathfrak{A}^{\#}} = \|\theta_1\|_e^{\mathfrak{A}^{\#}} \Rightarrow \|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}}$, $\|\theta_1\|_e^{\mathfrak{A}^{\#}} \leq \|\tilde{p}_{\mathbb{i}_1}(\bar{x})\|_e^{\mathfrak{A}^{\#}}$; concerning Tables 5 and 6, for every form of θ, for both s, by the corresponding laws in the column Laws, $\|\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow (\theta_1 \rightarrow \mathit{0})\|_e^{\mathfrak{A}^{\#}} = 1 \mid \|\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow Qx\,\theta_1\|_e^{\mathfrak{A}^{\#}} = 1 \mid \|(\theta_1 \rightarrow \mathit{0}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1 \mid \|Qx\,\theta_1 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1$, $\|\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta\|_e^{\mathfrak{A}^{\#}} = 1 \mid \|\theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = 1$; $\mathfrak{A}^{\#} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \mid \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} = \mathfrak{A}^{\#}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \mid \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})$. We put $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$, an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$. Then $\mathfrak{A} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \mid \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})$, $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$; (b), (c), (e), (f) hold.

Let $s = + \mid s = -$ and $\mathfrak{A}$ be an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J \overset{(78)}{=\!=} \{\mathbb{i}_1\} \cup J_1\}$ such that $\mathfrak{A} \models \psi^s$. We denote $\mathfrak{A}^{\#} = \mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}}$. Then $\mathfrak{A}^{\#} \models \mathit{Prefix}^s$, by (83) for θ, s, $\mathfrak{A}^{\#}$, $\mathfrak{A}^{\#} \models \mathit{ClPrefix}^s$, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}} \models \psi_1^{s_1^s}$, by the induction hypothesis (d) for θ_1, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}}$, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}} \models S_1^{s_1^s}$, $\mathfrak{A} \models \mathit{ClPrefix}^s, S_1^{s_1^s}$; $\mathfrak{A} \models S^s$.

Let $s = + \mid s = -$ and $\mathfrak{A}$ be an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J \overset{(78)}{=\!=} \{\mathbb{i}_1\} \cup J_1\}$ such that $\mathfrak{A} \models S^s$. We denote $\mathfrak{A}^{\#} = \mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}}$. Then $\mathfrak{A}^{\#} \models \mathit{ClPrefix}^s$, by (83) for θ, s, $\mathfrak{A}^{\#}$, $\mathfrak{A}^{\#} \models \mathit{Prefix}^s$, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}} \models S_1^{s_1^s}$, by the induction hypothesis (d) for θ_1, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}}$, $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}} \models \psi_1^{s_1^s}$, $\mathfrak{A} \models \mathit{Prefix}^s, \psi_1^{s_1^s}$; $\mathfrak{A} \models \psi^s$; (d) holds.

Concerning Tables 5 and 6, for every form of θ, for both s, $|\mathit{Prefix}^s| \leq 13 \cdot (1 + |\bar{x}|)$, Prefix^s can be built up from $\tilde{f}_0(\bar{x})$ with $\#\mathcal{O}(\tilde{f}_0(\bar{x})) \in O(|\mathit{Prefix}^s|) = O(1 + |\bar{x}|)$; we have, for both s, by the induction hypothesis (g) for θ_1, $|\psi_1^{s_1^s}| \leq 15 \cdot |\theta_1| \cdot (1 + |\bar{x}|)$, $\psi_1^{s_1^s}$ can be built up from θ_1 and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ_1 with $\#\mathcal{O}(\theta_1, \tilde{f}_0(\bar{x})) \in O(|\theta_1| \cdot (1 + |\bar{x}|))$; $|\theta| = 2 + |\theta_1|$, for both s, $|\psi^s| = 1 + |\mathit{Prefix}^s| + |\psi_1^{s_1^s}| \leq 1 + 13 \cdot (1 + |\bar{x}|) + 15 \cdot |\theta_1| \cdot (1 + |\bar{x}|) \leq 15 \cdot (2 + |\theta_1|) \cdot (1 + |\bar{x}|) = 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$, ψ^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ with $\#\mathcal{O}(\theta, \tilde{f}_0(\bar{x})) \in O(1 + |\bar{x}| + |\theta_1| \cdot (1 + |\bar{x}|)) = O((2 + |\theta_1|) \cdot (1 + |\bar{x}|)) = O(|\theta| \cdot (1 + |\bar{x}|))$; (g) holds.

Concerning Tables 5 and 6, for every form of θ, for both s, $|\mathit{ClPrefix}^s| \leq 15 \cdot (1 + |\bar{x}|)$, $\mathit{ClPrefix}^s$ can be built up from $\tilde{f}_0(\bar{x})$ with $\#\mathcal{O}(\tilde{f}_0(\bar{x})) \in O(|\mathit{ClPrefix}^s|) = O(1 + |\bar{x}|)$; we have, for both s, by the induction hypothesis (h) for θ_1, $|S_1^{s_1^s}| \leq 15 \cdot |\theta_1| \cdot (1 + |\bar{x}|)$, $S_1^{s_1^s}$ can be built up from θ_1 and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ_1 with $\#\mathcal{O}(\theta_1, \tilde{f}_0(\bar{x})) \in O(|\theta_1| \cdot (1 + |\bar{x}|))$; $|\theta| = 2 + |\theta_1|$, for both s, $|S^s| \overset{(80)}{=\!=} |\mathit{ClPrefix}^s| + |S_1^{s_1^s}| \leq 15 \cdot (1 + |\bar{x}|) + 15 \cdot |\theta_1| \cdot (1 + |\bar{x}|) \leq 15 \cdot (2 + |\theta_1|) \cdot (1 + |\bar{x}|) = 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$, S^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ with $\#\mathcal{O}(\theta, \tilde{f}_0(\bar{x})) \in O(1 + |\bar{x}| + |\theta_1| \cdot (1 + |\bar{x}|)) = O((2 + |\theta_1|) \cdot (1 + |\bar{x}|)) = O(|\theta| \cdot (1 + |\bar{x}|))$; (h) holds.

Concerning Tables 5 and 6, for every form of θ, for both s, $\mathit{Prefix}^s \overset{(81)}{=\!=} \bigvee_{j\leq m_0^s} l_{j,0}^s$, $l_{j,0}^s \in \mathit{Lit}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{i}_1}\}}$, $\bigvee_{j\leq m_0^s} l_{j,0}^s \neq \tilde{p}_{\mathbb{i}}(\bar{x})$ is a factor, $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(\bigvee_{j\leq m_0^s} l_{j,0}^s)$, $\emptyset \neq \mathit{preds}(\bigvee_{j\leq m_0^s} l_{j,0}^s) \cap \tilde{\mathbb{P}} = \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1 \overset{(78)}{=\!=} J\}$, trivially, for all $i < i' \leq 0$, $\mathit{lits}(\bigvee_{j\leq m_i^s} l_{j,i}^s) \neq \mathit{lits}(\bigvee_{j\leq m_{i'}^s} l_{j,i'}^s)$; for both s, by the induction hypothesis (i) for θ_1, $\psi_1^{s_{\mathbb{1}}^s} = \bigwedge_{k\leq n_{\psi_1^{s_{\mathbb{1}}^s}}} D_{k,1}^s$, for all $k \leq n_{\psi_1^{s_{\mathbb{1}}^s}}$, $D_{k,1}^s$ is a factor, $D_{k,1}^s \in \mathit{Form}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}_1}\}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J_1\}}$, $\emptyset \neq \mathit{preds}(D_{k,1}^s) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1 \overset{(78)}{=\!=} J\}$, for all $k < k' \leq n_{\psi_1^{s_{\mathbb{1}}^s}}$, $\mathit{lits}(D_{k,1}^s) \neq \mathit{lits}(D_{k',1}^s)$; $\tilde{p}_{\mathbb{i}} \underset{(79)}{\notin} \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}$, for all $k \leq n_{\psi_1^{s_{\mathbb{1}}^s}}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(D_{k,1}^s) \subseteq \mathit{Pred}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}_1}\}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J_1\}}$, $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(\tilde{p}_{\mathbb{i}}(\bar{x}))$, $\mathit{preds}(D_{k,1}^s) \neq \mathit{preds}(\tilde{p}_{\mathbb{i}}(\bar{x}))$, $D_{k,1}^s \neq \tilde{p}_{\mathbb{i}}(\bar{x})$. We put $n_{\psi^s} = n_{\psi_1^{s_{\mathbb{1}}^s}} + 1$ and

$$D_i^s = \begin{cases} \bigvee_{j\leq m_0^s} l_{j,0}^s & \mathit{if}\ i = 0, \\ D_{i-1,1}^s & \mathit{if}\ 1 \leq i \leq n_{\psi_1^{s_{\mathbb{1}}^s}} + 1, \end{cases} \qquad s = +, -.$$

Then, for both s, $\psi^s \overset{(81)}{=\!=} (\bigvee_{j\leq m_0^s} l_{j,0}^s) \wedge \bigwedge_{k\leq n_{\psi_1^{s_{\mathbb{1}}^s}}} D_{k,1}^s = \bigwedge_{i\leq n_{\psi^s}} D_i^s$, for all $i \leq n_{\psi^s}$, $D_i^s \neq \tilde{p}_{\mathbb{i}}(\bar{x})$ is a factor, $\emptyset \neq \mathit{preds}(D_i^s) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$; we have $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(\bigvee_{j\leq m_0^s} l_{j,0}^s)$, for all $k \leq n_{\psi_1^{s_{\mathbb{1}}^s}}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(D_{k,1}^s)$; for all $k \leq n_{\psi_1^{s_{\mathbb{1}}^s}}$, $\mathit{preds}(\bigvee_{j\leq m_0^s} l_{j,0}^s) \neq \mathit{preds}(D_{k,1}^s)$, $\mathit{lits}(\bigvee_{j\leq m_0^s} l_{j,0}^s) \neq \mathit{lits}(D_{k,1}^s)$; for all $i < i' \leq n_{\psi^s}$, $\mathit{lits}(D_i^s) \neq \mathit{lits}(D_{i'}^s)$; (i) holds.

Concerning Tables 5 and 6, for every form of θ, for both s, for all $C \in \mathit{ClPrefix}^s$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} = \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1 \overset{(78)}{=\!=} J\}$, $\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1} \notin \mathit{ClPrefix}^s$; for both s, by the induction hypothesis (j) for θ_1, for all $C \in S_1^{s_{\mathbb{1}}^s}$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}_1\} \cup J_1 \overset{(78)}{=\!=} J\}$; $\tilde{p}_{\mathbb{i}} \underset{(79)}{\notin} \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}$, $\tilde{p}_{\mathbb{i}} \in \tilde{\mathbb{P}}$, for all $C \in S_1^{s_{\mathbb{1}}^s}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(C)$, for both $\diamond \in \{\eqcirc, \prec\}$, $\tilde{p}_{\mathbb{i}} \in \mathit{preds}(\tilde{p}_{\mathbb{i}}(\bar{x}) \diamond \mathit{1})$, $\mathit{preds}(C) \neq \mathit{preds}(\tilde{p}_{\mathbb{i}}(\bar{x}) \diamond \mathit{1})$, $C \neq \tilde{p}_{\mathbb{i}}(\bar{x}) \diamond \mathit{1}$; $\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1} \notin S_1^{s_{\mathbb{1}}^s}$; for all $C \in S^s$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$, $\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1} \notin S^s$; (j) holds.

Concerning Tables 5 and 6, for every form of θ, for both s, for all $a \in \mathit{qatoms}(\mathit{Prefix}^s)$, $\mathbb{i}_1 \underset{(78)}{\in} J$, $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_1}\}$; for both s, by the induction hypothesis (k) for θ_1, for all $a \in \mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})$, there exists $\mathbb{j}^* \in J_1 \underset{(78)}{\subseteq} J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; for both s, for all $a \in \mathit{qatoms}(\psi^s) = \mathit{qatoms}(\mathit{Prefix}^s) \cup \mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})$, there exists $\mathbb{j}^* \in J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; (k) holds.

Concerning Tables 5 and 6, for every form of θ, for both s, for all $\mathbb{j} \in \{\mathbb{i}\} \cup \{\mathbb{i}_1\}$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\mathit{Prefix}^s)$ satisfying, for all $a \in \mathit{atoms}(\mathit{Prefix}^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\mathit{qatoms}(\mathit{Prefix}^s))$, either $\theta = \theta_1 \rightarrow \mathit{0}$, $\mathit{qatoms}(\mathit{Prefix}^s) = \emptyset$, or $\theta = Qx\,\theta_1$, $\mathit{qatoms}(\mathit{Prefix}^s) = \{Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x})\}$; we have, for both s, by the induction hypothesis (l) for θ_1, for all $\mathbb{j} \in \{\mathbb{i}_1\} \cup J_1$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\psi_1^{s_{\mathbb{1}}^s})$ satisfying, for all $a \in \mathit{atoms}(\psi_1^{s_{\mathbb{1}}^s})$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}_1} \notin \mathit{preds}(\mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s}))$, for all $\mathbb{j} \in J_1$, if there exists $a^* \in \mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Q^*x^*\,\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})$ satisfying, for all $a \in \mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Q^*x^*\,\tilde{p}_{\mathbb{j}}(\bar{x})$; we have, for both s, $\psi_1^{s_{\mathbb{1}}^s} \in \mathit{Form}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}_1}\}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J_1\}}$; $\tilde{p}_{\mathbb{i}} \underset{(79)}{\notin} \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}$, for both s, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\psi_1^{s_{\mathbb{1}}^s}) \subseteq \mathit{Pred}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}_1}\}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J_1\}}$; we have, for both s, $\mathit{Prefix}^s \in \mathit{Form}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{i}_1}\}}$; $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\} \cap (\{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}) \overset{(79)}{=\!=} \emptyset$, for both s, $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\} \cap \mathit{preds}(\mathit{Prefix}^s) \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\} \cap \mathit{Pred}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{i}_1}\}} = \emptyset$; $(q)\mathit{atoms}(\psi^s) = (q)\mathit{atoms}(\mathit{Prefix}^s) \cup (q)\mathit{atoms}(\psi_1^{s_{\mathbb{1}}^s})$; $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})) \subseteq \mathit{preds}(\psi_1^{s_{\mathbb{1}}^s})$, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\mathit{qatoms}(\mathit{Prefix}^s)) \cup \mathit{preds}(\mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})) = \mathit{preds}(\mathit{qatoms}(\mathit{Prefix}^s) \cup \mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})) = \mathit{preds}(\mathit{qatoms}(\psi^s))$. Let $\mathbb{j} \in \{\mathbb{i}\} \cup J \overset{(78)}{=\!=} \{\mathbb{i}\} \cup \{\mathbb{i}_1\} \cup J_1$. We distinguish three cases for $\mathbb{j}$.

Case 2.2.1: $\mathbb{j} = \mathbb{i}$. Then, for both s, $\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{atoms}(\mathit{Prefix}^s) \subseteq \mathit{atoms}(\psi^s)$; we have, for both s, $\tilde{p}_{\mathbb{i}} \notin \mathit{preds}(\psi_1^{s_{\mathbb{1}}^s})$; for both s, for all $a \in \mathit{atoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}}\}$, $a \notin \mathit{atoms}(\psi_1^{s_{\mathbb{1}}^s})$, $a \in \mathit{atoms}(\mathit{Prefix}^s)$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; (l) holds.

Case 2.2.2: $\mathbb{j} = \mathbb{i}_1$. Then, for both s, $\tilde{p}_{\mathbb{i}_1}(\bar{x}) \in \mathit{atoms}(\mathit{Prefix}^s), \mathit{atoms}(\psi_1^{s_{\mathbb{1}}^s}) \subseteq \mathit{atoms}(\psi^s)$, for all $a \in \mathit{atoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_1}\}$, $a \in \mathit{atoms}(\mathit{Prefix}^s) \cup \mathit{atoms}(\psi_1^{s_{\mathbb{1}}^s})$, for both the cases $a \in \mathit{atoms}(\mathit{Prefix}^s)$ and $a \in \mathit{atoms}(\psi_1^{s_{\mathbb{1}}^s})$, $a = \tilde{p}_{\mathbb{i}_1}(\bar{x})$; $a = \tilde{p}_{\mathbb{i}_1}(\bar{x})$; we have, for both s, $\tilde{p}_{\mathbb{i}_1} \notin \mathit{preds}(\mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s}))$, either $\theta = \theta_1 \rightarrow \mathit{0}$, $\mathit{qatoms}(\mathit{Prefix}^s) = \emptyset$, or $\theta = Qx\,\theta_1$, $\mathit{qatoms}(\mathit{Prefix}^s) = \{Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x})\}$; for both s, if there exists $a^* \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{i}_1}\}$, $a^* \notin \mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})$, $a^* \in \mathit{qatoms}(\mathit{Prefix}^s)$, $Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \in \mathit{qatoms}(\mathit{Prefix}^s) \subseteq \mathit{qatoms}(\psi^s)$, for all $a \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_1}\}$, $a \notin \mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})$, $a \in \mathit{qatoms}(\mathit{Prefix}^s) = \{Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x})\}$, $a = Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x})$; (l) holds.

Case 2.2.3: $\mathbb{j} \in J_1$. Then, for both s, $\tilde{p}_{\mathbb{j}} \notin \mathit{preds}(\mathit{Prefix}^s)$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\psi_1^{s_{\mathbb{1}}^s}) \subseteq \mathit{atoms}(\psi^s)$, for all $a \in \mathit{atoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \notin \mathit{atoms}(\mathit{Prefix}^s)$, $a \in \mathit{atoms}(\psi_1^{s_{\mathbb{1}}^s})$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; if there exists $a^* \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, $a^* \notin \mathit{qatoms}(\mathit{Prefix}^s)$, $a^* \in \mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})$, there exists $Q^*x^*\,\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s}) \subseteq \mathit{qatoms}(\psi^s)$, for all $a \in \mathit{qatoms}(\psi^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \notin \mathit{qatoms}(\mathit{Prefix}^s)$, $a \in \mathit{qatoms}(\psi_1^{s_{\mathbb{1}}^s})$, $a = Q^*x^*\,\tilde{p}_{\mathbb{j}}(\bar{x})$; (l) holds.

So, in all Cases 2.2.1–2.2.3, (l) holds; (l) holds.

Concerning Tables 5 and 6, for every form of θ, for both s, for all $a \in \mathit{qatoms}(\mathit{ClPrefix}^s)$, $\mathbb{i}_1 \underset{(78)}{\in} J$, $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_1}\}$; for both s, by the induction hypothesis (m) for θ_1, for all $a \in \mathit{qatoms}(S_1^{s_{\mathbb{1}}^s})$, there exists $\mathbb{j}^* \in J_1 \underset{(78)}{\subseteq} J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; for both s, for all $a \in \mathit{qatoms}(S^s) = \mathit{qatoms}(\mathit{ClPrefix}^s) \cup \mathit{qatoms}(S_1^{s_{\mathbb{1}}^s})$, there exists $\mathbb{j}^* \in J$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; (m) holds.

Concerning Tables 5 and 6, for every form of θ, for both s, for all $\mathbb{j} \in \{\mathbb{i}\} \cup \{\mathbb{i}_1\}$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\mathit{ClPrefix}^s)$ satisfying, for all $a \in \mathit{atoms}(\mathit{ClPrefix}^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$;

$\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(\mathit{qatoms}(\mathit{ClPrefix}^s))$, either $\theta = \theta_1 \rightarrow \mathit{0}$, $\mathit{qatoms}(\mathit{ClPrefix}^s) = \emptyset$, or $\theta = Qx\,\theta_1$, $\mathit{qatoms}(\mathit{ClPrefix}^s) = \{Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x})\}$; we have, for both s, by the induction hypothesis (n) for θ_1, for all $\mathbb{j} \in \{\mathbb{i}_1\} \cup J_1$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(S_1^{s_1^s})$ satisfying, for all $a \in \mathit{atoms}(S_1^{s_1^s})$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}_1} \not\in \mathit{preds}(\mathit{qatoms}(S_1^{s_1^s}))$, for all $\mathbb{j} \in J_1$, if there exists $a^* \in \mathit{qatoms}(S_1^{s_1^s})$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Q^*x^*\,\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S_1^{s_1^s})$ satisfying, for all $a \in \mathit{qatoms}(S_1^{s_1^s})$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Q^*x^*\,\tilde{p}_{\mathbb{j}}(\bar{x})$; we have, for both s, $S_1^{s_1^s} \subseteq \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}}$; $\tilde{p}_{\mathbb{i}} \underset{(79)}{\not\in} \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}$, for both s, $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(S_1^{s_1^s}) \subseteq \mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}_1}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\}}$; we have, for both s, $\mathit{ClPrefix}^s \subseteq \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}}$; $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\} \cap (\{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}) \overset{(79)}{=\!=} \emptyset$, for both s, $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\} \cap \mathit{preds}(\mathit{ClPrefix}^s) \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_1\} \cap \mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}} = \emptyset$; $(q)\mathit{atoms}(S^s) = (q)\mathit{atoms}(\mathit{ClPrefix}^s) \cup (q)\mathit{atoms}(S_1^{s_1^s})$; $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(\mathit{qatoms}(S_1^{s_1^s})) \subseteq \mathit{preds}(S_1^{s_1^s})$, $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(\mathit{qatoms}(\mathit{ClPrefix}^s)) \cup \mathit{preds}(\mathit{qatoms}(S_1^{s_1^s})) = \mathit{preds}(\mathit{qatoms}(\mathit{ClPrefix}^s) \cup \mathit{qatoms}(S_1^{s_1^s})) = \mathit{preds}(\mathit{qatoms}(S^s))$. Let $\mathbb{j} \in \{\mathbb{i}\} \cup J \overset{(78)}{=\!=} \{\mathbb{i}\} \cup \{\mathbb{i}_1\} \cup J_1$. We distinguish three cases for $\mathbb{j}$.

Case 2.2.4: $\mathbb{j} = \mathbb{i}$. Then, for both s, $\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{atoms}(\mathit{ClPrefix}^s) \subseteq \mathit{atoms}(S^s)$; we have, for both s, $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(S_1^{s_1^s})$; for all $a \in \mathit{atoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}}\}$, $a \not\in \mathit{atoms}(S_1^{s_1^s})$, $a \in \mathit{atoms}(\mathit{ClPrefix}^s)$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; (n) holds.

Case 2.2.5: $\mathbb{j} = \mathbb{i}_1$. Then, for both s, $\tilde{p}_{\mathbb{i}_1}(\bar{x}) \in \mathit{atoms}(\mathit{ClPrefix}^s), \mathit{atoms}(S_1^{s_1^s}) \subseteq \mathit{atoms}(S^s)$, for all $a \in \mathit{atoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_1}\}$, $a \in \mathit{atoms}(\mathit{ClPrefix}^s) \cup \mathit{atoms}(S_1^{s_1^s})$, for both the cases $a \in \mathit{atoms}(\mathit{ClPrefix}^s)$ and $a \in \mathit{atoms}(S_1^{s_1^s})$, $a = \tilde{p}_{\mathbb{i}_1}(\bar{x})$; $a = \tilde{p}_{\mathbb{i}_1}(\bar{x})$; we have, for both s, $\tilde{p}_{\mathbb{i}_1} \not\in \mathit{preds}(\mathit{qatoms}(S_1^{s_1^s}))$, either $\theta = \theta_1 \rightarrow \mathit{0}$, $\mathit{qatoms}(\mathit{ClPrefix}^s) = \emptyset$, or $\theta = Qx\,\theta_1$, $\mathit{qatoms}(\mathit{ClPrefix}^s) = \{Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x})\}$; for both s, if there exists $a^* \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{i}_1}\}$, $a^* \not\in \mathit{qatoms}(S_1^{s_1^s})$, $a^* \in \mathit{qatoms}(\mathit{ClPrefix}^s)$, $Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \in \mathit{qatoms}(\mathit{ClPrefix}^s) \subseteq \mathit{qatoms}(S^s)$, for all $a \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}_1}\}$, $a \not\in \mathit{qatoms}(S_1^{s_1^s})$, $a \in \mathit{qatoms}(\mathit{ClPrefix}^s) = \{Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x})\}$, $a = Qx\,\tilde{p}_{\mathbb{i}_1}(\bar{x})$; (n) holds.

Case 2.2.6: $\mathbb{j} \in J_1$. Then, for both s, $\tilde{p}_{\mathbb{j}} \not\in \mathit{preds}(\mathit{ClPrefix}^s)$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(S_1^{s_1^s}) \subseteq \mathit{atoms}(S^s)$, for all $a \in \mathit{atoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \not\in \mathit{atoms}(\mathit{ClPrefix}^s)$, $a \in \mathit{atoms}(S_1^{s_1^s})$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; if there exists $a^* \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, $a^* \not\in \mathit{qatoms}(\mathit{ClPrefix}^s)$, $a^* \in \mathit{qatoms}(S_1^{s_1^s})$, there exists $Q^*x^*\,\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S_1^{s_1^s}) \subseteq \mathit{qatoms}(S^s)$, for all $a \in \mathit{qatoms}(S^s)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \not\in \mathit{qatoms}(\mathit{ClPrefix}^s)$, $a \in \mathit{qatoms}(S_1^{s_1^s})$, $a = Q^*x^*\,\tilde{p}_{\mathbb{j}}(\bar{x})$; (n) holds.

So, in all Cases 2.2.4–2.2.6, (n) holds; (n) holds.

$\tilde{p}_{\mathbb{i}}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \in \mathit{Atom}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{i}_1}\}} - \mathit{Tcons}_{\mathcal{L}}$, concerning Tables 5 and 6, for every form of θ, for both s, $\mathit{tcons}(\mathit{Prefix}^s) = \mathit{tcons}(\mathit{ClPrefix}^s) = \{\mathit{0}, \mathit{1}\}$; we have, for both s, by the induction hypothesis (o) for θ_1, $\mathit{tcons}(\psi_1^{s_1^s}) = \mathit{tcons}(S_1^{s_1^s}) = \mathit{tcons}(\theta_1)$; for both s, $\mathit{tcons}(\psi^s) = \mathit{tcons}(\mathit{Prefix}^s) \cup \mathit{tcons}(\psi_1^{s_1^s}) = \{\mathit{0}, \mathit{1}\} \cup \mathit{tcons}(\theta_1) = \mathit{tcons}(\theta)$, $\mathit{tcons}(S^s) = \mathit{tcons}(\mathit{ClPrefix}^s) \cup \mathit{tcons}(S_1^{s_1^s}) = \{\mathit{0}, \mathit{1}\} \cup \mathit{tcons}(\theta_1) = \mathit{tcons}(\theta)$; (o) holds.

So, in all Cases 1, 2.1, 2.2, (a–o) hold; (a–o) hold. The induction is completed. Thus, (71) holds.

(I) By (69) for n_ϕ, ϕ, there exists $\phi' \in \mathit{Form}_{\mathcal{L}}$ such that (69a–e) hold for n_ϕ, ϕ, ϕ'. We distinguish three cases for ϕ'.

Case 1: $\phi' \in \mathit{Tcons}_{\mathcal{L}} - \{\mathit{1}\}$. We put $J_\phi = \emptyset \subseteq \{(n_\phi, j) \,|\, j \in \mathbb{N}\}$, $\psi = \mathit{0} \in \mathit{Form}_{\mathcal{L}}$, $S_\phi = \{\square\} \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}}$.

$\|J_\phi\| = 0 \leq 2 \cdot |\phi|$; (a) holds.

We have $J_\phi = \emptyset$, $S_\phi = \{\square\}$; (b) holds.

For every interpretation $\mathfrak{A}$ for $\mathcal{L}$, $\mathfrak{A} \not\models \phi' \overset{\text{(69a)}}{=\!=\!=} \phi$; trivially, there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models \phi$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L}$ and $\mathfrak{A}' \models \psi \mid S_\phi$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$; (c) and (d) hold.

$|\psi| = 1 \in O(|\phi|^2)$; the translation of ϕ to ψ uses the input ϕ, the output ψ, an auxiliary ϕ'; we have, by (69b), ϕ' can be built up from ϕ via a postorder traversal of ϕ with $\#\mathcal{O}(\phi) \in O(|\phi|)$; the test $\phi' \in \mathit{Tcons}_{\mathcal{L}} - \{\mathit{1}\}$ is with $\#\mathcal{O}(\phi') \in O(1)$; ψ can be built up with $\#\mathcal{O} \in O(1)$; the number of all elementary operations of the translation of ϕ to ψ $\#\mathcal{O}(\phi) \in O(|\phi|) \subseteq O(|\phi|^2)$; by (13) for n_ϕ, ϕ, $\emptyset$, ϕ', ψ, $q = 3$, $r = 2$, the time complexity of the translation of ϕ to ψ, is in $O(\#\mathcal{O}(\phi) \cdot (\log(1 + n_\phi) + \log(\#\mathcal{O}(\phi) + |\phi|))) \subseteq O(|\phi|^2 \cdot (\log(1 + n_\phi) + \log|\phi|))$; by (14) for n_ϕ, ϕ, $\emptyset$, ϕ', ψ, $q = 3$, $r = 2$, the space complexity of the translation of ϕ to ψ, is in $O((\#\mathcal{O}(\phi) + |\phi|^2) \cdot (\log(1 + n_\phi) + \log|\phi|)) \subseteq O(|\phi|^2 \cdot (\log(1 + n_\phi) + \log|\phi|))$; (e) holds.

$|S_\phi| = 0 \in O(|\phi|^2)$; the translation of ϕ to S_ϕ uses the input ϕ, the output S_ϕ, an auxiliary ϕ'; we have ϕ' can be built up from ϕ via a postorder traversal of ϕ with $\#\mathcal{O}(\phi) \in O(|\phi|)$; the test $\phi' \in \mathit{Tcons}_{\mathcal{L}} - \{\mathit{1}\}$ is with $\#\mathcal{O}(\phi') \in O(1)$; S_ϕ can be built up with $\#\mathcal{O} \in O(1)$; the number of all elementary operations of the translation of ϕ to S_ϕ $\#\mathcal{O}(\phi) \in O(|\phi|) \subseteq O(|\phi|^2)$; by (13) for n_ϕ, ϕ, $\emptyset$, ϕ', S_ϕ, $q = 3$, $r = 2$, the time complexity of the translation of ϕ to S_ϕ, is in $O(\#\mathcal{O}(\phi) \cdot (\log(1 + n_\phi) + \log(\#\mathcal{O}(\phi) + |\phi|))) \subseteq O(|\phi|^2 \cdot (\log(1 + n_\phi) + \log|\phi|))$; by (14) for n_ϕ, ϕ, $\emptyset$, ϕ', S_ϕ, $q = 3$, $r = 2$, the space complexity of the translation of ϕ to S_ϕ, is in $O((\#\mathcal{O}(\phi) + |\phi|^2) \cdot (\log(1 + n_\phi) + \log|\phi|)) \subseteq O(|\phi|^2 \cdot (\log(1 + n_\phi) + \log|\phi|))$; (f) holds.

We have $\psi = \mathit{0}$; (g) holds trivially.

We have $S_\phi = \{\square\}$; (h) holds trivially.

$\mathit{qatoms}(\psi) = \emptyset$; (i) holds trivially.

We have $J_\phi = \emptyset$; (j) holds trivially.

$\mathit{qatoms}(S_\phi) = \emptyset$; (k) holds trivially.

We have $J_\phi = \emptyset$; (l) holds trivially.

$\mathit{tcons}(\psi) = \mathit{tcons}(S_\phi) = \{\mathit{0}, \mathit{1}\} \subseteq \mathit{tcons}(\phi)$; (m) holds.

Case 2: $\phi' = \mathit{1}$. We put $J_\phi = \emptyset \subseteq \{(n_\phi, j) \,|\, j \in \mathbb{N}\}$, $\psi = \mathit{1} \in \mathit{Form}_{\mathcal{L}}$, $S_\phi = \emptyset \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}}$.

$\|J_\phi\| = 0 \leq 2 \cdot |\phi|$; (a) holds.

We have $J_\phi = S_\phi = \emptyset$; (b) holds.

For every interpretation $\mathfrak{A}$ for $\mathcal{L}$, $\mathfrak{A} \models \phi' \overset{(69a)}{=\!=\!=} \phi$; the class of interpretations for $\mathcal{L}$ is not empty; there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models \phi$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L}$ and $\mathfrak{A}' \models \psi \mid S_\phi$, satisfying $\mathfrak{A} = \mathfrak{A}' = \mathfrak{A}'|_{\mathcal{L}}$; (c) and (d) hold.

$|\psi| = 1 \in O(|\phi|^2)$; the translation of ϕ to ψ uses the input ϕ, the output ψ, an auxiliary ϕ'; we have ϕ' can be built up from ϕ via a postorder traversal of ϕ with $\#\mathcal{O}(\phi) \in O(|\phi|)$; the test $\phi' = \mathit{1}$ is with $\#\mathcal{O}(\phi') \in O(1)$; ψ can be built up with $\#\mathcal{O} \in O(1)$; the number of all elementary operations of the translation of ϕ to ψ $\#\mathcal{O}(\phi) \in O(|\phi|) \subseteq O(|\phi|^2)$; by (13) for n_ϕ, ϕ, $\emptyset$, ϕ', ψ, $q = 3$, $r = 2$, the time complexity of the translation of ϕ to ψ, is in $O(\#\mathcal{O}(\phi) \cdot (\log(1 + n_\phi) + \log(\#\mathcal{O}(\phi) + |\phi|))) \subseteq O(|\phi|^2 \cdot (\log(1 + n_\phi) + \log |\phi|))$; by (14) for n_ϕ, ϕ, $\emptyset$, ϕ', ψ, $q = 3$, $r = 2$, the space complexity of the translation of ϕ to ψ, is in $O((\#\mathcal{O}(\phi) + |\phi|^2) \cdot (\log(1 + n_\phi) + \log |\phi|)) \subseteq O(|\phi|^2 \cdot (\log(1 + n_\phi) + \log |\phi|))$; (e) holds.

$|S_\phi| = 0 \in O(|\phi|^2)$; the translation of ϕ to S_ϕ uses the input ϕ, the output S_ϕ, an auxiliary ϕ'; we have ϕ' can be built up from ϕ via a postorder traversal of ϕ with $\#\mathcal{O}(\phi) \in O(|\phi|)$; the test $\phi' = \mathit{1}$ is with $\#\mathcal{O}(\phi') \in O(1)$; S_ϕ can be built up with $\#\mathcal{O} \in O(1)$; the number of all elementary operations of the translation of ϕ to S_ϕ $\#\mathcal{O}(\phi) \in O(|\phi|) \subseteq O(|\phi|^2)$; by (13) for n_ϕ, ϕ, $\emptyset$, ϕ', S_ϕ, $q = 3$, $r = 2$, the time complexity of the translation of ϕ to S_ϕ, is in $O(\#\mathcal{O}(\phi) \cdot (\log(1 + n_\phi) + \log(\#\mathcal{O}(\phi) + |\phi|))) \subseteq O(|\phi|^2 \cdot (\log(1 + n_\phi) + \log |\phi|))$; by (14) for n_ϕ, ϕ, $\emptyset$, ϕ', S_ϕ, $q = 3$, $r = 2$, the space complexity of the translation of ϕ to S_ϕ, is in $O((\#\mathcal{O}(\phi) + |\phi|^2) \cdot (\log(1 + n_\phi) + \log |\phi|)) \subseteq O(|\phi|^2 \cdot (\log(1 + n_\phi) + \log |\phi|))$; (f) holds.

We have $\psi = \mathit{1}$; (g) holds trivially.

We have $S_\phi = \emptyset$; (h) holds trivially.

$\mathit{qatoms}(\psi) = \emptyset$; (i) holds trivially.

We have $J_\phi = \emptyset$; (j) holds trivially.

$\mathit{qatoms}(S_\phi) = \emptyset$; (k) holds trivially.

We have $J_\phi = \emptyset$; (l) holds trivially.

$\mathit{tcons}(\psi) = \mathit{tcons}(S_\phi) = \{\mathit{0}, \mathit{1}\} \subseteq \mathit{tcons}(\phi)$; (m) holds.

Case 3: $\phi' \not\in \mathit{Tcons}_{\mathcal{L}}$. We have $\phi' \in \mathit{Form}_{\mathcal{L}}$, (69c,d) hold for ϕ'. We put $\bar{x} = \mathit{varseq}(\phi')$. Then $\phi' \in \mathit{Form}_{\mathcal{L}} - \mathit{Tcons}_{\mathcal{L}} \subseteq \mathit{Form}_{\mathcal{L}} - \{\mathit{0}, \mathit{1}\}$, $\mathit{vars}(\bar{x}) = \mathit{vars}(\phi') \subseteq \mathit{Var}_{\mathcal{L}}$, $|\bar{x}| \leq |\phi'|$. We put $j_{\mathbb{i}} = 0$ and $\mathbb{i} = (n_\phi, j_{\mathbb{i}}) \in \{(n_\phi, j) \mid j \in \mathbb{N}\}$. $\tilde{p}_{\mathbb{i}} \in \tilde{\mathbb{P}}$. We put $\mathit{ar}(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$. We get by (71) for n_ϕ, ϕ', $\bar{x}$, $\mathbb{i}$, $\tilde{p}_{\mathbb{i}}$ that there exist $J = \{(n_\phi, j) \mid 1 \leq j \leq n_J\} \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, $j_{\mathbb{i}} \leq n_J$, $\mathbb{i} \not\in J$, a *CNF* $\psi^+ \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, $S^+ \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, and (71a,b,e,g–o) hold for ϕ', $\bar{x}$, $\tilde{p}_{\mathbb{i}}$, J, ψ^+, S^+. We put $n_{J_\phi} = n_J$ and $J_\phi = \{(n_\phi, j) \mid j \leq n_{J_\phi}\} \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$. Then

$$J_\phi = \{(n_\phi, j_{\mathbb{i}})\} \cup \{(n_\phi, j) \mid 1 \leq j \leq n_J\} = \{\mathbb{i}\} \cup J, \tag{84}$$

$$\{\mathbb{i}\} \cap J = \emptyset. \tag{85}$$

We put $\psi = \tilde{p}_{\mathbb{i}}(\bar{x}) \wedge \psi^+ \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in \{\mathbb{i}\} \cup J \overset{(84)}{=} J_\phi\}}$. Then $\tilde{p}_{\mathbb{i}}(\bar{x})$ is a literal, by (71i), $\psi^+ \neq \mathit{0}, \mathit{1}$; we have ψ^+ is a *CNF*; ψ is a *CNF*. We put $S_\phi = \{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}\} \cup S^+ \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in \{\mathbb{i}\} \cup J \overset{(84)}{=} J_\phi\}}$.

$$\{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}\} \cap S^+ \overset{(71j)}{=\!=\!=} \emptyset. \tag{86}$$

$\|J_\phi\| \underset{(85)}{\overset{(84)}{=\!=}} 1 + \|J\| \underset{(71a)}{\leq} 1 + |\phi'| - 1 \underset{(69b)}{\leq} 2 \cdot |\phi|$; (a) holds.

$J_\phi \neq \emptyset$, $S_\phi \neq \emptyset$, $\square \not\in \{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}\}$, for all $C \in S^+$, by (71j), $\mathit{preds}(\square) = \emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \mathit{preds}(C)$, $C \neq \square$; $\square \not\in S^+$; $\square \not\in S_\phi$; (b) holds.

Let $\mathfrak{A}$ be an interpretation for $\mathcal{L}$ such that $\mathfrak{A} \models \phi$. Then $\mathfrak{A} \models \phi \overset{(69a)}{=\!=\!=} \phi'$. We define an expansion $\mathfrak{A}^\#$ of $\mathfrak{A}$ to $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ as follows:

$$\tilde{p}_{\mathbb{i}}^{\mathfrak{A}^\#}(u_1, \dots, u_{|\bar{x}|}) = \begin{cases} \|\phi'\|_e^{\mathfrak{A}} & \text{if there exists } e \in \mathcal{S}_{\mathfrak{A}} \text{ such that } \|\bar{x}\|_e^{\mathfrak{A}} = u_1, \dots, u_{|\bar{x}|}, \\ 0 & \text{else.} \end{cases}$$

Then, for all $e \in \mathcal{S}_{\mathfrak{A}^\#}$, $\|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^\#} = \tilde{p}_{\mathbb{i}}^{\mathfrak{A}^\#}(\|\bar{x}\|_e^{\mathfrak{A}^\#}) = \tilde{p}_{\mathbb{i}}^{\mathfrak{A}^\#}(\|\bar{x}\|_e^{\mathfrak{A}}) = \|\phi'\|_e^{\mathfrak{A}} = \|\phi'\|_e^{\mathfrak{A}^\#} = 1$, $\|\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}\|_e^{\mathfrak{A}^\#} = \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^\#} \eqcirc 1 = 1 \eqcirc 1 = 1$, $\|\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \phi'\|_e^{\mathfrak{A}^\#} = \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^\#} \Rightarrow \|\phi'\|_e^{\mathfrak{A}^\#} = 1 \Rightarrow 1 = 1$; $\mathfrak{A}^\# \models \tilde{p}_{\mathbb{i}}(\bar{x})$, $\mathfrak{A}^\# \models \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}$, $\mathfrak{A}^\# \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \phi'$, by (71b) | (71e) for $\mathfrak{A}^\#$, there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in \{\mathbb{i}\} \cup J \overset{(84)}{=\!=} J_\phi\}$ and $\mathfrak{A}' \models \psi^+ \mid S^+$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} = \mathfrak{A}^\#$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} = \mathfrak{A}^\# \models \tilde{p}_{\mathbb{i}}(\bar{x})$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} = \mathfrak{A}^\# \models \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}$; $\mathfrak{A}' \models \psi \mid S_\phi$, $\mathfrak{A}'|_{\mathcal{L}} = \mathfrak{A}$.

Let $\mathfrak{A}'$ be an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi \overset{(84)}{=\!=} \{\mathbb{i}\} \cup J\}$ such that $\mathfrak{A}' \models \psi \mid S_\phi$. Then $\mathfrak{A}' \models \tilde{p}_{\mathbb{i}}(\bar{x}), \psi^+ \mid \mathfrak{A}' \models \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, S^+$, by (71b) | (71e) for $\mathfrak{A}'$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \phi'$, for all $e \in \mathcal{S}_{\mathfrak{A}'}$, $\|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}'} = 1 \mid 1 = \|\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}\|_e^{\mathfrak{A}'} = \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}'} \eqcirc 1 = \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}'}$; $\|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}'} = 1$, $1 = \|\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \phi'\|_e^{\mathfrak{A}'} = \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}'} \Rightarrow \|\phi'\|_e^{\mathfrak{A}'} = 1 \Rightarrow \|\phi'\|_e^{\mathfrak{A}'} = \|\phi'\|_e^{\mathfrak{A}'}$; $\mathfrak{A}' \models \phi' \overset{(69a)}{=\!=\!=} \phi$. We put $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$, an interpretation for $\mathcal{L}$. Then $\mathfrak{A} \models \phi$, $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$; (c) and (d) hold.

We have $|\bar{x}| \leq |\phi'|$. $|\psi| = 2 + |\bar{x}| + |\psi^+| \underset{(71g)}{\leq} 2 + |\bar{x}| + 15 \cdot |\phi'| \cdot (1 + |\bar{x}|) \leq 2 + |\phi'| + 15 \cdot |\phi'| \cdot (1 + |\phi'|) \leq 33 \cdot |\phi'|^2 \underset{(69b)}{\leq} 132 \cdot |\phi|^2 \in O(|\phi|^2)$; the translation of ϕ to ψ uses the input ϕ, the output ψ, auxiliary ϕ', $\tilde{f}_0(\bar{x})$, $\tilde{p}_{\mathbb{i}}(\bar{x})$, ψ^+; we have ϕ' can be built up from ϕ via a postorder traversal of ϕ with $\#\mathcal{O}(\phi) \in O(|\phi|)$; the test $\phi' \not\in \mathit{Tcons}_{\mathcal{L}}$ is with $\#\mathcal{O}(\phi') \in O(1)$; $\tilde{f}_0(\bar{x})$ can be built up from ϕ' via the left-right preorder traversal of ϕ' with $\#\mathcal{O}(\phi') \in O(|\phi'|) \underset{(69b)}{\subseteq} O(|\phi|)$; $\tilde{p}_{\mathbb{i}}(\bar{x})$ can be built up from $\tilde{f}_0(\bar{x})$ with $\#\mathcal{O}(\tilde{f}_0(\bar{x})) \in O(|\tilde{p}_{\mathbb{i}}(\bar{x})|) = O(1 + |\bar{x}|) \subseteq O(|\phi'|) \underset{(69b)}{\subseteq} O(|\phi|)$; by (71g), ψ^+ can be built up from ϕ' and $\tilde{f}_0(\bar{x})$ via a preorder traversal of ϕ' with $\#\mathcal{O}(\phi', \tilde{f}_0(\bar{x})) \in O(|\phi'| \cdot (1+|\bar{x}|)) \subseteq O(|\phi'| \cdot (1+|\phi'|)) = O(|\phi'|^2) \underset{(69b)}{\subseteq} O(|\phi|^2)$; ψ can be built up from $\tilde{p}_{\mathbb{i}}(\bar{x})$ and ψ^+ by copying and concatenating with $\#\mathcal{O}(\tilde{p}_{\mathbb{i}}(\bar{x}), \psi^+) \in O(|\psi|) \subseteq O(|\phi|^2)$; the number of all elementary operations of the translation of ϕ to ψ $\#\mathcal{O}(\phi) \in O(|\phi|^2)$; by (13) for n_ϕ, ϕ, $\emptyset$, ϕ', $\tilde{f}_0(\bar{x})$, $\tilde{p}_{\mathbb{i}}(\bar{x})$, ψ^+, ψ, $q=6$, $r=2$, the time complexity of the translation of ϕ to ψ, is in $O(\#\mathcal{O}(\phi) \cdot (\log(1+n_\phi) + \log(\#\mathcal{O}(\phi) + |\phi|))) \subseteq O(|\phi|^2 \cdot (\log(1+n_\phi) + \log|\phi|))$; by (14) for n_ϕ, ϕ, $\emptyset$, ϕ', $\tilde{f}_0(\bar{x})$, $\tilde{p}_{\mathbb{i}}(\bar{x})$, ψ^+, ψ, $q=6$, $r=2$, the space complexity of the translation of ϕ to ψ, is in $O((\#\mathcal{O}(\phi) + |\phi|^2) \cdot (\log(1+n_\phi) + \log|\phi|)) \subseteq O(|\phi|^2 \cdot (\log(1+n_\phi) + \log|\phi|))$; (e) holds.

We have $|\bar{x}| \leq |\phi'|$. $|S_\phi| \stackrel{(86)}{=} 3 + |\bar{x}| + |S^+| \underset{(71h)}{\leq} 3 + |\bar{x}| + 15 \cdot |\phi'| \cdot (1+|\bar{x}|) \leq 3 + |\phi'| + 15 \cdot |\phi'| \cdot (1+|\phi'|) \leq 34 \cdot |\phi'|^2 \underset{(69b)}{\leq} 136 \cdot |\phi|^2 \in O(|\phi|^2)$; the translation of ϕ to S_ϕ uses the input ϕ, the output S_ϕ, auxiliary ϕ', $\tilde{f}_0(\bar{x})$, $\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}$, S^+; we have ϕ' can be built up from ϕ via a postorder traversal of ϕ with $\#\mathcal{O}(\phi) \in O(|\phi|)$; the test $\phi' \not\in \mathit{Tcons}_{\mathcal{L}}$ is with $\#\mathcal{O}(\phi') \in O(1)$; we have $\tilde{f}_0(\bar{x})$ can be built up from ϕ' via the left-right preorder traversal of ϕ' with $\#\mathcal{O}(\phi') \in O(|\phi|)$; $\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}$ can be built up from $\tilde{f}_0(\bar{x})$ with $\#\mathcal{O}(\tilde{f}_0(\bar{x})) \in O(|\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}|) = O(1+|\bar{x}|) \subseteq O(|\phi'|) \underset{(69b)}{\subseteq} O(|\phi|)$; by (71h), S^+ can be built up from ϕ' and $\tilde{f}_0(\bar{x})$ via a preorder traversal of ϕ' with $\#\mathcal{O}(\phi', \tilde{f}_0(\bar{x})) \in O(|\phi'| \cdot (1+|\bar{x}|)) \subseteq O(|\phi'| \cdot (1+|\phi'|)) = O(|\phi'|^2) \underset{(69b)}{\subseteq} O(|\phi|^2)$; S_ϕ can be built up from $\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}$ and S^+ by copying and concatenating with $\#\mathcal{O}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}, S^+) \in O(|S_\phi|) \subseteq O(|\phi|^2)$; the number of all elementary operations of the translation of ϕ to S_ϕ $\#\mathcal{O}(\phi) \in O(|\phi|^2)$; by (13) for n_ϕ, ϕ, $\emptyset$, ϕ', $\tilde{f}_0(\bar{x})$, $\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}$, S^+, S_ϕ, $q=6$, $r=2$, the time complexity of the translation of ϕ to S_ϕ, is in $O(\#\mathcal{O}(\phi) \cdot (\log(1+n_\phi) + \log(\#\mathcal{O}(\phi) + |\phi|))) \subseteq O(|\phi|^2 \cdot (\log(1+n_\phi) + \log|\phi|))$; by (14) for n_ϕ, ϕ, $\emptyset$, ϕ', $\tilde{f}_0(\bar{x})$, $\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}$, S^+, S_ϕ, $q=6$, $r=2$, the space complexity of the translation of ϕ to S_ϕ, is in $O((\#\mathcal{O}(\phi) + |\phi|^2) \cdot (\log(1+n_\phi) + \log|\phi|)) \subseteq O(|\phi|^2 \cdot (\log(1+n_\phi) + \log|\phi|))$; (f) holds.

$\tilde{p}_{\mathbb{i}}(\bar{x})$ is a factor, $\emptyset \neq \mathit{preds}(\tilde{p}_{\mathbb{i}}(\bar{x})) \cap \tilde{\mathbb{P}} = \{\tilde{p}_{\mathbb{i}}\} \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}\} \cup J \stackrel{(84)}{=} J_\phi\}$; by (71i), $\psi^+ = \bigwedge_{i \leq n_{\psi^+}} D_i^+$, $D_i^+ \neq \tilde{p}_{\mathbb{i}}(\bar{x})$ is a factor, for all $i \leq n_{\psi^+}$, $\emptyset \neq \mathit{preds}(D_i^+) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\} \stackrel{(84)}{=} \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}$, for all $i < i' \leq n_{\psi^+}$, $\mathit{lits}(D_i^+) \neq \mathit{lits}(D_{i'}^+)$. We put $n_\psi = n_{\psi^+} + 1$ and

$$D_i = \begin{cases} \tilde{p}_{\mathbb{i}}(\bar{x}) & \mathit{if}\ i = 0, \\ D_{i-1}^+ & \mathit{if}\ 1 \leq i \leq n_{\psi^+} + 1. \end{cases}$$

Then $J_\phi \neq \emptyset$, $\psi \stackrel{(71i)}{=} \tilde{p}_{\mathbb{i}}(\bar{x}) \wedge \bigwedge_{i \leq n_{\psi^+}} D_i^+ = \bigwedge_{i \leq n_\psi} D_i$, for all $i \leq n_\psi$, D_i is a factor, $\emptyset \neq \mathit{preds}(D_i) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}$; for all $i \leq n_{\psi^+}$, $\mathit{lits}(\tilde{p}_{\mathbb{i}}(\bar{x})) = \{\tilde{p}_{\mathbb{i}}(\bar{x})\} \neq \mathit{lits}(D_i^+)$; for all $i < i' \leq n_\psi$, $\mathit{lits}(D_i) \neq \mathit{lits}(D_{i'})$; (g) holds.

$J_\phi \neq \emptyset$, $\emptyset \neq \mathit{preds}(\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}) \cap \tilde{\mathbb{P}} = \{\tilde{p}_{\mathbb{i}}\} \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in \{\mathbb{i}\} \cup J \stackrel{(84)}{=} J_\phi\}$, by (71j), for all $C \in S^+$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\} \stackrel{(84)}{=} \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}$; for all $C \in S_\phi$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}$; (h) holds.

Trivially, for all $a \in \emptyset = \mathit{qatoms}(\tilde{p}_{\mathbb{i}}(\bar{x}))$, there exists $\mathbb{j}^* \in J_\phi$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; by (71k), for all $a \in \mathit{qatoms}(\psi^+)$, there exists $\mathbb{j}^* \in J \underset{(84)}{\subseteq} J_\phi$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; for all $a \in \mathit{qatoms}(\psi) = \mathit{qatoms}(\tilde{p}_{\mathbb{i}}(\bar{x})) \cup \mathit{qatoms}(\psi^+)$, there exists $\mathbb{j}^* \in J_\phi$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; (i) holds.

$\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{atoms}(\tilde{p}_{\mathbb{i}}(\bar{x}))$ satisfying, for all $a \in \mathit{atoms}(\tilde{p}_{\mathbb{i}}(\bar{x}))$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}}\}$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; $\mathit{qatoms}(\tilde{p}_{\mathbb{i}}(\bar{x})) = \emptyset$, $\tilde{p}_{\mathbb{i}} \not\in \emptyset = \mathit{preds}(\mathit{qatoms}(\tilde{p}_{\mathbb{i}}(\bar{x})))$; we have, by (71l), for all $\mathbb{j} \in \{\mathbb{i}\} \cup J$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\psi^+)$ satisfying, for all $a \in \mathit{atoms}(\psi^+)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(\mathit{qatoms}(\psi^+))$, for all $\mathbb{j} \in J$, if there exists $a^* \in \mathit{qatoms}(\psi^+)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(\psi^+)$ satisfying, for all $a \in \mathit{qatoms}(\psi^+)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; $(q)\mathit{atoms}(\psi) = (q)\mathit{atoms}(\tilde{p}_{\mathbb{i}}(\bar{x})) \cup (q)\mathit{atoms}(\psi^+)$; $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(\mathit{qatoms}(\tilde{p}_{\mathbb{i}}(\bar{x}))) \cup \mathit{preds}(\mathit{qatoms}(\psi^+)) = \mathit{preds}(\mathit{qatoms}(\tilde{p}_{\mathbb{i}}(\bar{x})) \cup \mathit{qatoms}(\psi^+)) = \mathit{preds}(\mathit{qatoms}(\psi))$. Let $\mathbb{j} \in J_\phi \stackrel{(84)}{=} \{\mathbb{i}\} \cup J$. We distinguish two cases for $\mathbb{j}$.

Case 3.1: $\mathbb{j} = \mathbb{i}$. Then $\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{atoms}(\tilde{p}_{\mathbb{i}}(\bar{x})), \mathit{atoms}(\psi^+) \subseteq \mathit{atoms}(\psi)$, for all $a \in \mathit{atoms}(\psi)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}}\}$, $a \in \mathit{atoms}(\tilde{p}_{\mathbb{i}}(\bar{x})) \cup \mathit{atoms}(\psi^+)$, for both the cases $a \in \mathit{atoms}(\tilde{p}_{\mathbb{i}}(\bar{x}))$ and $a \in \mathit{atoms}(\psi^+)$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; (j) holds.

Case 3.2: $\mathbb{j} \in J$. Then $\tilde{p}_{\mathbb{j}} \underset{(85)}{\not\in} \{\tilde{p}_{\mathbb{i}}\} = \mathit{preds}(\tilde{p}_{\mathbb{i}}(\bar{x}))$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(\psi^+) \subseteq \mathit{atoms}(\psi)$, for all $a \in \mathit{atoms}(\psi)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \not\in \mathit{atoms}(\tilde{p}_{\mathbb{i}}(\bar{x}))$, $a \in \mathit{atoms}(\psi^+)$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; if there exists $a^* \in \mathit{qatoms}(\psi)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, $a^* \not\in \emptyset = \mathit{qatoms}(\tilde{p}_{\mathbb{i}}(\bar{x}))$, $a^* \in \mathit{qatoms}(\psi^+)$, there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(\psi^+) \subseteq \mathit{qatoms}(\psi)$, for all $a \in \mathit{qatoms}(\psi)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \not\in \emptyset = \mathit{qatoms}(\tilde{p}_{\mathbb{i}}(\bar{x}))$, $a \in \mathit{qatoms}(\psi^+)$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; (j) holds.

So, in both Cases 3.1 and 3.2, (j) holds; (j) holds.

Trivially, for all $a \in \emptyset = \mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\})$, there exists $\mathbb{j}^* \in J_\phi$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; by (71m), for all $a \in \mathit{qatoms}(S^+)$, there exists $\mathbb{j}^* \in J \underset{(84)}{\subseteq} J_\phi$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; for all $a \in \mathit{qatoms}(S_\phi) = \mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}) \cup \mathit{qatoms}(S^+)$, there exists $\mathbb{j}^* \in J_\phi$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; (k) holds.

$\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{atoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\})$ satisfying, for all $a \in \mathit{atoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}) = \{\mathit{1}, \tilde{p}_{\mathbb{i}}(\bar{x})\}$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}}\}$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; $\mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}) = \emptyset$, $\tilde{p}_{\mathbb{i}} \not\in \emptyset = \mathit{preds}(\mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}))$; we have, by (71n), for all $\mathbb{j} \in \{\mathbb{i}\} \cup J$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(S^+)$ satisfying, for all $a \in \mathit{atoms}(S^+)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(\mathit{qatoms}(S^+))$, for all $\mathbb{j} \in J$, if there exists $a^* \in \mathit{qatoms}(S^+)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S^+)$ satisfying, for all $a \in \mathit{qatoms}(S^+)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; $(q)\mathit{atoms}(S_\phi) = (q)\mathit{atoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}) \cup (q)\mathit{atoms}(S^+)$; $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(\mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\})) \cup \mathit{preds}(\mathit{qatoms}(S^+)) = \mathit{preds}(\mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}) \cup \mathit{qatoms}(S^+)) = \mathit{preds}(\mathit{qatoms}(S_\phi))$. Let $\mathbb{j} \in J_\phi \stackrel{(84)}{=} \{\mathbb{i}\} \cup J$. We distinguish two cases for $\mathbb{j}$.

Case 3.3: $\mathbb{j} = \mathbb{i}$. Then $\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{atoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}), \mathit{atoms}(S^+) \subseteq \mathit{atoms}(S_\phi)$, for all $a \in \mathit{atoms}(S_\phi)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}}\}$, $a \in \mathit{atoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\}) \cup \mathit{atoms}(S^+)$, for both the cases $a \in \mathit{atoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) = \mathit{1}\})$ and $a \in \mathit{atoms}(S^+)$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; (l) holds.

Case 3.4: $\mathbb{j} \in J$. Then $\tilde{p}_{\mathbb{j}} \underset{(85)}{\not\in} \{\mathit{1}, \tilde{p}_{\mathbb{i}}\} = \mathit{preds}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}\})$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(S^+) \subseteq \mathit{atoms}(S_\phi)$, for all $a \in \mathit{atoms}(S_\phi)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \not\in \mathit{atoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}\})$, $a \in \mathit{atoms}(S^+)$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; if there exists $a^* \in \mathit{qatoms}(S_\phi)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, $a^* \not\in \emptyset = \mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}\})$, $a^* \in \mathit{qatoms}(S^+)$, there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S^+) \subseteq \mathit{qatoms}(S_\phi)$, for all $a \in \mathit{qatoms}(S_\phi)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \not\in \emptyset = \mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}\})$, $a \in \mathit{qatoms}(S^+)$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; (l) holds.

So, in both Cases 3.3 and 3.4, (l) holds; (l) holds.

$\mathit{tcons}(\psi) = \mathit{tcons}(\tilde{p}_{\mathbb{i}}(\bar{x})) \cup \mathit{tcons}(\psi^+) = \{\mathit{0}, \mathit{1}\} \cup \mathit{tcons}(\psi^+) \overset{(71o)}{=\!=} \mathit{tcons}(\phi') \underset{(69e)}{\subseteq} \mathit{tcons}(\phi)$, $\mathit{tcons}(S_\phi) = \mathit{tcons}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}\}) \cup \mathit{tcons}(S^+) = \{\mathit{0}, \mathit{1}\} \cup \mathit{tcons}(S^+) \overset{(71o)}{=\!=} \mathit{tcons}(\phi') \underset{(69e)}{\subseteq} \mathit{tcons}(\phi)$; (m) holds.

So, in all Cases 1–3, (a–m) hold; (a–m) hold. Thus, (I) holds.

(II) We have $\mathcal{L}$ is a countable first-order language; $\mathbb{I}$, $\tilde{\mathbb{P}}$ are countable. Then $\mathit{Form}_{\mathcal{L}}$ is countable; $T \subseteq \mathit{Form}_{\mathcal{L}}$ is countable; $\Gamma = \{J \mid J \subseteq_{\mathcal{F}} \mathbb{I}\}$ is countable; $\mathcal{L} \cup \tilde{\mathbb{P}}$ is a countable first-order language; $\mathit{SimOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{P}}}$ is countable; $\Delta = \{S \mid S \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{P}}}\}$ is countable; $\Gamma \times \Delta$ is countable; there exists a well order $\trianglelefteq \subseteq (\Gamma \times \Delta)^2$. Let $\emptyset \neq K \subseteq \Gamma \times \Delta$. By $\mathit{least}(K) \in K$ we denote the least element of K with respect to $\trianglelefteq$. Let $\theta \in \mathit{Form}_{\mathcal{L}}$, $n_\theta \in \mathbb{N}$, $(J, S) \in \Gamma \times \Delta$. (J, S) is a clausal translation of θ with respect to n_θ iff either $J = \emptyset$ or $J = \{(n_\theta, j) \mid j \leq n_J\}$, $J \subseteq \{(n_\theta, j) \mid j \in \mathbb{N}\}$, $S \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$, (I a,b,d,f,h,k–m) hold for n_θ, θ, J, S. We put $K^\theta_{n_\theta} = \{(J, S) \mid (J, S) \text{ is a clausal translation of } \theta \text{ with respect to } n_\theta\} \subseteq \Gamma \times \Delta$. We have T is countable. Hence, there exist $\gamma \leq \omega$ and a sequence $\delta : \gamma \longrightarrow T$ of T. We put $n_\alpha = n_0 + \alpha$, $\alpha < \gamma$. Then, for all $\alpha < \gamma$, $\delta(\alpha) \in T \subseteq \mathit{Form}_{\mathcal{L}}$, $n_\alpha \geq n_0$, by (I) for n_α, $\delta(\alpha)$, $K^{\delta(\alpha)}_{n_\alpha} \neq \emptyset$. We put $(J_\alpha, S_\alpha) = \mathit{least}(K^{\delta(\alpha)}_{n_\alpha}) \in \Gamma \times \Delta$, $\alpha < \gamma$. Then, for all $\alpha < \gamma$, either $J_\alpha = \emptyset$ or $J_\alpha = \{(n_\alpha, j) \mid j \leq n_{J_\alpha}\}$, $J_\alpha \subseteq \{(n_\alpha, j) \mid j \in \mathbb{N}\}$, $S_\alpha \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\alpha\}}$, (I a,b,d,f,h,k–m) hold for n_α, $\delta(\alpha)$, J_α, S_α; for all $\alpha < \alpha' < \gamma$, $n_\alpha < n_{\alpha'}$;

$$\text{for all } \alpha < \alpha' < \gamma, J_\alpha \cap J_{\alpha'} \subseteq \{(n_\alpha, j) \mid j \in \mathbb{N}\} \cap \{(n_{\alpha'}, j) \mid j \in \mathbb{N}\} = \emptyset. \tag{87}$$

We put

$$J_T = \begin{cases} \emptyset & \text{if there exists } \alpha^* < \gamma \text{ such that } S_{\alpha^*} = \{\square\}, \\ \bigcup_{\alpha < \gamma} J_\alpha & \text{else;} \end{cases}$$

$$S_T = \begin{cases} \{\square\} & \text{if there exists } \alpha^* < \gamma \text{ such that } S_{\alpha^*} = \{\square\}, \\ \bigcup_{\alpha < \gamma} S_\alpha & \text{else.} \end{cases}$$

We distinguish two cases.

Case 1: There exists $\alpha^* < \gamma$ such that $S_{\alpha^*} = \{\square\}$. Then $J_T = \emptyset \subseteq \{(i, j) \mid i \geq n_0\}$ and $S_T = \{\square\} \subseteq \mathit{SimOrdCl}_{\mathcal{L}}$.

We have $J_T = \emptyset$, $S_T = \{\square\}$; (a) holds.

By (I b) for J_{α^*}, S_{α^*}, $J_{\alpha^*} = \emptyset$, $S_{\alpha^*} \subseteq \mathit{SimOrdCl}_{\mathcal{L}}$; for every interpretation $\mathfrak{A}$ for $\mathcal{L}$, $\mathfrak{A} \not\models \{\square\} = S_{\alpha^*}$, by (I d) for $\delta(\alpha^*)$, J_{α^*}, S_{α^*}, $\mathfrak{A}$, $\mathfrak{A} \not\models \delta(\alpha^*) \in T$, $\mathfrak{A} \not\models T$, $\mathfrak{A} \not\models S_T$; trivially, there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models T$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L}$ and $\mathfrak{A}' \models S_T$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$; (b) holds.

$J_T \subseteq_{\mathcal{F}} \{(i, j) \mid i \geq n_0\}$, $\|J_T\| = 0 \leq 2 \cdot |T|$, $S_T \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}}$, $|S_T| = 0 \in O(|T|^2)$.

We have $S_T = \{\square\}$; (d) holds trivially.

$\mathit{qatoms}(S_T) = \emptyset$; (e) holds trivially.

We have $J_T = \emptyset$; (f) holds trivially.

$\mathit{tcons}(S_T) = \{\mathit{0}, \mathit{1}\} \subseteq \mathit{tcons}(T)$; (g) holds.

Case 2: For all $\alpha < \gamma$, $S_\alpha \neq \{\square\}$. We have, for all $\alpha < \gamma$, $n_\alpha \geq n_0$. Then $J_T = \bigcup_{\alpha<\gamma} J_\alpha \subseteq \bigcup_{\alpha<\gamma} \{(n_\alpha, j) \mid j \in \mathbb{N}\} \subseteq \{(i, j) \mid i \geq n_0\}$ and $S_T = \bigcup_{\alpha<\gamma} S_\alpha \subseteq \bigcup_{\alpha<\gamma} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\alpha\}} \subseteq \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$.

Hence, for all $\alpha < \gamma$, either $S_\alpha = \emptyset$, $\square \not\in \emptyset = S_\alpha$, or $S_\alpha \neq \emptyset$, by (I b), $\square \not\in S_\alpha$; $\square \not\in S_\alpha$; $\square \not\in S_T$. We distinguish two cases.

Case 2.1: There exists $\alpha^* < \gamma$ such that $S_{\alpha^*} \neq \emptyset$. Then $S_{\alpha^*} \neq \{\square\}$, by (I b) for J_{α^*}, S_{α^*}, $\emptyset \neq J_{\alpha^*} \subseteq J_T$, $\emptyset \neq S_{\alpha^*} \subseteq S_T$; (a) holds.

Case 2.2: For all $\alpha < \gamma$, $S_\alpha = \emptyset$. Then, for all $\alpha < \gamma$, by (I b), $J_\alpha = \emptyset$; $J_T = S_T = \emptyset$; (a) holds.

So, in both Cases 2.1 and 2.2, (a) holds; (a) holds.

Let $\mathfrak{A}$ be an interpretation for $\mathcal{L}$ such that $\mathfrak{A} \models T$. Then, for all $\alpha < \gamma$, $\mathfrak{A} \models \delta(\alpha) \in T$, by (I d) for $\mathfrak{A}$, there exists an interpretation $\mathfrak{A}_\alpha$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\alpha\}$ and $\mathfrak{A}_\alpha \models S_\alpha$, $\mathfrak{A}_\alpha|_{\mathcal{L}} = \mathfrak{A}$. For all $\alpha < \alpha' < \gamma$, $\{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\alpha\} \cap \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_{\alpha'}\} \overset{(87)}{=\!=} \emptyset$. We define an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}$ as follows:

$$\tilde{p}_{\mathbb{j}}^{\mathfrak{A}'} = \tilde{p}_{\mathbb{j}}^{\mathfrak{A}_\alpha}, \quad \mathbb{j} \in J_\alpha, \alpha < \gamma.$$

We get, for all $\alpha < \gamma$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\alpha\}} = \mathfrak{A}_\alpha \models S_\alpha$; $\mathfrak{A}' \models S_T$, $\mathfrak{A}'|_{\mathcal{L}} = \mathfrak{A}$.

Let $\mathfrak{A}'$ be an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}$ such that $\mathfrak{A}' \models S_T$. Then, for all $\alpha < \gamma$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\alpha\}} \models S_\alpha$, by (I d) for $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\alpha\}}$, $\mathfrak{A}'|_{\mathcal{L}} \models \delta(\alpha)$. We put $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$, an interpretation for $\mathcal{L}$. We get, for all $\alpha < \gamma$, $\mathfrak{A} \models \delta(\alpha)$; $\mathfrak{A} \models T$, $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$; (b) holds.

Let $\alpha < \alpha' < \gamma$. We distinguish two cases for S_α, $S_{\alpha'}$.

Case 2.3: $S_\alpha = \emptyset$ or $S_{\alpha'} = \emptyset$. Then $S_\alpha \cap S_{\alpha'} = \emptyset$.

Case 2.4: $S_\alpha, S_{\alpha'} \neq \emptyset$. We have, for all $\alpha < \gamma$, $S_\alpha \neq \{\square\}$. Then, by (I h), for all $C \in S_\alpha$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\alpha\}$, $S_{\alpha'} \neq \{\square\}$, by (I h) for $J_{\alpha'}$, $S_{\alpha'}$, for all $C \in S_{\alpha'}$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_{\alpha'}\}$; for all $C \in S_\alpha$ and $C' \in S_{\alpha'}$, $(\mathit{preds}(C) \cap \tilde{\mathbb{P}}) \cap (\mathit{preds}(C') \cap \tilde{\mathbb{P}}) \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\alpha\} \cap \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_{\alpha'}\} \overset{(87)}{=} \emptyset$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}}$, $\emptyset \neq \mathit{preds}(C') \cap \tilde{\mathbb{P}}$, $\mathit{preds}(C) \cap \tilde{\mathbb{P}} \neq \mathit{preds}(C') \cap \tilde{\mathbb{P}}$, $\mathit{preds}(C) \neq \mathit{preds}(C')$, $C \neq C'$; $S_\alpha \cap S_{\alpha'} = \emptyset$.

So, in both Cases 2.3 and 2.4, $S_\alpha \cap S_{\alpha'} = \emptyset$;

$$\text{for all } \alpha < \alpha' < \gamma, S_\alpha \cap S_{\alpha'} = \emptyset. \tag{88}$$

Let $T \subseteq_{\mathcal{F}} \mathit{Form}_{\mathcal{L}}$. Then $\gamma < \omega$, $J_T = \bigcup_{\alpha < \gamma < \omega} J_\alpha \subseteq_{\mathcal{F}} \{(i, j) \,|\, i \geq n_0\}$; we have, for all $\alpha < \gamma$, by (I a), $\|J_\alpha\| \leq 2 \cdot |\delta(\alpha)|$; $\|J_T\| \overset{(87)}{=} \sum_{\alpha < \gamma < \omega} \|J_\alpha\| \leq \sum_{\alpha < \gamma < \omega} 2 \cdot |\delta(\alpha)| = 2 \cdot \sum_{\alpha < \gamma < \omega} |\delta(\alpha)| = 2 \cdot |T|$; $S_T = \bigcup_{\alpha < \gamma < \omega} S_\alpha \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$; we have, for all $\alpha < \gamma$, by (I f), $|S_\alpha| \in O(|\delta(\alpha)|^2)$; $|S_T| \overset{(88)}{=} \sum_{\alpha < \gamma < \omega} |S_\alpha| \in O(\sum_{\alpha < \gamma < \omega} |\delta(\alpha)|^2) = O((\sum_{\alpha < \gamma < \omega} |\delta(\alpha)|)^2) = O(|T|^2)$.

Let $S_T \neq \emptyset$. We have, for all $\alpha < \gamma$, $S_\alpha \neq \{\square\}$. Then there exists $\alpha^* < \gamma$ and $S_{\alpha^*} \neq \emptyset, \{\square\}$, by (I b) for J_{α^*}, S_{α^*}, $\emptyset \neq J_{\alpha^*} \subseteq J_T$; for all $C \in S_T$, there exists $\alpha^* < \gamma$ and $C \in S_{\alpha^*}$, $S_{\alpha^*} \neq \emptyset, \{\square\}$, by (I h) for J_{α^*}, S_{α^*}, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_{\alpha^*}\} \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}$; (d) holds.

For all $a \in \mathit{qatoms}(S_T) = \bigcup_{\alpha < \gamma} \mathit{qatoms}(S_\alpha)$, there exists $\alpha^* < \gamma$ and $a \in \mathit{qatoms}(S_{\alpha^*})$; by (I k) for J_{α^*}, S_{α^*}, there exists $\mathbb{j}^* \in J_{\alpha^*} \subseteq J_T$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; (e) holds.

For all $\mathbb{j} \in J_T$, there exists $\alpha^* < \gamma$ and $\mathbb{j} \in J_{\alpha^*}$; by (I l) for J_{α^*}, S_{α^*}, there exist a sequence $\bar{x}$ of variables of $\mathcal{L}$ and $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(S_{\alpha^*}) \subseteq \bigcup_{\alpha < \gamma} \mathit{atoms}(S_\alpha) = \mathit{atoms}(S_T)$ satisfying, for all $a \in \mathit{atoms}(S_{\alpha^*})$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; if there exists $a^* \in \mathit{qatoms}(S_{\alpha^*})$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S_{\alpha^*}) \subseteq \bigcup_{\alpha < \gamma} \mathit{qatoms}(S_\alpha) = \mathit{qatoms}(S_T)$ satisfying, for all $a \in \mathit{qatoms}(S_{\alpha^*})$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; for all $\alpha < \gamma$ and $\alpha \neq \alpha^*$, $J_\alpha \cap J_{\alpha^*} \overset{(87)}{=} \emptyset$, $\mathbb{j} \not\in J_\alpha$, $\tilde{p}_{\mathbb{j}} \not\in \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\alpha\}$, $\tilde{p}_{\mathbb{j}} \not\in \mathit{preds}(S_\alpha) \subseteq \mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\alpha\}}$; for all $a \in \mathit{atoms}(S_T) = \bigcup_{\alpha < \gamma} \mathit{atoms}(S_\alpha)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, for all $\alpha < \gamma$ and $\alpha \neq \alpha^*$, $a \not\in \mathit{atoms}(S_\alpha)$, $a \in \mathit{atoms}(S_{\alpha^*})$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; if there exists $a^* \in \mathit{qatoms}(S_T) = \bigcup_{\alpha < \gamma} \mathit{qatoms}(S_\alpha)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, for all $\alpha < \gamma$ and $\alpha \neq \alpha^*$, $a^* \not\in \mathit{qatoms}(S_\alpha)$, $a^* \in \mathit{qatoms}(S_{\alpha^*})$, there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S_T)$, for all $a \in \mathit{qatoms}(S_T) = \bigcup_{\alpha < \gamma} \mathit{qatoms}(S_\alpha)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, for all $\alpha < \gamma$ and $\alpha \neq \alpha^*$, $a \not\in \mathit{qatoms}(S_\alpha)$, $a \in \mathit{qatoms}(S_{\alpha^*})$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; (f) holds.

$\mathit{tcons}(S_T) = \bigcup_{\alpha < \gamma} \mathit{tcons}(S_\alpha) \underset{\text{(I m)}}{\subseteq} \bigcup_{\alpha < \gamma} \mathit{tcons}(\delta(\alpha)) = \mathit{tcons}(T)$; (g) holds.

Let $T \subseteq_{\mathcal{F}} \mathit{Form}_{\mathcal{L}}$. Then $\gamma < \omega$; we have, in both Cases 1 and 2, $J_T \subseteq_{\mathcal{F}} \{(i, j) \,|\, i \geq n_0\}$, $\|J_T\| \leq 2 \cdot |T|$, $S_T \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$, $|S_T| \in O(|T|^2)$; the translation of T to S_T uses the input T and the output S_T; we have, for all $\alpha < \gamma < \omega$, by (I f), the time and space complexity of the translation of $\delta(\alpha)$ to S_α, is in $O(|\delta(\alpha)|^2 \cdot (\log(1 + n_\alpha) + \log |\delta(\alpha)|))$; for all $\alpha < \gamma < \omega$, $\gamma = \|T\| \leq |T|$, $n_\alpha < n_0 + \gamma \leq n_0 + |T|$, $|\delta(\alpha)| \leq \sum_{\alpha < \gamma < \omega} |\delta(\alpha)| = |T|$; the translation of T to S_T uses a constant number of auxiliary data structures of constant size with respect to input and data structures, including *index generator*, values of which are of the form $(n_\alpha, j) \in \mathbb{I}$, $j \leq n_{J_\alpha} = \|J_\alpha\| + 1 \underset{\text{(I a)}}{\in} O(|\delta(\alpha)|)$; the time and space complexity of an elementary operation on an auxiliary data structure is of the order of its size, $O(1)$ for data structures of constant size with respect to input, including the test *if* $S_\alpha = \{\square\}$, $O(\log(1 + n_\alpha) + \log |\delta(\alpha)|)$ for data structures of the form $(n_\alpha, j) \in \mathbb{I}$, $j \in O(|\delta(\alpha)|)$; the translation of T to S_T executes a constant number of elementary operations on auxiliary data structures; the time and space complexity is in $O(\log(1 + n_\alpha) + \log |\delta(\alpha)|)$; the translation of T to S_T also executes updating the partial output $\bigcup_{\beta < \alpha} S_\beta$, $S_\beta \neq \{\square\}$, by S_α, including appending S_α to $\bigcup_{\beta < \alpha} S_\beta$, $S_\beta \neq \{\square\}$, which uses the input S_α and the output S_α (a copy), with $\#\mathcal{O}(S_\alpha) \in O(|S_\alpha|) \underset{\text{(I f)}}{\subseteq} O(|\delta(\alpha)|^2)$; by (13) for n_α, S_α, $\emptyset$, S_α, $q = 2$, $r = 1$, the time complexity is in $O(\#\mathcal{O}(S_\alpha) \cdot (\log(1 + n_\alpha) + \log(\#\mathcal{O}(S_\alpha) + |S_\alpha|))) \subseteq O(|\delta(\alpha)|^2 \cdot (\log(1 + n_\alpha) + \log |\delta(\alpha)|))$; by (14) for n_α, S_α, $\emptyset$, S_α, $q = 2$, $r = 1$, the space complexity is in $O((\#\mathcal{O}(S_\alpha) + |S_\alpha|) \cdot (\log(1 + n_\alpha) + \log(1 + |S_\alpha|))) \subseteq O(|\delta(\alpha)|^2 \cdot (\log(1 + n_\alpha) + \log |\delta(\alpha)|))$; the total time and space complexity of the translation of T to S_T at the α-th stage, is in $O(|\delta(\alpha)|^2 \cdot (\log(1 + n_\alpha) + \log |\delta(\alpha)|))$; the time and space complexity of the translation of T to S_T, is in

$$\begin{aligned}
&O(\sum_{\alpha < \gamma < \omega} |\delta(\alpha)|^2 \cdot (\log(1 + n_\alpha) + \log |\delta(\alpha)|)) \subseteq \\
&O((\sum_{\alpha < \gamma < \omega} |\delta(\alpha)|)^2 \cdot (\log(1 + n_0 + |T|) + \log(1 + |T|))) = \\
&O(|T|^2 \cdot \log(1 + n_0 + |T|));
\end{aligned}$$

(c) holds.

So, in both Cases 1 and 2, (a–g) hold; (a–g) hold. Thus, (II) holds. The lemma is proved. ☐

9. Full proof of Theorem 4

Proof. We get by Lemma 3(II) for n_0+1, T that there exist $J_T \subseteq \{(i,j) \mid i \geq n_0+1\}$, $S_T \subseteq \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \mid \mathbb{j}\in J_T\}}$, and Lemma 3(II a–g) hold for n_0+1, T, J_T, S_T. By (15) for n_0, ϕ, there exists $\phi' \in \mathit{Form}_{\mathcal{L}}$ such that (15a–e) hold for n_0, ϕ, ϕ'. We distinguish three cases for ϕ'.

Case 1: $\phi' \in \mathit{Tcons}_{\mathcal{L}} - \{1\}$. We put $J_T^\phi = J_T \subseteq \{(i,j) \mid i \geq n_0+1\} \subseteq \{(i,j) \mid i \geq n_0\}$ and $S_T^\phi = S_T \subseteq \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \mid \mathbb{j}\in J_T^\phi\}}$.

For every interpretation $\mathfrak{A}$ for $\mathcal{L}$, $\mathfrak{A} \not\models \phi' \overset{(15a)}{\equiv} \phi$; by Lemma 3(II b), there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models T$, $\mathfrak{A} \not\models \phi$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}$ and $\mathfrak{A}' \models S_T^\phi$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$; (i) holds.

Let $T \subseteq_{\mathcal{F}} \mathit{Form}_{\mathcal{L}}$. Then, by Lemma 3(II c), $J_T^\phi \subseteq_{\mathcal{F}} \{(i,j) \mid i \geq n_0\}$, $\|J_T^\phi\| \leq 2 \cdot |T| \in O(|T| + |\phi|)$, $S_T^\phi \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \mid \mathbb{j}\in J_T^\phi\}}$, $|S_T^\phi| \in O(|T|^2) \subseteq O(|T|^2 + |\phi|^2)$; the translation of T and ϕ to S_T^ϕ uses the input T, ϕ, the output S_T^ϕ, an auxiliary ϕ'; we have, by (15b), ϕ' can be built up from ϕ via a postorder traversal of ϕ with $\#\mathcal{O}(\phi) \in O(|\phi|)$ and the time, space complexity in $O(|\phi| \cdot (\log(1+n_0) + \log|\phi|))$; the test $\phi' \in \mathit{Tcons}_{\mathcal{L}} - \{1\}$ is with $\#\mathcal{O}(\phi') \in O(1)$ and the time, space complexity in $O(1)$; by Lemma 3(II c), the number of all elementary operations of the translation of T to S_T^ϕ, is in $O(|T|^2)$; the time and space complexity of the translation of T to S_T^ϕ, is in $O(|T|^2 \cdot \log(1+n_0+|T|))$; the number of all elementary operations of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 + |\phi|^2)$; the time and space complexity of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 \cdot \log(1+n_0+|T|) + |\phi|^2 \cdot (\log(1+n_0) + \log|\phi|))$; (ii) holds.

By Lemma 3(II e), for all $a \in \mathit{qatoms}(S_T^\phi)$, there exists $\mathbb{j}^* \in J_T^\phi$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; by Lemma 3(II f), for all $\mathbb{j} \in J_T^\phi$, there exist a sequence $\bar{x}$ of variables of $\mathcal{L}$ and $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(S_T^\phi)$ satisfying, for all $a \in \mathit{atoms}(S_T^\phi)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; if there exists $a^* \in \mathit{qatoms}(S_T^\phi)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S_T^\phi)$ satisfying, for all $a \in \mathit{qatoms}(S_T^\phi)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; (iii) holds.

$\mathit{tcons}(S_T^\phi) \underset{\text{(Lemma 3(II g))}}{\subseteq} \mathit{tcons}(\phi) \cup \mathit{tcons}(T)$; (iv) holds.

Case 2: $\phi' = 1$. We put $J_T^\phi = \emptyset \subseteq \{(i,j) \mid i \geq n_0\}$ and $S_T^\phi = \{\square\} \subseteq \mathit{SimOrdCl}_{\mathcal{L}}$.

For every interpretation $\mathfrak{A}$ for $\mathcal{L}$, $\mathfrak{A} \models \phi' \overset{(15a)}{\equiv} \phi$; trivially, there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models T$, $\mathfrak{A} \not\models \phi$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L}$ and $\mathfrak{A}' \models S_T^\phi$, satisfying $\mathfrak{A} = \mathfrak{A}' = \mathfrak{A}'|_{\mathcal{L}}$; (i) holds.

Let $T \subseteq_{\mathcal{F}} \mathit{Form}_{\mathcal{L}}$. Then $J_T^\phi \subseteq_{\mathcal{F}} \{(i,j) \mid i \geq n_0\}$, $\|J_T^\phi\| = 0 \in O(|T| + |\phi|)$, $S_T^\phi \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}}$, $|S_T^\phi| = 0 \in O(|T|^2 + |\phi|^2)$; the translation of T and ϕ to S_T^ϕ uses the input T, ϕ, the output S_T^ϕ, an auxiliary ϕ'; we have ϕ' can be built up from ϕ via a postorder traversal of ϕ with $\#\mathcal{O}(\phi) \in O(|\phi|)$ and the time, space complexity in $O(|\phi| \cdot (\log(1+n_0) + \log|\phi|))$; the test $\phi' = 1$ is with $\#\mathcal{O}(\phi') \in O(1)$ and the time, space complexity in $O(1)$; S_T^ϕ can be built up with $\#\mathcal{O} \in O(1)$ and the time, space complexity in $O(1)$; the number of all elementary operations of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 + |\phi|^2)$; the time and space complexity of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 \cdot \log(1+n_0+|T|) + |\phi|^2 \cdot (\log(1+n_0) + \log|\phi|))$; (ii) holds.

$\mathit{qatoms}(S_T^\phi) = \emptyset$; we have $J_T^\phi = \emptyset$; (iii) holds trivially.

$\mathit{tcons}(S_T^\phi) = \{0, 1\} \subseteq \mathit{tcons}(\phi) \cup \mathit{tcons}(T)$; (iv) holds.

Case 3: $\phi' \notin \mathit{Tcons}_{\mathcal{L}}$. We have $\phi' \in \mathit{Form}_{\mathcal{L}}$, (15c,d) hold for ϕ'. We put $\bar{x} = \mathit{varseq}(\phi')$. Then $\phi' \in \mathit{Form}_{\mathcal{L}} - \mathit{Tcons}_{\mathcal{L}} \subseteq \mathit{Form}_{\mathcal{L}} - \{0, 1\}$, $\mathit{vars}(\bar{x}) = \mathit{vars}(\phi') \subseteq \mathit{Var}_{\mathcal{L}}$, $\forall \bar{x}\, \phi' \in \mathit{Form}_{\mathcal{L}} - \{0, 1\}$, (15c,d) hold for $\forall \bar{x}\, \phi'$, $\mathit{vars}(\forall \bar{x}\, \phi') = \mathit{vars}(\bar{x}) \cup \mathit{vars}(\phi') = \mathit{vars}(\bar{x}) \subseteq \mathit{Var}_{\mathcal{L}}$, $|\bar{x}| \leq |\phi'|, |\forall \bar{x}\, \phi'|$. We put $j_{\mathbb{i}} = 0$ and $\mathbb{i} = (n_0, j_{\mathbb{i}}) \in \{(n_0, j) \mid j \in \mathbb{N}\}$. $\tilde{p}_{\mathbb{i}} \in \tilde{\mathbb{P}}$. We put $\mathit{ar}(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$. We get by (17) for n_0, $\forall \bar{x}\, \phi'$, $\bar{x}$, $\mathbb{i}$, $\tilde{p}_{\mathbb{i}}$ that there exist $J = \{(n_0, j) \mid 1 \leq j \leq n_J\} \subseteq \{(n_0, j) \mid j \in \mathbb{N}\}$, $j_{\mathbb{i}} \leq n_J$, $\mathbb{i} \notin J$, $S^- \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{j}} \mid \mathbb{j}\in J\}}$, and (17f,h,j,m–o) hold for $\forall \bar{x}\, \phi'$, $\bar{x}$, $\tilde{p}_{\mathbb{i}}$, J, S^-. We put $J_T^\phi = J_T \cup \{\mathbb{i}\} \cup J \subseteq \{(i,j) \mid i \geq n_0\}$. Then $J_T \cap (\{\mathbb{i}\} \cup J) \subseteq \{(i,j) \mid i \geq n_0+1\} \cap \{(n_0, j) \mid j \in \mathbb{N}\} = \emptyset$,

$$J_T, \{\mathbb{i}\}, J \text{ are pairwise disjoint.} \tag{89}$$

We put $S_T^\phi = S_T \cup \{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\} \cup S^- \subseteq \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \mid \mathbb{j}\in J_T^\phi\}}$. Then, by (17j), for all $C \in S^-$, $\mathit{preds}(\square) = \emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \mathit{preds}(C)$, $C \neq \square$; $\square \notin S^-$, by (17j), $\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\} \cap S^- = \emptyset$. We distinguish three cases for S_T.

Case 3.1: $S_T = \emptyset$. Then $S_T \cap (\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\} \cup S^-) = \emptyset$.

Case 3.2: $S_T = \{\square\}$. We have $\square \notin S^-$. Then $\square \notin \{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\}$, $S_T \cap (\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\} \cup S^-) = \emptyset$.

Case 3.3: $S_T \neq \emptyset, \{\square\}$. Then, by Lemma 3(II d), for all $C \in S_T$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}$, by (17j), for all $C \in S^-$, $\emptyset \neq \mathit{preds}(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}$; for all $C_1 \in S_T$ and $C_2 \in \{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\} \cup S^-$, $\emptyset \neq \mathit{preds}(C_1) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}$, either $C_2 = \tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1$, $\emptyset \neq \{\tilde{p}_{\mathbb{i}}\} = \mathit{preds}(C_2) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}$ or $C_2 \in S^-$, $\emptyset \neq \mathit{preds}(C_2) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}$; $\emptyset \neq \mathit{preds}(C_2) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}$, $(\mathit{preds}(C_1) \cap \tilde{\mathbb{P}}) \cap (\mathit{preds}(C_2) \cap \tilde{\mathbb{P}}) \subseteq \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\} \cap (\{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}) \overset{(89)}{=} \emptyset$, for both i, $\emptyset \neq \mathit{preds}(C_i) \cap \tilde{\mathbb{P}}$, $\mathit{preds}(C_1) \cap \tilde{\mathbb{P}} \neq \mathit{preds}(C_2) \cap \tilde{\mathbb{P}}$, $\mathit{preds}(C_1) \neq \mathit{preds}(C_2)$, $C_1 \neq C_2$; $S_T \cap (\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\} \cup S^-) = \emptyset$.

So, in all Cases 3.1–3.3, $S_T \cap (\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\} \cup S^-) = \emptyset$; $S_T \cap (\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\} \cup S^-) = \emptyset$;

$$S_T, \{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\}, S^- \text{ are pairwise disjoint.} \tag{90}$$

Let $\mathfrak{A}$ be an interpretation for $\mathcal{L}$ such that $\mathfrak{A} \models T$, $\mathfrak{A} \not\models \phi$. Then, by Lemma 3(II b) for $\mathfrak{A}$, there exists an interpretation $\mathfrak{A}_T$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}$ and $\mathfrak{A}_T \models S_T$, $\mathfrak{A}_T|_{\mathcal{L}} = \mathfrak{A}$; we have $vars(\bar{x}) = vars(\phi')$; $\forall \bar{x}\, \phi'$ is closed, $\mathfrak{A} \not\models \phi \overset{(15a)}{=\!=} \phi'$, $\mathfrak{A} \not\models \forall \bar{x}\, \phi'$, $\|\forall \bar{x}\, \phi'\|^{\mathfrak{A}} < 1$. We define an expansion $\mathfrak{A}^{\#}$ of $\mathfrak{A}$ to $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ as follows:

$$\tilde{p}_{\mathbb{i}}^{\mathfrak{A}^{\#}}(u_1, \dots, u_{|\bar{x}|}) = \|\forall \bar{x}\, \phi'\|^{\mathfrak{A}}.$$

Then, for all $e \in \mathcal{S}_{\mathfrak{A}^{\#}}$, $\|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = \tilde{p}_{\mathbb{i}}^{\mathfrak{A}^{\#}}(\|\bar{x}\|_e^{\mathfrak{A}^{\#}}) = \|\forall \bar{x}\, \phi'\|^{\mathfrak{A}} = \|\forall \bar{x}\, \phi'\|^{\mathfrak{A}^{\#}} < 1$, $\|\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\|_e^{\mathfrak{A}^{\#}} = \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} \prec 1 = 1$, $\|\forall \bar{x}\, \phi' \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = \|\forall \bar{x}\, \phi'\|^{\mathfrak{A}^{\#}} \Rightarrow \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}^{\#}} = \|\forall \bar{x}\, \phi'\|^{\mathfrak{A}^{\#}} \Rightarrow \|\forall \bar{x}\, \phi'\|^{\mathfrak{A}^{\#}} = 1$; $\mathfrak{A}^{\#} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}$, $\mathfrak{A}^{\#} \models \forall \bar{x}\, \phi' \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})$, by (17f) for $\mathfrak{A}^{\#}$, there exists an interpretation $\mathfrak{A}_\phi$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}_\phi \models S^-$, $\mathfrak{A}_\phi|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} = \mathfrak{A}^{\#}$. $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\} \cap (\{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}) \overset{(89)}{=\!=} \emptyset$. We define an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^\phi\}$ as follows:

$$\tilde{p}_{\mathbb{j}}^{\mathfrak{A}'} = \begin{cases} \tilde{p}_{\mathbb{j}}^{\mathfrak{A}_T} & \mathit{if}\ \mathbb{j} \in J_T, \\ \tilde{p}_{\mathbb{j}}^{\mathfrak{A}_\phi} & \mathit{if}\ \mathbb{j} \in \{\mathbb{i}\} \cup J. \end{cases}$$

We get $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}} = \mathfrak{A}_T \models S_T$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} = \mathfrak{A}^{\#} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}} = \mathfrak{A}_\phi \models S^-$; $\mathfrak{A}' \models S_T^\phi$, $\mathfrak{A}'|_{\mathcal{L}} = \mathfrak{A}$.

Let $\mathfrak{A}'$ be an interpretation for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^\phi\}$ such that $\mathfrak{A}' \models S_T^\phi$. Then $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}} \models S_T$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}} \models S^-$, by Lemma 3(II b) for $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$, $\mathfrak{A}'|_{\mathcal{L}} \models T$, by (17f) for $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$, $\mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} \models \forall \bar{x}\, \phi' \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})$; for all $e \in \mathcal{S}_{\mathfrak{A}'}$, $1 = \|\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\|_e^{\mathfrak{A}'} = \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}'} \prec 1$, $\|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}'} < 1$, $1 = \|\forall \bar{x}\, \phi' \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}'} = \|\forall \bar{x}\, \phi'\|^{\mathfrak{A}'} \Rightarrow \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}'}$, $\|\forall \bar{x}\, \phi'\|^{\mathfrak{A}'} \leq \|\tilde{p}_{\mathbb{i}}(\bar{x})\|_e^{\mathfrak{A}'} < 1$; $\mathfrak{A}'|_{\mathcal{L}} \not\models \forall \bar{x}\, \phi'$, $\mathfrak{A}'|_{\mathcal{L}} \not\models \phi' \overset{(15a)}{=\!=} \phi$. We put $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$, an interpretation for $\mathcal{L}$. Then $\mathfrak{A} \models T$ and $\mathfrak{A} \not\models \phi$, $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$; (i) holds.

Let $T \subseteq_{\mathcal{F}} \mathit{Form}_{\mathcal{L}}$. Then, by Lemma 3(II c), $J_T \subseteq_{\mathcal{F}} \{(i,j) \,|\, i \geq n_0 + 1\} \subseteq \{(i,j) \,|\, i \geq n_0\}$, $\|J_T\| \leq 2 \cdot |T|$, $S_T \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$, $|S_T| \in O(|T|^2)$; we have $\mathbb{i} \in \{(n_0, j) \,|\, j \in \mathbb{N}\}$, $|\bar{x}| \leq |\phi'|$; $J \subseteq_{\mathcal{F}} \{(n_0, j) \,|\, j \in \mathbb{N}\}$, $\|J\| \underset{(17a)}{\leq} |\forall \bar{x}\, \phi'| - 1 < |\forall \bar{x}\, \phi'| = 2 \cdot |\bar{x}| + |\phi'| \leq 3 \cdot |\phi'| \underset{(15b)}{\leq} 6 \cdot |\phi|$; we have $S^- \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$; $|S^-|, |\forall \bar{x}\, \phi'| \cdot (1 + |\bar{x}|) \underset{(17h)}{\leq} 15 \cdot |\forall \bar{x}\, \phi'| \cdot (1 + |\bar{x}|) \leq 15 \cdot 3 \cdot |\phi'| \cdot 2 \cdot |\phi'| = 90 \cdot |\phi'|^2 \underset{(15b)}{\leq} 360 \cdot |\phi|^2 \in O(|\phi|^2)$; $J_T^\phi \subseteq_{\mathcal{F}} \{(i,j) \,|\, i \geq n_0\}$, $\|J_T^\phi\| \overset{(89)}{=\!=} \|J_T\| + \|\{\mathbb{i}\}\| + \|J\| \leq 2 \cdot |T| + 1 + 6 \cdot |\phi| \in O(|T| + |\phi|)$, $S_T^\phi \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^\phi\}}$, $|S_T^\phi| \overset{(90)}{=\!=} |S_T| + |\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}| + |S^-| = |S_T| + |\bar{x}| + 3 + |S^-| \leq |S_T| + 4 \cdot |\phi'| + |S^-| \underset{(15b)}{\leq} |S_T| + 8 \cdot |\phi| + |S^-| \in O(|T|^2 + |\phi|^2)$; the translation of T and ϕ to S_T^ϕ uses the input T, ϕ, the output S_T^ϕ, auxiliary S_T, ϕ', $\tilde{f}_0(\bar{x})$, $\forall \bar{x}\, \phi'$, $\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}$, S^-; we have ϕ' can be built up from ϕ via a postorder traversal of ϕ with $\#\mathcal{O}_1(\phi) \in O(|\phi|)$; the test $\phi' \not\in \mathit{Tcons}_{\mathcal{L}}$ is with $\#\mathcal{O}_2(\phi') \in O(1)$; by Lemma 3(II c), the number of all elementary operations of the translation of T to S_T, is in $O(|T|^2)$; the time and space complexity of the translation of T to S_T, is in $O(|T|^2 \cdot \log(1 + n_0 + |T|))$; $\tilde{f}_0(\bar{x})$ can be built up from ϕ' via the left-right preorder traversal of ϕ' with $\#\mathcal{O}_3(\phi') \in O(|\phi'|) \underset{(15b)}{\subseteq} O(|\phi|)$; $\forall \bar{x}\, \phi'$ can be built up from ϕ' and $\tilde{f}_0(\bar{x})$ with $\#\mathcal{O}_4(\phi', \tilde{f}_0(\bar{x})) \in O(|\forall \bar{x}\, \phi'|) \subseteq O(|\phi|)$; $\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}$ can be built up from $\tilde{f}_0(\bar{x})$ with $\#\mathcal{O}_5(\tilde{f}_0(\bar{x})) \in O(|\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}|) = O(1 + |\bar{x}|) \subseteq O(|\phi'|) \underset{(15b)}{\subseteq} O(|\phi|)$; by (17h), S^- can be built up from $\forall \bar{x}\, \phi'$ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of $\forall \bar{x}\, \phi'$ with $\#\mathcal{O}_6(\forall \bar{x}\, \phi', \tilde{f}_0(\bar{x})) \in O(|\forall \bar{x}\, \phi'| \cdot (1 + |\bar{x}|)) \subseteq O(|\phi|^2)$; S_T^ϕ can be built up from $\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}$ and S^- by copying and appending to S_T with $\#\mathcal{O}_7(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}, S^-) \in O(|\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}| + |S^-|) \subseteq O(|\phi|^2)$; $\sum_{i=1}^{7} \#\mathcal{O}_i \in O(|\phi|^2)$, by (13) for n_0, ϕ, $\emptyset$, ϕ', $\tilde{f}_0(\bar{x})$, $\forall \bar{x}\, \phi'$, $\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}$, S^-, $\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}$ (a copy), S^- (a copy), $q = 8$, $r = 2$, the total time complexity of elementary operations at the stages $1, \dots, 7$, is in $O((\sum_{i=1}^{7} \#\mathcal{O}_i) \cdot (\log(1 + n_0) + \log((\sum_{i=1}^{7} \#\mathcal{O}_i) + |\phi|))) \subseteq O(|\phi|^2 \cdot (\log(1 + n_0) + \log |\phi|))$; by (14) for n_0, ϕ, $\emptyset$, ϕ', $\tilde{f}_0(\bar{x})$, $\forall \bar{x}\, \phi'$, $\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}$, S^-, $\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}$ (a copy), S^- (a copy), $q = 8$, $r = 2$, the total space complexity of elementary operations at the stages $1, \dots, 7$, is in $O(((\sum_{i=1}^{7} \#\mathcal{O}_i) + |\phi|^2) \cdot (\log(1 + n_0) + \log |\phi|)) \subseteq O(|\phi|^2 \cdot (\log(1 + n_0) + \log |\phi|))$; the number of all elementary operations of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 + |\phi|^2)$; the time and space complexity of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 \cdot \log(1 + n_0 + |T|) + |\phi|^2 \cdot (\log(1 + n_0) + \log |\phi|))$; (ii) holds.

By Lemma 3(II e), for all $a \in \mathit{qatoms}(S_T)$, there exists $\mathbb{j}^* \in J_T \subseteq J_T^\phi$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; trivially, for all $a \in \emptyset = \mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\})$, there exists $\mathbb{j}^* \in J_T^\phi$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; by (17m), for all $a \in \mathit{qatoms}(S^-)$, there exists $\mathbb{j}^* \in J \subseteq J_T^\phi$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; for all $a \in \mathit{qatoms}(S_T^\phi) = \mathit{qatoms}(S_T) \cup \mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}) \cup \mathit{qatoms}(S^-)$, there exists $\mathbb{j}^* \in J_T^\phi$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$; (a) holds.

We have, by Lemma 3(II f), for all $\mathbb{j} \in J_T$, there exist a sequence $\bar{x}^*$ of variables of $\mathcal{L}$ and $\tilde{p}_{\mathbb{j}}(\bar{x}^*) \in \mathit{atoms}(S_T)$ satisfying, for all $a \in \mathit{atoms}(S_T)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x}^*)$; if there exists $a^* \in \mathit{qatoms}(S_T)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Q^* x^*\, \tilde{p}_{\mathbb{j}}(\bar{x}^*) \in \mathit{qatoms}(S_T)$ satisfying, for all $a \in \mathit{qatoms}(S_T)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Q^* x^*\, \tilde{p}_{\mathbb{j}}(\bar{x}^*)$; $S_T \subseteq \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$. Then $\tilde{p}_{\mathbb{i}} \underset{(89)}{\not\in} \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}$, $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(S_T) \subseteq \mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$, $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(\mathit{qatoms}(S_T)) \subseteq \mathit{preds}(S_T)$; $\tilde{p}_{\mathbb{i}}(\bar{x}) \in \mathit{atoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\})$ satisfying, for all $a \in \mathit{atoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}) = \{\mathit{1}, \tilde{p}_{\mathbb{i}}(\bar{x})\}$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{i}}\}$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; $\mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}) = \emptyset$, $\tilde{p}_{\mathbb{i}} \not\in \emptyset = \mathit{preds}(\mathit{qatoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}))$; we have, by (17n), for all $\mathbb{j} \in \{\mathbb{i}\} \cup J$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{atoms}(S^-)$ satisfying, for all $a \in \mathit{atoms}(S^-)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}} \not\in \mathit{preds}(\mathit{qatoms}(S^-))$, for all $\mathbb{j} \in J$, if there exists $a^* \in \mathit{qatoms}(S^-)$ and $\mathit{preds}(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in \mathit{qatoms}(S^-)$ satisfying, for all $a \in \mathit{qatoms}(S^-)$ and $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$; $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T \mid J\} \cap \{\tilde{p}_{\mathbb{i}}\} \overset{(89)}{=\!=} \emptyset$, $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T \mid J\} \cap \mathit{preds}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\}) \subseteq \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T \mid J\} \cap \mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}} = \emptyset$; we have $S^- \subseteq \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$; $(\mathit{preds}(S_T) \cap \tilde{\mathbb{P}}) \cap (\mathit{preds}(S^-) \cap \tilde{\mathbb{P}}) \subseteq (\mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}} \cap \tilde{\mathbb{P}}) \cap (\mathit{Pred}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}} \cap \tilde{\mathbb{P}}) = \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\} \cap (\{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}) \overset{(89)}{=\!=} \emptyset$; $(q)\mathit{atoms}(S_T^\phi) = (q)\mathit{atoms}(S_T) \cup (q)\mathit{atoms}(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec$

$1\}) \cup (q)atoms(S^-)$; $\tilde{p}_{\mathbb{i}} \notin preds(qatoms(S_T)) \cup preds(qatoms(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\})) \cup preds(qatoms(S^-)) = preds(qatoms(S_T) \cup qatoms(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\}) \cup qatoms(S^-)) = preds(qatoms(S_T^{\phi}))$. Let $\mathbb{j} \in J_T^{\phi} = J_T \cup \{\mathbb{i}\} \cup J$. We distinguish three cases for $\mathbb{j}$.

Case 3.4: $\mathbb{j} \in J_T$. Then $\tilde{p}_{\mathbb{j}} \notin preds(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\})$, $\tilde{p}_{\mathbb{j}}(\bar{x}^*) \in atoms(S_T) \subseteq atoms(S_T^{\phi})$, $\tilde{p}_{\mathbb{j}} \in preds(S_T)$, $\tilde{p}_{\mathbb{j}} \in \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \in preds(S_T) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \notin preds(S^-) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \notin preds(S^-)$; for all $a \in atoms(S_T^{\phi})$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \notin atoms(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\})$, $a \notin atoms(S^-)$, $a \in atoms(S_T)$, $a = \tilde{p}_{\mathbb{j}}(\bar{x}^*)$; if there exists $a^* \in qatoms(S_T^{\phi})$ and $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, $a^* \notin \emptyset = qatoms(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\})$, $a^* \notin qatoms(S^-)$, $a^* \in qatoms(S_T)$, there exists $Q^* x^* \, \tilde{p}_{\mathbb{j}}(\bar{x}^*) \in qatoms(S_T) \subseteq qatoms(S_T^{\phi})$, for all $a \in qatoms(S_T^{\phi})$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \notin \emptyset = qatoms(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\})$, $a \notin qatoms(S^-)$, $a \in qatoms(S_T)$, $a = Q^* x^* \, \tilde{p}_{\mathbb{j}}(\bar{x}^*)$; (b) holds.

Case 3.5: $\mathbb{j} = \mathbb{i}$. Then $\tilde{p}_{\mathbb{i}}(\bar{x}) \in atoms(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\}), atoms(S^-) \subseteq atoms(S_T^{\phi})$; we have $\tilde{p}_{\mathbb{i}} \notin preds(S_T)$; for all $a \in atoms(S_T^{\phi})$ and $preds(a) = \{\tilde{p}_{\mathbb{i}}\}$, $a \notin atoms(S_T)$, $a \in atoms(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\}) \cup atoms(S^-)$, for both the cases $a \in atoms(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\})$ and $a \in atoms(S^-)$, $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; $a = \tilde{p}_{\mathbb{i}}(\bar{x})$; (b) holds.

Case 3.6: $\mathbb{j} \in J$. Then $\tilde{p}_{\mathbb{j}} \notin preds(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\})$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in atoms(S^-) \subseteq atoms(S_T^{\phi})$, $\tilde{p}_{\mathbb{j}} \in preds(S^-)$, $\tilde{p}_{\mathbb{j}} \in \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \in preds(S^-) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \notin preds(S_T) \cap \tilde{\mathbb{P}}$, $\tilde{p}_{\mathbb{j}} \notin preds(S_T)$; for all $a \in atoms(S_T^{\phi})$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \notin atoms(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\})$, $a \notin atoms(S_T)$, $a \in atoms(S^-)$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; if there exists $a^* \in qatoms(S_T^{\phi})$ and $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, $a^* \notin \emptyset = qatoms(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\})$, $a^* \notin qatoms(S_T)$, $a^* \in qatoms(S^-)$, there exists $Qx \, \tilde{p}_{\mathbb{j}}(\bar{x}) \in qatoms(S^-) \subseteq qatoms(S_T^{\phi})$, for all $a \in qatoms(S_T^{\phi})$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a \notin \emptyset = qatoms(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\})$, $a \notin qatoms(S_T)$, $a \in qatoms(S^-)$, $a = Qx \, \tilde{p}_{\mathbb{j}}(\bar{x})$; (b) holds.

So, in all Cases 3.4–3.6, (b) holds; (b) holds; (iii) holds.

$tcons(S_T^{\phi}) = tcons(S_T) \cup tcons(\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\}) \cup tcons(S^-) = tcons(S_T) \cup \{0, 1\} \cup tcons(S^-) \underset{(17o)}{\overset{(\text{Lemma 3(II g)})}{\subseteq}} tcons(T) \cup tcons(\forall \bar{x}\, \phi') = tcons(T) \cup tcons(\phi') \underset{(15e)}{\subseteq} tcons(T) \cup tcons(\phi)$; (iv) holds.

Thus, in all Cases 1–3, (i–iv) hold; (i–iv) hold. The theorem is proved. □

10. Full proof of Lemma 6

PROOF. Let $C \in \mathit{clo}^{\mathcal{BH}}(S)$. Then there exist a deduction $\mathcal{D} = C_1^{\mathcal{B}}, \dots, C_n^{\mathcal{B}}$, $C_\kappa^{\mathcal{B}} \in \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $C = C_n^{\mathcal{B}}$, $n \geq 1$, of C from S by basic order hyperresolution, associated $\mathcal{L}_\kappa^{\mathcal{B}}$, $S_\kappa^{\mathcal{B}}$, $\kappa = 0, \dots, n$. At first, we prove the following statement:

$$\text{For all } 1 \leq \sigma \leq n, \text{ there exist a deduction } \mathcal{D}_\sigma = C_1, \dots, C_\sigma,\ C_\kappa \in \mathit{OrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}, \text{ of } C_\sigma \text{ from } S \text{ by order hyperresolution, associated } \mathcal{L}_\kappa,\ S_\kappa,\ \kappa = 0, \dots, \sigma, \text{ such that } \mathcal{L}_\kappa = \mathcal{L}_\kappa^{\mathcal{B}}; \text{ for all } 1 \leq \kappa \leq \sigma, \text{ there exists } \vartheta_\kappa \in \mathit{Subst}_{\mathcal{L}_\kappa},\ \mathit{dom}(\vartheta_\kappa) = \mathit{freevars}(C_\kappa), \text{ and } C_\kappa^{\mathcal{B}} = C_\kappa \vartheta_\kappa. \tag{91}$$

We proceed by induction on $1 \leq \sigma \leq n$.

Case 1 (the base case): $\sigma = 1$. $\mathcal{L}_0^{\mathcal{B}} = \mathcal{L} \cup P$, $S_0^{\mathcal{B}} = \emptyset \subseteq \mathit{GOrdCl}_{\mathcal{L}_0^{\mathcal{B}}}$, $(q)\mathit{atoms}(S_0^{\mathcal{B}}) = \emptyset$, Rules (42)–(48) are not applicable to $S_0^{\mathcal{B}}$, $C_1^{\mathcal{B}} \in \mathit{ordtcons}(S) \cup \mathit{GInst}_{\mathcal{L}_0^{\mathcal{B}}}(S)$; there exist $C_1 \in \mathit{ordtcons}(S) \cup S$, $\vartheta_1 \in \mathit{Subst}_{\mathcal{L}_0^{\mathcal{B}}}$, $\mathit{dom}(\vartheta_1) = \mathit{freevars}(C_1)$, and $C_1^{\mathcal{B}} = C_1 \vartheta_1$; $\mathcal{L}_1^{\mathcal{B}} = \mathcal{L}_0^{\mathcal{B}} = \mathcal{L} \cup P$. We put $\mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L} \cup P$, $S_0 = \emptyset \subseteq \mathit{OrdCl}_{\mathcal{L}_0}$, $\mathcal{D}_1 = C_1$, $C_1 \in \mathit{ordtcons}(S) \cup S \subseteq \mathit{OrdCl}_{\mathcal{L} \cup P} \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $S_1 = \{C_1\} \subseteq \mathit{OrdCl}_{\mathcal{L} \cup P} = \mathit{OrdCl}_{\mathcal{L}_1}$; $\mathcal{D}_1$ is a deduction of C_1 from S by order hyperresolution. Then $\mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L}_0^{\mathcal{B}} = \mathcal{L}_1^{\mathcal{B}}$, $\vartheta_1 \in \mathit{Subst}_{\mathcal{L}_0^{\mathcal{B}}} = \mathit{Subst}_{\mathcal{L}_1}$; (91) holds.

Case 2 (the induction case): $1 < \sigma \leq n$. By induction hypothesis for $\sigma - 1$, there exist a deduction $\mathcal{D}_{\sigma-1} = C_1, \dots, C_{\sigma-1}$, $C_\kappa \in \mathit{OrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, of $C_{\sigma-1}$ from S by order hyperresolution, associated $\mathcal{L}_\kappa$, S_κ, $\kappa = 0, \dots, \sigma - 1$, such that $\mathcal{L}_\kappa = \mathcal{L}_\kappa^{\mathcal{B}}$, $C_\kappa \in S_\kappa \subseteq S_{\sigma-1}$; for all $1 \leq \kappa \leq \sigma - 1$, there exists $\vartheta_\kappa \in \mathit{Subst}_{\mathcal{L}_\kappa}$, $\mathit{dom}(\vartheta_\kappa) = \mathit{freevars}(C_\kappa)$, and $C_\kappa^{\mathcal{B}} = C_\kappa \vartheta_\kappa$. We distinguish eight cases for $C_\sigma^{\mathcal{B}}$.

Case 2.1: $C_\sigma^{\mathcal{B}} \in \mathit{ordtcons}(S) \cup \mathit{GInst}_{\mathcal{L}_{\sigma-1}^{\mathcal{B}}}(S)$. Then $\mathcal{L}_\sigma^{\mathcal{B}} = \mathcal{L}_{\sigma-1}^{\mathcal{B}}$, there exist $C_\sigma \in \mathit{ordtcons}(S) \cup S$, $\vartheta_\sigma \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}^{\mathcal{B}}}$, $\mathit{dom}(\vartheta_\sigma) = \mathit{freevars}(C_\sigma)$, $\mathit{range}(\vartheta_\sigma) = \mathit{freevars}(C_\sigma^{\mathcal{B}}) = \emptyset$, and $C_\sigma^{\mathcal{B}} = C_\sigma \vartheta_\sigma$, $\mathcal{L} \cup P = \mathcal{L}_0 \subseteq \mathcal{L}_{\sigma-1}$. We put $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, C_\sigma$, $C_\sigma \in \mathit{ordtcons}(S) \cup S \subseteq \mathit{OrdCl}_{\mathcal{L} \cup P} \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $S_\sigma = S_{\sigma-1} \cup \{C_\sigma\} \subseteq \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} \cup \mathit{OrdCl}_{\mathcal{L} \cup P} = \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} = \mathit{OrdCl}_{\mathcal{L}_\sigma}$; $\mathcal{D}_\sigma$ is a deduction of C_σ from S by order hyperresolution. Hence, $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1} = \mathcal{L}_{\sigma-1}^{\mathcal{B}} = \mathcal{L}_\sigma^{\mathcal{B}}$, $\vartheta_\sigma \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}^{\mathcal{B}}} = \mathit{Subst}_{\mathcal{L}_\sigma}$; (91) holds.

Case 2.2: There exist $1 \leq j_k^* \leq \sigma - 1$, $k = 0, \dots, m$, such that $C_\sigma^{\mathcal{B}} \in S_\sigma^{\mathcal{B}}$ is a basic order resolvent of $C_{j_1^*}^{\mathcal{B}}, \dots, C_{j_m^*}^{\mathcal{B}} \in S_{\sigma-1}^{\mathcal{B}}$ using Rule (42) with respect to $\mathcal{L}_{\sigma-1}^{\mathcal{B}}$, $S_{\sigma-1}^{\mathcal{B}}$. Then $\mathcal{L}_\sigma^{\mathcal{B}} = \mathcal{L}_{\sigma-1}^{\mathcal{B}}$, for all $k \leq m$, $C_{j_k^*}^{\mathcal{B}} = \varepsilon_{j_k^*} \diamond_{j_k^*} \upsilon_{j_k^*} \vee \overline{C_{j_k^*}}$, $\varepsilon_{j_k^*} \diamond_{j_k^*} \upsilon_{j_k^*} \notin \overline{C_{j_k^*}}$, $\varepsilon_{j_0^*} \diamond_{j_0^*} \upsilon_{j_0^*}, \dots, \varepsilon_{j_m^*} \diamond_{j_m^*} \upsilon_{j_m^*}$ is a contradiction of $\mathcal{L}_{\sigma-1}^{\mathcal{B}}$; $\varepsilon_{j_0^*} = \mathit{1}$ or $\upsilon_{j_m^*} = \mathit{0}$ or $\varepsilon_{j_0^*} = \upsilon_{j_m^*}$, there exists $k^* \leq m$ and $\diamond_{j_{k^*}^*} = \prec$; $C_\sigma^{\mathcal{B}} = \bigvee_{k=0}^m \overline{C_{j_k^*}}$; there exists $\vartheta_{j_k^*} \in \mathit{Subst}_{\mathcal{L}_{j_k^*}} \subseteq \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $\mathit{dom}(\vartheta_{j_k^*}) = \mathit{freevars}(C_{j_k^*})$, $\mathit{range}(\vartheta_{j_k^*}) = \mathit{freevars}(C_{j_k^*}^{\mathcal{B}}) = \emptyset$, and $\vartheta_{j_k^*}$ is applicable to $C_{j_k^*}$, $C_{j_k^*}^{\mathcal{B}} = C_{j_k^*} \vartheta_{j_k^*}$, $C_{j_k^*} \in S_{\sigma-1}$; there exist variable renamings $\rho_{j_k^*} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $\mathit{dom}(\rho_{j_k^*}) = \mathit{freevars}(C_{j_k^*})$, $\mathit{range}(\rho_{j_k^*}) = \mathit{freevars}(C_{j_k^*} \rho_{j_k^*})$, $k = 0, \dots, m$, and $\rho_{j_k^*}$ is applicable to $C_{j_k^*}$, for all $k < k' \leq m$, $\mathit{range}(\rho_{j_k^*}) \cap \mathit{range}(\rho_{j_{k'}^*}) = \mathit{freevars}(C_{j_k^*} \rho_{j_k^*}) \cap \mathit{freevars}(C_{j_{k'}^*} \rho_{j_{k'}^*}) = \emptyset$, $C_{j_k^*} \rho_{j_k^*} \in S_{\sigma-1}^{\mathit{Vr}}$; $\rho_{j_k^*}^{-1} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$ is a variable renaming, $\mathit{dom}(\rho_{j_k^*}^{-1}) = \mathit{range}(\rho_{j_k^*}) = \mathit{freevars}(C_{j_k^*} \rho_{j_k^*})$, $\mathit{range}(\rho_{j_k^*}^{-1}) = \mathit{dom}(\rho_{j_k^*}) = \mathit{freevars}(C_{j_k^*}) = \mathit{dom}(\vartheta_{j_k^*})$; $\rho_{j_k^*} \circ \rho_{j_k^*}^{-1} = \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{dom}(\rho_{j_k^*})} = \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(C_{j_k^*})} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$; for all $Qx\, a \in \mathit{qatoms}(C_{j_k^*})$, $\mathit{freevars}(Qx\, a) \subseteq \mathit{freevars}(C_{j_k^*})$, $x \notin \mathit{freevars}(Qx\, a) = \mathit{range}(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(Qx\, a)})$, $\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(C_{j_k^*})}$ is applicable to $Qx\, a$; $\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(C_{j_k^*})}$ is applicable to $C_{j_k^*}$; $C_{j_k^*}(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(C_{j_k^*})}) = C_{j_k^*}(\rho_{j_k^*} \circ \rho_{j_k^*}^{-1}) = (C_{j_k^*} \rho_{j_k^*}) \rho_{j_k^*}^{-1}$, $\rho_{j_k^*}^{-1}$ is applicable to $C_{j_k^*} \rho_{j_k^*}$; for all $k < k' \leq m$, $\mathit{dom}(\rho_{j_k^*}^{-1}) \cap \mathit{dom}(\rho_{j_{k'}^*}^{-1}) = \mathit{range}(\rho_{j_k^*}) \cap \mathit{range}(\rho_{j_{k'}^*}) = \emptyset$. We put $\eta = \bigcup_{k=0}^m \rho_{j_k^*}^{-1} \circ \vartheta_{j_k^*} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $\mathit{dom}(\eta) = \bigcup_{k=0}^m \mathit{dom}(\rho_{j_k^*}^{-1} \circ \vartheta_{j_k^*}) = \bigcup_{k=0}^m \mathit{dom}(\rho_{j_k^*}^{-1}) = \bigcup_{k=0}^m \mathit{freevars}(C_{j_k^*} \rho_{j_k^*})$, $\mathit{range}(\eta) = \bigcup_{k=0}^m \mathit{range}(\rho_{j_k^*}^{-1} \circ \vartheta_{j_k^*}) = \bigcup_{k=0}^m \mathit{range}(\vartheta_{j_k^*}) = \bigcup_{k \leq m} \mathit{freevars}(C_{j_k^*}^{\mathcal{B}}) = \emptyset$. Hence, for all $k \leq m$, $((C_{j_k^*} \rho_{j_k^*}) \rho_{j_k^*}^{-1}) \vartheta_{j_k^*} = (C_{j_k^*}(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(C_{j_k^*})})) \vartheta_{j_k^*} = C_{j_k^*} \vartheta_{j_k^*}$, $\rho_{j_k^*}^{-1} \circ \vartheta_{j_k^*}$ is applicable to $C_{j_k^*} \rho_{j_k^*}$; η is applicable to $C_{j_k^*} \rho_{j_k^*}$; $(C_{j_k^*} \rho_{j_k^*}) \eta = (C_{j_k^*} \rho_{j_k^*})(\rho_{j_k^*}^{-1} \circ \vartheta_{j_k^*}) = ((C_{j_k^*} \rho_{j_k^*}) \rho_{j_k^*}^{-1}) \vartheta_{j_k^*} = C_{j_k^*} \vartheta_{j_k^*} = C_{j_k^*}^{\mathcal{B}}$, $C_{j_k^*} \rho_{j_k^*} = \bigvee_{q=0}^{r_k} \varepsilon_q^k \diamond_q^k \upsilon_q^k \vee \bigvee_{q=1}^{s_k} l_q^k$, $(\bigvee_{q=0}^{r_k} \varepsilon_q^k \diamond_q^k \upsilon_q^k) \cap (\bigvee_{q=1}^{s_k} l_q^k) = \emptyset$, $(\bigvee_{q=0}^{r_k} \varepsilon_q^k \diamond_q^k \upsilon_q^k) \eta = \varepsilon_{j_k^*} \diamond_{j_k^*} \upsilon_{j_k^*}$, $(\bigvee_{q=1}^{s_k} l_q^k) \eta = \overline{C_{j_k^*}}$; η is applicable to and a unifier for

$$\overline{E} = \Big(\bigvee_{q=0}^{r_0} \varepsilon_q^0 \diamond_q^0 \upsilon_q^0, l_1^0, \dots, l_{s_0}^0, \dots, \bigvee_{q=0}^{r_m} \varepsilon_q^m \diamond_q^m \upsilon_q^m, l_1^m, \dots, l_{s_m}^m, \{\upsilon_0^0, \varepsilon_0^1\}, \dots, \{\upsilon_0^{m-1}, \varepsilon_0^m\}, \{a, b\} \Big),$$

$a = \varepsilon_0^0$, $b = \mathit{1}$ or $a = \upsilon_0^m$, $b = \mathit{0}$ or $a = \varepsilon_0^0$, $b = \upsilon_0^m$, $\mathit{dom}(\eta) = \bigcup_{k \leq m} \mathit{freevars}(C_{j_k^*} \rho_{j_k^*}) = \mathit{freevars}\big(\{\varepsilon_q^k \diamond_q^k \upsilon_q^k \,|\, q \leq r_k, k \leq m\}, \{l_q^k \,|\, 1 \leq q \leq s_k, k \leq m\}\big) = \mathit{freevars}(\overline{E})$, $\diamond_0^{k^*} = \diamond_{j_{k^*}^*} = \prec$; by Theorem 2 for $\overline{E}$, η, there exists $\theta^* \in \mathit{mgu}_{\mathcal{L}_{\sigma-1}}(\overline{E})$, $\mathit{dom}(\theta^*) = \mathit{freevars}(\overline{E}) = \mathit{dom}(\eta)$, and θ^* is applicable to $\overline{E}$, for all $k \leq m$, to $C_{j_k^*} \rho_{j_k^*}$; there exists $\gamma^* \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $\mathit{dom}(\gamma^*) = \mathit{range}(\theta^*)$, and $\eta = \theta^* \circ \gamma^*$; for all $k \leq m$, $(C_{j_k^*} \rho_{j_k^*}) \eta = (C_{j_k^*} \rho_{j_k^*})(\theta^* \circ \gamma^*) = ((C_{j_k^*} \rho_{j_k^*}) \theta^*) \gamma^*$, γ^* is applicable to $(C_{j_k^*} \rho_{j_k^*}) \theta^*$; γ^* is applicable to $(\bigvee_{q=1}^{s_k} l_q^k) \theta^* \sqsubseteq (C_{j_k^*} \rho_{j_k^*}) \theta^*$; using Rule (49) with respect to $\mathcal{L}_{\sigma-1}$, $S_{\sigma-1}$, we derive $(\bigvee_{k=0}^m \bigvee_{q=1}^{s_k} l_q^k) \theta^* \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}}$. We put $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1}$, $C_\sigma = (\bigvee_{k=0}^m \bigvee_{q=1}^{s_k} l_q^k) \theta^* \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, C_\sigma$, $C_\sigma \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $S_\sigma = S_{\sigma-1} \cup \{C_\sigma\} \subseteq \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} = \mathit{OrdCl}_{\mathcal{L}_\sigma}$, $\vartheta_\sigma = \gamma^*|_{\mathit{freevars}(C_\sigma)} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}} = \mathit{Subst}_{\mathcal{L}_\sigma}$, $\mathit{dom}(\vartheta_\sigma) = \mathit{freevars}(C_\sigma)$; $\mathcal{D}_\sigma$ is a deduction of C_σ from S by order hyperresolution. Hence, $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1} = \mathcal{L}_{\sigma-1}^{\mathcal{B}} = \mathcal{L}_\sigma^{\mathcal{B}}$, ϑ_σ is applicable to C_σ, $C_\sigma \vartheta_\sigma = ((\bigvee_{k=0}^m \bigvee_{q=1}^{s_k} l_q^k) \theta^*) \gamma^* = \bigvee_{k=0}^m (\bigvee_{q=1}^{s_k} l_q^k)(\theta^* \circ \gamma^*) = \bigvee_{k=0}^m (\bigvee_{q=1}^{s_k} l_q^k) \eta = \bigvee_{k=0}^m \overline{C_{j_k^*}} = C_\sigma^{\mathcal{B}}$, $\mathit{range}(\vartheta_\sigma) = \mathit{freevars}(C_\sigma^{\mathcal{B}}) = \emptyset$; (91) holds.

Case 2.3: There exist $a, b \in \mathit{atoms}(S_{\sigma-1}^{\mathcal{B}}) \subseteq \mathit{Atom}_{\mathcal{L}_{\sigma-1}^{\mathcal{B}}}$, $a \in \overline{C}_{\mathcal{L}}$, $b \notin \mathit{Tcons}_{\mathcal{L}}$, $\mathit{qatoms}(S) = \emptyset$, such that $C_\sigma^{\mathcal{B}} = a \prec b \vee a \eqcirc b \vee b \prec a \in S_\sigma^{\mathcal{B}}$ is a basic order trichotomy resolvent of a and b using Rule (43) with respect to $\mathcal{L}_{\sigma-1}^{\mathcal{B}}$, $S_{\sigma-1}^{\mathcal{B}}$. Then $\mathcal{L}_\sigma^{\mathcal{B}} = \mathcal{L}_{\sigma-1}^{\mathcal{B}}$, there exist $1 \leq j_1^*, j_2^* \leq \sigma - 1$ and $a \in \mathit{atoms}(C_{j_1^*}^{\mathcal{B}})$, $b \in \mathit{atoms}(C_{j_2^*}^{\mathcal{B}})$, $C_{j_1^*}^{\mathcal{B}}, C_{j_2^*}^{\mathcal{B}} \in S_{\sigma-1}^{\mathcal{B}}$; there exists $\vartheta_{j_k^*} \in \mathit{Subst}_{\mathcal{L}_{j_k^*}} \subseteq \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $k = 1, 2$, $\mathit{dom}(\vartheta_{j_k^*}) = \mathit{freevars}(C_{j_k^*})$, $\mathit{range}(\vartheta_{j_k^*}) = \mathit{freevars}(C_{j_k^*}^{\mathcal{B}}) = \emptyset$, and $\vartheta_{j_k^*}$ is applicable to $C_{j_k^*}$, $C_{j_k^*}^{\mathcal{B}} = C_{j_k^*} \vartheta_{j_k^*}$, $C_{j_k^*} \in S_{\sigma-1}$; we have $a \in \overline{C}_{\mathcal{L}}$; $a \in \mathit{atoms}(C_{j_1^*}) \subseteq \mathit{atoms}(S_{\sigma-1}) \subseteq \mathit{Atom}_{\mathcal{L}_{\sigma-1}}$, there exists $b' \in \mathit{atoms}(C_{j_2^*}) \subseteq \mathit{atoms}(S_{\sigma-1}) \subseteq \mathit{Atom}_{\mathcal{L}_{\sigma-1}}$, $\mathit{vars}(b') \subseteq \mathit{freevars}(C_{j_2^*}) = \mathit{dom}(\vartheta_{j_2^*})$, $b' \notin \mathit{Tcons}_{\mathcal{L}}$, and $b = b' \vartheta_{j_2^*}$; using Rule (50) with respect to $\mathcal{L}_{\sigma-1}$, $S_{\sigma-1}$, we derive $a \prec b' \vee a \eqcirc b' \vee b' \prec a \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}}$. We put $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1}$, $C_\sigma = a \prec b' \vee a \eqcirc b' \vee b' \prec a \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, C_\sigma$, $C_\sigma \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $S_\sigma = S_{\sigma-1} \cup \{C_\sigma\} \subseteq \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} = \mathit{OrdCl}_{\mathcal{L}_\sigma}$,

$\vartheta_\sigma = \vartheta_{j_2^*}|_{vars(b')} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}} = \mathit{Subst}_{\mathcal{L}_\sigma}$, $dom(\vartheta_\sigma) = vars(b') = \mathit{freevars}(C_\sigma)$, $range(\vartheta_\sigma) = \emptyset$; $\mathcal{D}_\sigma$ is a deduction of C_σ from S by order hyperresolution. Hence, $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1} = \mathcal{L}^{\mathcal{B}}_{\sigma-1} = \mathcal{L}^{\mathcal{B}}_\sigma$, $a\vartheta_\sigma = a$, $b'\vartheta_\sigma = b'\vartheta_{j_2^*} = b$, $C_\sigma\vartheta_\sigma = a\vartheta_\sigma \prec b'\vartheta_\sigma \vee a\vartheta_\sigma \eqcirc b'\vartheta_\sigma \vee b'\vartheta_\sigma \prec a\vartheta_\sigma = a \prec b \vee a \eqcirc b \vee b \prec a = C^{\mathcal{B}}_\sigma$; (91) holds.

Case 2.4: There exist $a, b \in atoms(S^{\mathcal{B}}_{\sigma-1}) - \{0,1\} \subseteq atoms(S^{\mathcal{B}}_{\sigma-1}) \subseteq \mathit{Atom}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}}$, $\{a,b\} \not\subseteq \mathit{Tcons}_{\mathcal{L}}$, $qatoms(S) \neq \emptyset$, such that $C^{\mathcal{B}}_\sigma = a \prec b \vee a \eqcirc b \vee b \prec a \in S^{\mathcal{B}}_\sigma$ is a basic order trichotomy resolvent of a and b using Rule (44) with respect to $\mathcal{L}^{\mathcal{B}}_{\sigma-1}$, $S^{\mathcal{B}}_{\sigma-1}$. Then $\mathcal{L}^{\mathcal{B}}_\sigma = \mathcal{L}^{\mathcal{B}}_{\sigma-1}$, there exist $1 \le j_1^*, j_2^* \le \sigma - 1$ and $a \in atoms(C^{\mathcal{B}}_{j_1^*}) - \{0,1\}$, $b \in atoms(C^{\mathcal{B}}_{j_2^*}) - \{0,1\}$, $C^{\mathcal{B}}_{j_1^*}, C^{\mathcal{B}}_{j_2^*} \in S^{\mathcal{B}}_{\sigma-1}$; there exists $\vartheta_{j_k^*} \in \mathit{Subst}_{\mathcal{L}_{j_k^*}} \subseteq \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $k = 1,2$, $dom(\vartheta_{j_k^*}) = \mathit{freevars}(C_{j_k^*})$, $range(\vartheta_{j_k^*}) = \mathit{freevars}(C^{\mathcal{B}}_{j_k^*}) = \emptyset$, and $\vartheta_{j_k^*}$ is applicable to $C_{j_k^*}$, $C^{\mathcal{B}}_{j_k^*} = C_{j_k^*}\vartheta_{j_k^*}$, $C_{j_k^*} \in S_{\sigma-1}$; there exist $a' \in atoms(C_{j_1^*}) - \{0,1\} \subseteq atoms(S_{\sigma-1}) - \{0,1\} \subseteq atoms(S_{\sigma-1}) \subseteq \mathit{Atom}_{\mathcal{L}_{\sigma-1}}$, $vars(a') \subseteq \mathit{freevars}(C_{j_1^*}) = dom(\vartheta_{j_1^*})$, $b' \in atoms(C_{j_2^*}) - \{0,1\} \subseteq atoms(S_{\sigma-1}) - \{0,1\} \subseteq \mathit{Atom}_{\mathcal{L}_{\sigma-1}}$, $vars(b') \subseteq \mathit{freevars}(C_{j_2^*}) = dom(\vartheta_{j_2^*})$, $\{a', b'\} \not\subseteq \mathit{Tcons}_{\mathcal{L}}$, and $a = a'\vartheta_{j_1^*}$, $b = b'\vartheta_{j_2^*}$; there exist variable renamings $\rho_{j_1^*}, \rho_{j_2^*} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $dom(\rho_{j_1^*}) = vars(a')$, $range(\rho_{j_1^*}) = vars(a'\rho_{j_1^*})$, $dom(\rho_{j_2^*}) = vars(b')$, $range(\rho_{j_2^*}) = vars(b'\rho_{j_2^*})$, and $range(\rho_{j_1^*}) \cap range(\rho_{j_2^*}) = vars(a'\rho_{j_1^*}) \cap vars(b'\rho_{j_2^*}) = \emptyset$, $a'\rho_{j_1^*}, b'\rho_{j_2^*} \in atoms(S^{Vr}_{\sigma-1}) - \{0,1\} \subseteq \mathit{Atom}_{\mathcal{L}_{\sigma-1}}$, $\{a'\rho_{j_1^*}, b'\rho_{j_2^*}\} \not\subseteq \mathit{Tcons}_{\mathcal{L}}$; for both k, $\rho^{-1}_{j_k^*} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$ is a variable renaming, $dom(\rho^{-1}_{j_1^*}) = range(\rho_{j_1^*}) = vars(a'\rho_{j_1^*})$, $range(\rho^{-1}_{j_1^*}) = dom(\rho_{j_1^*}) = vars(a') \subseteq dom(\vartheta_{j_1^*})$, $dom(\rho^{-1}_{j_2^*}) = range(\rho_{j_2^*}) = vars(b'\rho_{j_2^*})$, $range(\rho^{-1}_{j_2^*}) = dom(\rho_{j_2^*}) = vars(b') \subseteq dom(\vartheta_{j_2^*})$, $dom(\rho^{-1}_{j_1^*}) \cap dom(\rho^{-1}_{j_2^*}) = range(\rho_{j_1^*}) \cap range(\rho_{j_2^*}) = \emptyset$; using Rule (51) with respect to $\mathcal{L}_{\sigma-1}$, $S_{\sigma-1}$, we derive $a'\rho_{j_1^*} \prec b'\rho_{j_2^*} \vee a'\rho_{j_1^*} \eqcirc b'\rho_{j_2^*} \vee b'\rho_{j_2^*} \prec a'\rho_{j_1^*} \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}}$. We put $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1}$, $C_\sigma = a'\rho_{j_1^*} \prec b'\rho_{j_2^*} \vee a'\rho_{j_1^*} \eqcirc b'\rho_{j_2^*} \vee b'\rho_{j_2^*} \prec a'\rho_{j_1^*} \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, C_\sigma$, $C_\sigma \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} \subseteq \mathit{OrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $S_\sigma = S_{\sigma-1} \cup \{C_\sigma\} \subseteq \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} = \mathit{OrdCl}_{\mathcal{L}_\sigma}$, $\vartheta_\sigma = \rho^{-1}_{j_1^*} \circ \vartheta_{j_1^*} \cup \rho^{-1}_{j_2^*} \circ \vartheta_{j_2^*} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}} = \mathit{Subst}_{\mathcal{L}_\sigma}$, $dom(\vartheta_\sigma) = dom(\rho^{-1}_{j_1^*}) \cup dom(\rho^{-1}_{j_2^*}) = vars(a'\rho_{j_1^*}) \cup vars(b'\rho_{j_2^*}) = \mathit{freevars}(C_\sigma)$, $range(\vartheta_\sigma) = range(\rho^{-1}_{j_1^*} \circ \vartheta_{j_1^*}) \cup range(\rho^{-1}_{j_2^*} \circ \vartheta_{j_2^*}) = range(\vartheta_{j_1^*}|_{range(\rho^{-1}_{j_1^*})}) \cup range(\vartheta_{j_2^*}|_{range(\rho^{-1}_{j_2^*})}) = \emptyset$; $\mathcal{D}_\sigma$ is a deduction of C_σ from S by order hyperresolution. Hence, $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1} = \mathcal{L}^{\mathcal{B}}_{\sigma-1} = \mathcal{L}^{\mathcal{B}}_\sigma$, $(a'\rho_{j_1^*})\vartheta_\sigma = (a'\rho_{j_1^*})(\rho^{-1}_{j_1^*} \circ \vartheta_{j_1^*}) = a'(\rho_{j_1^*} \circ \rho^{-1}_{j_1^*} \circ \vartheta_{j_1^*}) = a'(id_{\mathcal{L}_{\sigma-1}}|_{vars(a')} \circ \vartheta_{j_1^*}) = a'\vartheta_{j_1^*} = a$, $(b'\rho_{j_2^*})\vartheta_\sigma = (b'\rho_{j_2^*})(\rho^{-1}_{j_2^*} \circ \vartheta_{j_2^*}) = b'(\rho_{j_2^*} \circ \rho^{-1}_{j_2^*} \circ \vartheta_{j_2^*}) = b'(id_{\mathcal{L}_{\sigma-1}}|_{vars(b')} \circ \vartheta_{j_2^*}) = b'\vartheta_{j_2^*} = b$, $C_\sigma\vartheta_\sigma = (a'\rho_{j_1^*})\vartheta_\sigma \prec (b'\rho_{j_2^*})\vartheta_\sigma \vee (a'\rho_{j_1^*})\vartheta_\sigma \eqcirc (b'\rho_{j_2^*})\vartheta_\sigma \vee (b'\rho_{j_2^*})\vartheta_\sigma \prec (a'\rho_{j_1^*})\vartheta_\sigma = a \prec b \vee a \eqcirc b \vee b \prec a = C^{\mathcal{B}}_\sigma$; (91) holds.

Case 2.5: There exist $\forall x\, a \in qatoms^{\forall}(S^{\mathcal{B}}_{\sigma-1}) \subseteq \mathit{QAtom}^{\forall}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}}$, $x \in vars(a)$, $t \in \mathit{GTerm}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}} = \mathit{GTerm}_{\mathcal{L}_{\sigma-1}}$, $\gamma = x/t \in \mathit{Subst}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}}$, $dom(\gamma) = \{x\} = vars(a)$, such that $C^{\mathcal{B}}_\sigma = \forall x\, a \prec a\gamma \vee \forall x\, a \eqcirc a\gamma \in S^{\mathcal{B}}_\sigma$ is a basic order $\forall$-quantification resolvent of $\forall x\, a$ using Rule (45) with respect to $\mathcal{L}^{\mathcal{B}}_{\sigma-1}$, $S^{\mathcal{B}}_{\sigma-1}$. Then $\mathcal{L}^{\mathcal{B}}_\sigma = \mathcal{L}^{\mathcal{B}}_{\sigma-1}$, there exist $1 \le j^* \le \sigma - 1$ and $\forall x\, a \in qatoms^{\forall}(C^{\mathcal{B}}_{j^*})$, $C^{\mathcal{B}}_{j^*} \in S^{\mathcal{B}}_{\sigma-1}$; there exists $\vartheta_{j^*} \in \mathit{Subst}_{\mathcal{L}_{j^*}} \subseteq \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $dom(\vartheta_{j^*}) = \mathit{freevars}(C_{j^*})$, $range(\vartheta_{j^*}) = \mathit{freevars}(C^{\mathcal{B}}_{j^*}) = \emptyset$, and ϑ_{j^*} is applicable to C_{j^*}, $C^{\mathcal{B}}_{j^*} = C_{j^*}\vartheta_{j^*}$, $C_{j^*} \in S_{\sigma-1}$; there exists $\forall x\, a' \in qatoms^{\forall}(C_{j^*}) \subseteq qatoms^{\forall}(S_{\sigma-1}) \subseteq \mathit{QAtom}^{\forall}_{\mathcal{L}_{\sigma-1}}$, $a' \in \mathit{Atom}_{\mathcal{L}_{\sigma-1}}$, $x \in vars(a')$, $vars(a') - \{x\} = \mathit{freevars}(\forall x\, a') \subseteq \mathit{freevars}(C_{j^*}) = dom(\vartheta_{j^*})$, and ϑ_{j^*} is applicable to $\forall x\, a'$, $\forall x\, a = (\forall x\, a')\vartheta_{j^*}$, $range(\vartheta_{j^*}|_{vars(a')-\{x\}}) = \emptyset$, $a = a'(\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/x)$; using Rule (52) with respect to $\mathcal{L}_{\sigma-1}$, $S_{\sigma-1}$, we derive $\forall x\, a' \prec a' \vee \forall x\, a' \eqcirc a' \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}}$. We put $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1}$, $C_\sigma = \forall x\, a' \prec a' \vee \forall x\, a' \eqcirc a' \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, C_\sigma$, $C_\sigma \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} \subseteq \mathit{OrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $S_\sigma = S_{\sigma-1} \cup \{C_\sigma\} \subseteq \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} = \mathit{OrdCl}_{\mathcal{L}_\sigma}$, $\vartheta_\sigma = \vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/t \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}} = \mathit{Subst}_{\mathcal{L}_\sigma}$, $dom(\vartheta_\sigma) = vars(a') = \mathit{freevars}(C_\sigma)$, $range(\vartheta_\sigma) = range(\vartheta_{j^*}|_{vars(a')-\{x\}}) \cup range(x/t) = \emptyset \cup vars(t) = \emptyset$; $\mathcal{D}_\sigma$ is a deduction of C_σ from S by order hyperresolution. Hence, $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1} = \mathcal{L}^{\mathcal{B}}_{\sigma-1} = \mathcal{L}^{\mathcal{B}}_\sigma$, ϑ_σ is applicable to C_σ, $(\forall x\, a')\vartheta_\sigma = \forall x\, a'(\vartheta_\sigma|_{\mathit{freevars}(\forall x\, a')} \cup x/x) = \forall x\, a'(\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/x) = \forall x\, a$, $a'\vartheta_\sigma = a'(\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/t) = a'(\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/\gamma(x)) = a'((\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/x) \circ \gamma) = (a'(\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/x))\gamma = a\gamma$, $C_\sigma\vartheta_\sigma = (\forall x\, a')\vartheta_\sigma \prec a'\vartheta_\sigma \vee (\forall x\, a')\vartheta_\sigma \eqcirc a'\vartheta_\sigma = \forall x\, a \prec a\gamma \vee \forall x\, a \eqcirc a\gamma = C^{\mathcal{B}}_\sigma$; (91) holds.

Case 2.6: There exist $\exists x\, a \in qatoms^{\exists}(S^{\mathcal{B}}_{\sigma-1}) \subseteq \mathit{QAtom}^{\exists}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}}$, $x \in vars(a)$, $t \in \mathit{GTerm}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}} = \mathit{GTerm}_{\mathcal{L}_{\sigma-1}}$, $\gamma = x/t \in \mathit{Subst}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}}$, $dom(\gamma) = \{x\} = vars(a)$, such that $C^{\mathcal{B}}_\sigma = a\gamma \prec \exists x\, a \vee a\gamma \eqcirc \exists x\, a \in S^{\mathcal{B}}_\sigma$ is a basic order $\exists$-quantification resolvent of $\exists x\, a$ using Rule (46) with respect to $\mathcal{L}^{\mathcal{B}}_{\sigma-1}$, $S^{\mathcal{B}}_{\sigma-1}$. Then $\mathcal{L}^{\mathcal{B}}_\sigma = \mathcal{L}^{\mathcal{B}}_{\sigma-1}$, there exist $1 \le j^* \le \sigma - 1$ and $\exists x\, a \in qatoms^{\exists}(C^{\mathcal{B}}_{j^*})$, $C^{\mathcal{B}}_{j^*} \in S^{\mathcal{B}}_{\sigma-1}$; there exists $\vartheta_{j^*} \in \mathit{Subst}_{\mathcal{L}_{j^*}} \subseteq \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $dom(\vartheta_{j^*}) = \mathit{freevars}(C_{j^*})$, $range(\vartheta_{j^*}) = \mathit{freevars}(C^{\mathcal{B}}_{j^*}) = \emptyset$, and ϑ_{j^*} is applicable to C_{j^*}, $C^{\mathcal{B}}_{j^*} = C_{j^*}\vartheta_{j^*}$, $C_{j^*} \in S_{\sigma-1}$; there exists $\exists x\, a' \in qatoms^{\exists}(C_{j^*}) \subseteq qatoms^{\exists}(S_{\sigma-1}) \subseteq \mathit{QAtom}^{\exists}_{\mathcal{L}_{\sigma-1}}$, $a' \in \mathit{Atom}_{\mathcal{L}_{\sigma-1}}$, $x \in vars(a')$, $vars(a') - \{x\} = \mathit{freevars}(\exists x\, a') \subseteq \mathit{freevars}(C_{j^*}) = dom(\vartheta_{j^*})$, and ϑ_{j^*} is applicable to $\exists x\, a'$, $\exists x\, a = (\exists x\, a')\vartheta_{j^*}$, $range(\vartheta_{j^*}|_{vars(a')-\{x\}}) = \emptyset$, $a = a'(\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/x)$; using Rule (53) with respect to $\mathcal{L}_{\sigma-1}$, $S_{\sigma-1}$, we derive $a' \prec \exists x\, a' \vee a' \eqcirc \exists x\, a' \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}}$. We put $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1}$, $C_\sigma = a' \prec \exists x\, a' \vee a' \eqcirc \exists x\, a' \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, C_\sigma$, $C_\sigma \in \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} \subseteq \mathit{OrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $S_\sigma = S_{\sigma-1} \cup \{C_\sigma\} \subseteq \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} = \mathit{OrdCl}_{\mathcal{L}_\sigma}$, $\vartheta_\sigma = \vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/t \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}} = \mathit{Subst}_{\mathcal{L}_\sigma}$, $dom(\vartheta_\sigma) = vars(a') = \mathit{freevars}(C_\sigma)$, $range(\vartheta_\sigma) = range(\vartheta_{j^*}|_{vars(a')-\{x\}}) \cup range(x/t) = \emptyset \cup vars(t) = \emptyset$; $\mathcal{D}_\sigma$ is a deduction of C_σ from S by order hyperresolution. Hence, $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1} = \mathcal{L}^{\mathcal{B}}_{\sigma-1} = \mathcal{L}^{\mathcal{B}}_\sigma$, ϑ_σ is applicable to C_σ, $(\exists x\, a')\vartheta_\sigma = \exists x\, a'(\vartheta_\sigma|_{\mathit{freevars}(\exists x\, a')} \cup x/x) = \exists x\, a'(\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/x) = \exists x\, a$, $a'\vartheta_\sigma = a'(\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/t) = a'(\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/\gamma(x)) = a'((\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/x) \circ \gamma) = (a'(\vartheta_{j^*}|_{vars(a')-\{x\}} \cup x/x))\gamma = a\gamma$, $C_\sigma\vartheta_\sigma = a'\vartheta_\sigma \prec (\exists x\, a')\vartheta_\sigma \vee a'\vartheta_\sigma \eqcirc (\exists x\, a')\vartheta_\sigma = a\gamma \prec \exists x\, a \vee a\gamma \eqcirc \exists x\, a = C^{\mathcal{B}}_\sigma$; (91) holds.

Case 2.7: There exist $\forall x\, a \in qatoms^{\forall}(S^{\mathcal{B}}_{\sigma-1}) \subseteq \mathit{QAtom}^{\forall}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}}$, $x \in vars(a)$, $\mathit{freevars}(\forall x\, a) = \emptyset$, $b \in atoms(S^{\mathcal{B}}_{\sigma-1}) \cup qatoms(S^{\mathcal{B}}_{\sigma-1}) \subseteq \mathit{Atom}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}} \cup \mathit{QAtom}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}}$, $\mathit{freevars}(b) = \emptyset$, $\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}}$, $ar(\tilde{w}) = |\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b)|$, $\gamma = x/\tilde{w}(\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b)) \in \mathit{Subst}_{\mathcal{L}^{\mathcal{B}}_\sigma}$, $dom(\gamma) = \{x\} = vars(a)$, $range(\gamma) = vars(\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b)) = \mathit{freevars}(\forall x\, a) \cup \mathit{freevars}(b) = \emptyset$, such that $C^{\mathcal{B}}_\sigma = a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a \in S^{\mathcal{B}}_\sigma$ is a basic order $\forall$-witnessing resolvent of $\forall x\, a$ and b using Rule (47) with respect to $\mathcal{L}^{\mathcal{B}}_{\sigma-1}$, $S^{\mathcal{B}}_{\sigma-1}$. Then $\mathcal{L}^{\mathcal{B}}_\sigma = \mathcal{L}^{\mathcal{B}}_{\sigma-1} \cup \{\tilde{w}\}$, there exist $1 \le j_1^*, j_2^* \le \sigma - 1$ and $\forall x\, a \in qatoms^{\forall}(C^{\mathcal{B}}_{j_1^*})$, $b \in atoms(C^{\mathcal{B}}_{j_2^*}) \cup qatoms(C^{\mathcal{B}}_{j_2^*})$, $C^{\mathcal{B}}_{j_1^*}, C^{\mathcal{B}}_{j_2^*} \in S^{\mathcal{B}}_{\sigma-1}$; there exists $\vartheta_{j_k^*} \in \mathit{Subst}_{\mathcal{L}_{j_k^*}} \subseteq \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $k = 1,2$, $dom(\vartheta_{j_k^*}) = \mathit{freevars}(C_{j_k^*})$, $range(\vartheta_{j_k^*}) = \mathit{freevars}(C^{\mathcal{B}}_{j_k^*}) = \emptyset$, and $\vartheta_{j_k^*}$ is applicable to $C_{j_k^*}$, $C^{\mathcal{B}}_{j_k^*} = C_{j_k^*}\vartheta_{j_k^*}$, $C_{j_k^*} \in S_{\sigma-1}$; there exists $\forall x\, a' \in qatoms^{\forall}(C_{j_1^*}) \subseteq qatoms^{\forall}(S_{\sigma-1}) \subseteq \mathit{QAtom}^{\forall}_{\mathcal{L}_{\sigma-1}}$, $a' \in \mathit{Atom}_{\mathcal{L}_{\sigma-1}}$, $x \in vars(a')$, $vars(a') - \{x\} = \mathit{freevars}(\forall x\, a') \subseteq \mathit{freevars}(C_{j_1^*}) = dom(\vartheta_{j_1^*})$, and $\vartheta_{j_1^*}$ is applicable to $\forall x\, a'$, $\forall x\, a = (\forall x\, a')\vartheta_{j_1^*}$, $range(\vartheta_{j_1^*}|_{vars(a')-\{x\}}) = \emptyset$, $a = a'(\vartheta_{j_1^*}|_{vars(a')-\{x\}} \cup x/x)$; there exists $b' \in atoms(C_{j_2^*}) \cup qatoms(C_{j_2^*}) \subseteq atoms(S_{\sigma-1}) \cup qatoms(S_{\sigma-1}) \subseteq \mathit{Atom}_{\mathcal{L}_{\sigma-1}} \cup \mathit{QAtom}_{\mathcal{L}_{\sigma-1}}$, $\mathit{freevars}(b') \subseteq \mathit{freevars}(C_{j_2^*}) = dom(\vartheta_{j_2^*})$, and $\vartheta_{j_2^*}$ is applicable to b', $b = b'\vartheta_{j_2^*}$; there exist variable renamings $\rho_{j_1^*}, \rho_{j_2^*} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $dom(\rho_{j_1^*}) = \mathit{freevars}(\forall x\, a') = vars(a') - \{x\}$, $dom(\rho_{j_2^*}) = \mathit{freevars}(b')$, and $x \notin range(\rho_{j_1^*})$, $range(\rho_{j_2^*}) \cap (\mathit{boundvars}(b') \cup range(\rho_{j_1^*})) = \emptyset$; $\rho_{j_1^*}$ is applicable to $\forall x\, a'$, $(\forall x\, a')\rho_{j_1^*} = \forall x\, a'(\rho_{j_1^*} \cup x/x)$, $x \in vars(a'(\rho_{j_1^*} \cup x/x))$, $range(\rho_{j_1^*}) = \mathit{freevars}((\forall x\, a')\rho_{j_1^*}) = vars(a'(\rho_{j_1^*} \cup x/x)) - \{x\}$; $\rho_{j_2^*}$ is applicable to b', $range(\rho_{j_2^*}) = \mathit{freevars}(b'\rho_{j_2^*})$; $(\forall x\, a')\rho_{j_1^*} \in qatoms^{\forall}(S^{Vr}_{\sigma-1})$, $b'\rho_{j_2^*} \in atoms(S^{Vr}_{\sigma-1}) \cup qatoms(S^{Vr}_{\sigma-1})$; for both k, $\rho^{-1}_{j_k^*} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$ is a variable renaming, $dom(\rho^{-1}_{j_1^*}) =$

$\mathit{range}(\rho_{j_1^*}) = \mathit{freevars}((\forall x\, a')\rho_{j_1^*})$, $\mathit{range}(\rho_{j_1^*}^{-1}) = \mathit{dom}(\rho_{j_1^*}) = \mathit{freevars}(\forall x\, a') \subseteq \mathit{dom}(\vartheta_{j_1^*})$, $\mathit{dom}(\rho_{j_2^*}^{-1}) = \mathit{range}(\rho_{j_2^*}) = \mathit{freevars}(b'\rho_{j_2^*})$, $\mathit{range}(\rho_{j_2^*}^{-1}) = \mathit{dom}(\rho_{j_2^*}) = \mathit{freevars}(b') \subseteq \mathit{dom}(\vartheta_{j_2^*})$, $\mathit{dom}(\rho_{j_1^*}^{-1}) \cap \mathit{dom}(\rho_{j_2^*}^{-1}) = \mathit{range}(\rho_{j_1^*}) \cap \mathit{range}(\rho_{j_2^*}) = \emptyset$; $\rho_{j_1^*} \circ \rho_{j_1^*}^{-1} = \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{dom}(\rho_{j_1^*})} = \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\forall x\, a')} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $x \not\in \mathit{freevars}(\forall x\, a') = \mathit{range}(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\forall x\, a')})$, $\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\forall x\, a')}$ is applicable to $\forall x\, a'$; $(\forall x\, a')\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\forall x\, a')} = (\forall x\, a')(\rho_{j_1^*} \circ \rho_{j_1^*}^{-1}) = ((\forall x\, a')\rho_{j_1^*})\rho_{j_1^*}^{-1}$, $\rho_{j_1^*}^{-1}$ is applicable to $(\forall x\, a')\rho_{j_1^*}$; $\rho_{j_2^*} \circ \rho_{j_2^*}^{-1} = \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{dom}(\rho_{j_2^*})} = \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(b')} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $\mathit{boundvars}(b') \cap \mathit{range}(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(b')}) = \mathit{boundvars}(b') \cap \mathit{freevars}(b') = \emptyset$, $\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(b')}$ is applicable to b'; $b'\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(b')} = b'(\rho_{j_2^*} \circ \rho_{j_2^*}^{-1}) = (b'\rho_{j_2^*})\rho_{j_2^*}^{-1}$, $\rho_{j_2^*}^{-1}$ is applicable to $b'\rho_{j_2^*}$; $\mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b)| = |\mathit{freetermseq}((\forall x\, a')\vartheta_{j_1^*}), \mathit{freetermseq}(b'\vartheta_{j_2^*})| = |\mathit{freetermseq}(\forall x\, a'), \mathit{freetermseq}(b')| = |\mathit{freetermseq}((\forall x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*})|$. We put $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1} \cup \{\tilde{w}\} = \mathcal{L}^{\mathcal{B}}_{\sigma-1} \cup \{\tilde{w}\} = \mathcal{L}^{\mathcal{B}}_\sigma$, $\gamma' = x/\tilde{w}(\mathit{freetermseq}((\forall x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*})) \cup \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{vars}(a'(\rho_{j_1^*} \cup x/x)) - \{x\}} \in \mathit{Subst}_{\mathcal{L}_\sigma}$, $\mathit{dom}(\gamma') = \{x\} \cup (\mathit{vars}(a'(\rho_{j_1^*} \cup x/x)) - \{x\}) = \mathit{vars}(a'(\rho_{j_1^*} \cup x/x)) = \{x\} \cup \mathit{range}(\rho_{j_1^*}) = \mathit{range}(\rho_{j_1^*} \cup x/x)$, $\mathit{range}(\gamma') = \mathit{vars}(\mathit{freetermseq}((\forall x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*})) \cup (\mathit{vars}(a'(\rho_{j_1^*} \cup x/x)) - \{x\}) = \mathit{freevars}((\forall x\, a')\rho_{j_1^*}) \cup \mathit{freevars}(b'\rho_{j_2^*})$. Using Rule (54) with respect to $\mathcal{L}_{\sigma-1}$, $S_{\sigma-1}$, we derive $(a'(\rho_{j_1^*} \cup x/x))\gamma' \prec b'\rho_{j_2^*} \vee b'\rho_{j_2^*} \eqcirc (\forall x\, a')\rho_{j_1^*} \vee b'\rho_{j_2^*} \prec (\forall x\, a')\rho_{j_1^*} \in \mathit{OrdCl}_{\mathcal{L}_\sigma}$. We put $C_\sigma = (a'(\rho_{j_1^*} \cup x/x))\gamma' \prec b'\rho_{j_2^*} \vee b'\rho_{j_2^*} \eqcirc (\forall x\, a')\rho_{j_1^*} \vee b'\rho_{j_2^*} \prec (\forall x\, a')\rho_{j_1^*} \in \mathit{OrdCl}_{\mathcal{L}_\sigma}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, C_\sigma$, $C_\sigma \in \mathit{OrdCl}_{\mathcal{L}_\sigma} \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $S_\sigma = S_{\sigma-1} \cup \{C_\sigma\} \subseteq \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} \cup \mathit{OrdCl}_{\mathcal{L}_\sigma} = \mathit{OrdCl}_{\mathcal{L}_\sigma}$, $\vartheta_\sigma = \rho_{j_1^*}^{-1} \circ \vartheta_{j_1^*} \cup \rho_{j_2^*}^{-1} \circ \vartheta_{j_2^*} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}} \subseteq \mathit{Subst}_{\mathcal{L}_\sigma}$, $\mathit{dom}(\vartheta_\sigma) = \mathit{dom}(\rho_{j_1^*}^{-1}) \cup \mathit{dom}(\rho_{j_2^*}^{-1}) = \mathit{range}(\rho_{j_1^*}) \cup \mathit{range}(\rho_{j_2^*}) = \mathit{freevars}((\forall x\, a')\rho_{j_1^*}) \cup \mathit{freevars}(b'\rho_{j_2^*}) = \mathit{range}(\gamma') \cup \mathit{freevars}((\forall x\, a')\rho_{j_1^*}) \cup \mathit{freevars}(b'\rho_{j_2^*}) = \mathit{freevars}(C_\sigma)$, $\mathit{range}(\vartheta_\sigma) = \mathit{range}(\rho_{j_1^*}^{-1} \circ \vartheta_{j_1^*}) \cup \mathit{range}(\rho_{j_2^*}^{-1} \circ \vartheta_{j_2^*}) = \mathit{range}(\vartheta_{j_1^*}|_{\mathit{range}(\rho_{j_1^*}^{-1})}) \cup \mathit{range}(\vartheta_{j_2^*}|_{\mathit{range}(\rho_{j_2^*}^{-1})}) = \emptyset$; $\mathcal{D}_\sigma$ is a deduction of C_σ from S by order hyperresolution. Hence, ϑ_σ is applicable to C_σ, $((\forall x\, a')\rho_{j_1^*})\vartheta_\sigma = ((\forall x\, a')\rho_{j_1^*})(\rho_{j_1^*}^{-1} \circ \vartheta_{j_1^*}) = (\forall x\, a')(\rho_{j_1^*} \circ \rho_{j_1^*}^{-1} \circ \vartheta_{j_1^*}) = (\forall x\, a')(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\forall x\, a')} \circ \vartheta_{j_1^*}) = (\forall x\, a')\vartheta_{j_1^*} = \forall x\, a$, $(b'\rho_{j_2^*})\vartheta_\sigma = (b'\rho_{j_2^*})(\rho_{j_2^*}^{-1} \circ \vartheta_{j_2^*}) = b'(\rho_{j_2^*} \circ \rho_{j_2^*}^{-1} \circ \vartheta_{j_2^*}) = b'(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(b')} \circ \vartheta_{j_2^*}) = b'\vartheta_{j_2^*} = b$,

$$\begin{aligned}
&((a'(\rho_{j_1^*} \cup x/x))\gamma')\vartheta_\sigma = \\
&(a'((\rho_{j_1^*} \cup x/x) \circ \gamma'))\vartheta_\sigma = \\
&(a'(\rho_{j_1^*} \circ \mathit{id}|_{\mathit{range}(\rho_{j_1^*})} \cup x/\tilde{w}(\mathit{freetermseq}((\forall x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*}))))\vartheta_\sigma = \\
&(a'(\rho_{j_1^*} \cup x/\tilde{w}(\mathit{freetermseq}((\forall x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*}))))\vartheta_\sigma = \\
&a'((\rho_{j_1^*} \cup x/\tilde{w}(\mathit{freetermseq}((\forall x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*}))) \circ \vartheta_\sigma) = \\
&a'(\rho_{j_1^*} \circ \vartheta_\sigma \cup x/\tilde{w}(\mathit{freetermseq}((\forall x\, a')\rho_{j_1^*})\vartheta_\sigma, \mathit{freetermseq}(b'\rho_{j_2^*})\vartheta_\sigma)) = \\
&a'(\rho_{j_1^*} \circ \vartheta_\sigma \cup x/\tilde{w}(\mathit{freetermseq}(((\forall x\, a')\rho_{j_1^*})\vartheta_\sigma), \mathit{freetermseq}((b'\rho_{j_2^*})\vartheta_\sigma))) = \\
&a'(\rho_{j_1^*} \circ \vartheta_\sigma \cup x/\tilde{w}(\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b))) = \\
&a'(\rho_{j_1^*} \circ \rho_{j_1^*}^{-1} \circ \vartheta_{j_1^*} \cup x/\tilde{w}(\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b))) = \\
&a'(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\forall x\, a')} \circ \vartheta_{j_1^*} \cup x/\tilde{w}(\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b))) = \\
&a'(\vartheta_{j_1^*}|_{\mathit{freevars}(\forall x\, a')} \cup x/\tilde{w}(\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b))) = \\
&a'(\vartheta_{j_1^*}|_{\mathit{vars}(a') - \{x\}} \cup x/\gamma(x)) = \\
&a'((\vartheta_{j_1^*}|_{\mathit{vars}(a') - \{x\}} \cup x/x) \circ \gamma) = \\
&(a'(\vartheta_{j_1^*}|_{\mathit{vars}(a') - \{x\}} \cup x/x))\gamma = \\
&a\gamma,
\end{aligned}$$

$C_\sigma\vartheta_\sigma = ((a'(\rho_{j_1^*} \cup x/x))\gamma')\vartheta_\sigma \prec (b'\rho_{j_2^*})\vartheta_\sigma \vee (b'\rho_{j_2^*})\vartheta_\sigma \eqcirc ((\forall x\, a')\rho_{j_1^*})\vartheta_\sigma \vee (b'\rho_{j_2^*})\vartheta_\sigma \prec ((\forall x\, a')\rho_{j_1^*})\vartheta_\sigma = a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a = C^{\mathcal{B}}_\sigma$; (91) holds.

Case 2.8: There exist $\exists x\, a \in \mathit{qatoms}^{\exists}(S^{\mathcal{B}}_{\sigma-1}) \subseteq \mathit{QAtom}^{\exists}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}}$, $x \in \mathit{vars}(a)$, $\mathit{freevars}(\exists x\, a) = \emptyset$, $b \in \mathit{atoms}(S^{\mathcal{B}}_{\sigma-1}) \cup \mathit{qatoms}(S^{\mathcal{B}}_{\sigma-1}) \subseteq \mathit{Atom}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}} \cup \mathit{QAtom}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}}$, $\mathit{freevars}(b) = \emptyset$, $\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}^{\mathcal{B}}_{\sigma-1}}$, $\mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b)|$, $\gamma = x/\tilde{w}(\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b)) \in \mathit{Subst}_{\mathcal{L}^{\mathcal{B}}_\sigma}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(a)$, $\mathit{range}(\gamma) = \mathit{vars}(\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b)) = \mathit{freevars}(\exists x\, a) \cup \mathit{freevars}(b) = \emptyset$, such that $C^{\mathcal{B}}_\sigma = b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b \in S^{\mathcal{B}}_\sigma$ is a basic order $\exists$-witnessing resolvent of $\exists x\, a$ and b using Rule (48) with respect to $\mathcal{L}^{\mathcal{B}}_{\sigma-1}$, $S^{\mathcal{B}}_{\sigma-1}$. Then $\mathcal{L}^{\mathcal{B}}_\sigma = \mathcal{L}^{\mathcal{B}}_{\sigma-1} \cup \{\tilde{w}\}$, there exist $1 \leq j_1^*, j_2^* \leq \sigma - 1$ and $\exists x\, a \in \mathit{qatoms}^{\exists}(C^{\mathcal{B}}_{j_1^*})$, $b \in \mathit{atoms}(C^{\mathcal{B}}_{j_2^*}) \cup \mathit{qatoms}(C^{\mathcal{B}}_{j_2^*})$, $C^{\mathcal{B}}_{j_1^*}, C^{\mathcal{B}}_{j_2^*} \in S^{\mathcal{B}}_{\sigma-1}$; there exists $\vartheta_{j_k^*} \in \mathit{Subst}_{\mathcal{L}_{j_k^*}} \subseteq \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $k = 1, 2$, $\mathit{dom}(\vartheta_{j_k^*}) = \mathit{freevars}(C_{j_k^*})$, $\mathit{range}(\vartheta_{j_k^*}) = \mathit{freevars}(C^{\mathcal{B}}_{j_k^*}) = \emptyset$, and $\vartheta_{j_k^*}$ is applicable to $C_{j_k^*}$, $C^{\mathcal{B}}_{j_k^*} = C_{j_k^*}\vartheta_{j_k^*}$, $C_{j_k^*} \in S_{\sigma-1}$; there exists $\exists x\, a' \in \mathit{qatoms}^{\exists}(C_{j_1^*}) \subseteq \mathit{qatoms}^{\exists}(S_{\sigma-1}) \subseteq \mathit{QAtom}^{\exists}_{\mathcal{L}_{\sigma-1}}$, $a' \in \mathit{Atom}_{\mathcal{L}_{\sigma-1}}$, $x \in \mathit{vars}(a')$, $\mathit{vars}(a') - \{x\} = \mathit{freevars}(\exists x\, a') \subseteq \mathit{freevars}(C_{j_1^*}) = \mathit{dom}(\vartheta_{j_1^*})$, and $\vartheta_{j_1^*}$ is applicable to $\exists x\, a'$, $\exists x\, a = (\exists x\, a')\vartheta_{j_1^*}$, $\mathit{range}(\vartheta_{j_1^*}|_{\mathit{vars}(a') - \{x\}}) = \emptyset$, $a = a'(\vartheta_{j_1^*}|_{\mathit{vars}(a') - \{x\}} \cup x/x)$; there exists $b' \in \mathit{atoms}(C_{j_2^*}) \cup \mathit{qatoms}(C_{j_2^*}) \subseteq \mathit{atoms}(S_{\sigma-1}) \cup \mathit{qatoms}(S_{\sigma-1}) \subseteq \mathit{Atom}_{\mathcal{L}_{\sigma-1}} \cup \mathit{QAtom}_{\mathcal{L}_{\sigma-1}}$, $\mathit{freevars}(b') \subseteq \mathit{freevars}(C_{j_2^*}) = \mathit{dom}(\vartheta_{j_2^*})$, and $\vartheta_{j_2^*}$ is applicable to b', $b = b'\vartheta_{j_2^*}$; there exist variable renamings $\rho_{j_1^*}, \rho_{j_2^*} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $\mathit{dom}(\rho_{j_1^*}) = \mathit{freevars}(\exists x\, a') = \mathit{vars}(a') - \{x\}$, $\mathit{dom}(\rho_{j_2^*}) = \mathit{freevars}(b')$, and $x \not\in \mathit{range}(\rho_{j_1^*})$, $\mathit{range}(\rho_{j_2^*}) \cap (\mathit{boundvars}(b') \cup \mathit{range}(\rho_{j_1^*})) = \emptyset$; $\rho_{j_1^*}$ is applicable to $\exists x\, a'$, $(\exists x\, a')\rho_{j_1^*} = \exists x\, a'(\rho_{j_1^*} \cup x/x)$, $x \in \mathit{vars}(a'(\rho_{j_1^*} \cup x/x))$, $\mathit{range}(\rho_{j_1^*}) = \mathit{freevars}((\exists x\, a')\rho_{j_1^*}) = \mathit{vars}(a'(\rho_{j_1^*} \cup x/x)) - \{x\}$; $\rho_{j_2^*}$ is applicable to b', $\mathit{range}(\rho_{j_2^*}) = \mathit{freevars}(b'\rho_{j_2^*})$; $(\exists x\, a')\rho_{j_1^*} \in \mathit{qatoms}^{\exists}(S^{Vr}_{\sigma-1})$, $b'\rho_{j_2^*} \in \mathit{atoms}(S^{Vr}_{\sigma-1}) \cup \mathit{qatoms}(S^{Vr}_{\sigma-1})$; for both k, $\rho_{j_k^*}^{-1} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$ is a variable renaming, $\mathit{dom}(\rho_{j_1^*}^{-1}) = \mathit{range}(\rho_{j_1^*}) = \mathit{freevars}((\exists x\, a')\rho_{j_1^*})$, $\mathit{range}(\rho_{j_1^*}^{-1}) = \mathit{dom}(\rho_{j_1^*}) = \mathit{freevars}(\exists x\, a') \subseteq \mathit{dom}(\vartheta_{j_1^*})$, $\mathit{dom}(\rho_{j_2^*}^{-1}) = \mathit{range}(\rho_{j_2^*}) = \mathit{freevars}(b'\rho_{j_2^*})$, $\mathit{range}(\rho_{j_2^*}^{-1}) = \mathit{dom}(\rho_{j_2^*}) = \mathit{freevars}(b') \subseteq \mathit{dom}(\vartheta_{j_2^*})$,

$\mathit{dom}(\rho_{j_1^*}^{-1}) \cap \mathit{dom}(\rho_{j_2^*}^{-1}) = \mathit{range}(\rho_{j_1^*}) \cap \mathit{range}(\rho_{j_2^*}) = \emptyset$; $\rho_{j_1^*} \circ \rho_{j_1^*}^{-1} = \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{dom}(\rho_{j_1^*})} = \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\exists x\, a')} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $x \not\in \mathit{freevars}(\exists x\, a') = \mathit{range}(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\exists x\, a')})$, $\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\exists x\, a')}$ is applicable to $\exists x\, a'$; $(\exists x\, a')\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\exists x\, a')} = (\exists x\, a')(\rho_{j_1^*} \circ \rho_{j_1^*}^{-1}) = ((\exists x\, a')\rho_{j_1^*})\rho_{j_1^*}^{-1}$, $\rho_{j_1^*}^{-1}$ is applicable to $(\exists x\, a')\rho_{j_1^*}$; $\rho_{j_2^*} \circ \rho_{j_2^*}^{-1} = \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{dom}(\rho_{j_2^*})} = \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(b')} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}}$, $\mathit{boundvars}(b') \cap \mathit{range}(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(b')}) = \mathit{boundvars}(b') \cap \mathit{freevars}(b') = \emptyset$, $\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(b')}$ is applicable to b'; $b'\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(b')} = b'(\rho_{j_2^*} \circ \rho_{j_2^*}^{-1}) = (b'\rho_{j_2^*})\rho_{j_2^*}^{-1}$, $\rho_{j_2^*}^{-1}$ is applicable to $b'\rho_{j_2^*}$; $\mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b)| = |\mathit{freetermseq}((\exists x\, a')\vartheta_{j_1^*}), \mathit{freetermseq}(b'\vartheta_{j_2^*})| = |\mathit{freetermseq}(\exists x\, a'), \mathit{freetermseq}(b')| = |\mathit{freetermseq}((\exists x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*})|$. We put $\mathcal{L}_\sigma = \mathcal{L}_{\sigma-1} \cup \{\tilde{w}\} = \mathcal{L}_{\sigma-1}^{\mathcal{B}} \cup \{\tilde{w}\} = \mathcal{L}_\sigma^{\mathcal{B}}$, $\gamma' = x/\tilde{w}(\mathit{freetermseq}((\exists x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*})) \cup \mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{vars}(a'(\rho_{j_1^*} \cup x/x)) - \{x\}} \in \mathit{Subst}_{\mathcal{L}_\sigma}$, $\mathit{dom}(\gamma') = \{x\} \cup (\mathit{vars}(a'(\rho_{j_1^*} \cup x/x)) - \{x\}) = \mathit{vars}(a'(\rho_{j_1^*} \cup x/x)) = \{x\} \cup \mathit{range}(\rho_{j_1^*}) = \mathit{range}(\rho_{j_1^*} \cup x/x)$, $\mathit{range}(\gamma') = \mathit{vars}(\mathit{freetermseq}((\exists x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*})) \cup (\mathit{vars}(a'(\rho_{j_1^*} \cup x/x)) - \{x\}) = \mathit{freevars}((\exists x\, a')\rho_{j_1^*}) \cup \mathit{freevars}(b'\rho_{j_2^*})$. Using Rule (55) with respect to $\mathcal{L}_{\sigma-1}$, $S_{\sigma-1}$, we derive $b'\rho_{j_2^*} \prec (a'(\rho_{j_1^*} \cup x/x))\gamma' \vee (\exists x\, a')\rho_{j_1^*} \doteq b'\rho_{j_2^*} \vee (\exists x\, a')\rho_{j_1^*} \prec b'\rho_{j_2^*} \in \mathit{OrdCl}_{\mathcal{L}_\sigma}$. We put $C_\sigma = b'\rho_{j_2^*} \prec (a'(\rho_{j_1^*} \cup x/x))\gamma' \vee (\exists x\, a')\rho_{j_1^*} \doteq b'\rho_{j_2^*} \vee (\exists x\, a')\rho_{j_1^*} \prec b'\rho_{j_2^*} \in \mathit{OrdCl}_{\mathcal{L}_\sigma}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, C_\sigma$, $C_\sigma \in \mathit{OrdCl}_{\mathcal{L}_\sigma} \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $S_\sigma = S_{\sigma-1} \cup \{C_\sigma\} \subseteq \mathit{OrdCl}_{\mathcal{L}_{\sigma-1}} \cup \mathit{OrdCl}_{\mathcal{L}_\sigma} = \mathit{OrdCl}_{\mathcal{L}_\sigma}$, $\vartheta_\sigma = \rho_{j_1^*}^{-1} \circ \vartheta_{j_1^*} \cup \rho_{j_2^*}^{-1} \circ \vartheta_{j_2^*} \in \mathit{Subst}_{\mathcal{L}_{\sigma-1}} \subseteq \mathit{Subst}_{\mathcal{L}_\sigma}$, $\mathit{dom}(\vartheta_\sigma) = \mathit{dom}(\rho_{j_1^*}^{-1}) \cup \mathit{dom}(\rho_{j_2^*}^{-1}) = \mathit{range}(\rho_{j_1^*}) \cup \mathit{range}(\rho_{j_2^*}) = \mathit{freevars}((\exists x\, a')\rho_{j_1^*}) \cup \mathit{freevars}(b'\rho_{j_2^*}) = \mathit{range}(\gamma') \cup \mathit{freevars}((\exists x\, a')\rho_{j_1^*}) \cup \mathit{freevars}(b'\rho_{j_2^*}) = \mathit{freevars}(C_\sigma)$, $\mathit{range}(\vartheta_\sigma) = \mathit{range}(\rho_{j_1^*}^{-1} \circ \vartheta_{j_1^*}) \cup \mathit{range}(\rho_{j_2^*}^{-1} \circ \vartheta_{j_2^*}) = \mathit{range}(\vartheta_{j_1^*}|_{\mathit{range}(\rho_{j_1^*}^{-1})}) \cup \mathit{range}(\vartheta_{j_2^*}|_{\mathit{range}(\rho_{j_2^*}^{-1})}) = \emptyset$; $\mathcal{D}_\sigma$ is a deduction of C_σ from S by order hyperresolution. Hence, ϑ_σ is applicable to C_σ, $((\exists x\, a')\rho_{j_1^*})\vartheta_\sigma = ((\exists x\, a')\rho_{j_1^*})(\rho_{j_1^*}^{-1} \circ \vartheta_{j_1^*}) = (\exists x\, a')(\rho_{j_1^*} \circ \rho_{j_1^*}^{-1} \circ \vartheta_{j_1^*}) = (\exists x\, a')(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\exists x\, a')} \circ \vartheta_{j_1^*}) = (\exists x\, a')\vartheta_{j_1^*} = \exists x\, a$, $(b'\rho_{j_2^*})\vartheta_\sigma = (b'\rho_{j_2^*})(\rho_{j_2^*}^{-1} \circ \vartheta_{j_2^*}) = b'(\rho_{j_2^*} \circ \rho_{j_2^*}^{-1} \circ \vartheta_{j_2^*}) = b'(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(b')} \circ \vartheta_{j_2^*}) = b'\vartheta_{j_2^*} = b$,

$$
\begin{aligned}
&((a'(\rho_{j_1^*} \cup x/x))\gamma')\vartheta_\sigma = \\
&(a'((\rho_{j_1^*} \cup x/x) \circ \gamma'))\vartheta_\sigma = \\
&(a'(\rho_{j_1^*} \circ \mathit{id}|_{\mathit{range}(\rho_{j_1^*})} \cup x/\tilde{w}(\mathit{freetermseq}((\exists x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*}))))\vartheta_\sigma = \\
&(a'(\rho_{j_1^*} \cup x/\tilde{w}(\mathit{freetermseq}((\exists x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*}))))\vartheta_\sigma = \\
&a'((\rho_{j_1^*} \cup x/\tilde{w}(\mathit{freetermseq}((\exists x\, a')\rho_{j_1^*}), \mathit{freetermseq}(b'\rho_{j_2^*}))) \circ \vartheta_\sigma) = \\
&a'(\rho_{j_1^*} \circ \vartheta_\sigma \cup x/\tilde{w}(\mathit{freetermseq}((\exists x\, a')\rho_{j_1^*})\vartheta_\sigma, \mathit{freetermseq}(b'\rho_{j_2^*})\vartheta_\sigma)) = \\
&a'(\rho_{j_1^*} \circ \vartheta_\sigma \cup x/\tilde{w}(\mathit{freetermseq}(((\exists x\, a')\rho_{j_1^*})\vartheta_\sigma), \mathit{freetermseq}((b'\rho_{j_2^*})\vartheta_\sigma))) = \\
&a'(\rho_{j_1^*} \circ \vartheta_\sigma \cup x/\tilde{w}(\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b))) = \\
&a'(\rho_{j_1^*} \circ \rho_{j_1^*}^{-1} \circ \vartheta_{j_1^*} \cup x/\tilde{w}(\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b))) = \\
&a'(\mathit{id}_{\mathcal{L}_{\sigma-1}}|_{\mathit{freevars}(\exists x\, a')} \circ \vartheta_{j_1^*} \cup x/\tilde{w}(\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b))) = \\
&a'(\vartheta_{j_1^*}|_{\mathit{freevars}(\exists x\, a')} \cup x/\tilde{w}(\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b))) = \\
&a'(\vartheta_{j_1^*}|_{\mathit{vars}(a') - \{x\}} \cup x/\gamma(x)) = \\
&a'((\vartheta_{j_1^*}|_{\mathit{vars}(a') - \{x\}} \cup x/x) \circ \gamma) = \\
&(a'(\vartheta_{j_1^*}|_{\mathit{vars}(a') - \{x\}} \cup x/x))\gamma = \\
&a\gamma,
\end{aligned}
$$

$C_\sigma\vartheta_\sigma = (b'\rho_{j_2^*})\vartheta_\sigma \prec ((a'(\rho_{j_1^*} \cup x/x))\gamma')\vartheta_\sigma \vee ((\exists x\, a')\rho_{j_1^*})\vartheta_\sigma \doteq (b'\rho_{j_2^*})\vartheta_\sigma \vee ((\exists x\, a')\rho_{j_1^*})\vartheta_\sigma \prec (b'\rho_{j_2^*})\vartheta_\sigma = b \prec a\gamma \vee \exists x\, a \doteq b \vee \exists x\, a \prec b = C_\sigma^{\mathcal{B}}$; (91) holds.

So, in all Cases 2.1–2.8, (91) holds; (91) holds. So, in both Cases 1 and 2, (91) holds. The induction is completed. Thus, (91) holds.

We conclude. By (91) for $\sigma = n$, there exist a deduction $\mathcal{D}_n = C_1, \dots, C_n$, $C_\kappa \in \mathit{OrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, of C_n from S by order hyperresolution, $\vartheta_n \in \mathit{Subst}_{\mathcal{L}_n} \subseteq \mathit{Subst}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathit{dom}(\vartheta_n) = \mathit{freevars}(C_n)$, and $C = C_n^{\mathcal{B}} = C_n\vartheta_n$. We put $C^* = C_n$, $C^* \in \mathit{OrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$. Then $C^* \in \mathit{clo}^{\mathcal{H}}(S)$, $C = C^*\vartheta_n$, C is an instance of C^* of $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$. The lemma is proved. □

11. Full proof of Lemma 7

Proof. Let $r = max\{k_i \,|\, i \le n\}$. We proceed by induction on r.

Case 1 (the base case): $r = 0$. Then, for all $i \le n$, $k_i = 0$; there exists a contradiction of $\{\varepsilon_0^i \diamond_0^i \upsilon_0^i \,|\, i \le n\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$; there exists $\emptyset \ne I^* = \{i_j \,|\, j \le m\} \subseteq \{i \,|\, i \le n\}$, and $\varepsilon_0^{i_j} \diamond_0^{i_j} \upsilon_0^{i_j}$, $j = 0, \dots, m$, is a contradiction of $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$; using Rule (42), $\bigvee_{j=0}^{m} C_{i_j} \sqsubseteq \bigvee_{i=0}^{n} C_i$ is a basic order hyperresolvent of $\varepsilon_0^{i_j} \diamond_0^{i_j} \upsilon_0^{i_j} \vee C_{i_j} \in clo^{\mathcal{BH}}(S)$, $j = 0, \dots, m$; $\bigvee_{i\in I^*} C_i = \bigvee_{j=0}^{m} C_{i_j} \in clo^{\mathcal{BH}}(S)$; the statement holds.

Case 2 (the induction case): $r > 0$. We put $A = \{i \,|\, i \le n, k_i = r\}$ and $B = \{i \,|\, i \le n, k_i < r\}$, $A \cup B = \{i \,|\, i \le n\}$, $A \cap B = \emptyset$. Then $\{\bigvee_{j=0}^{r-1} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee \varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i \,|\, i \in A\} \cup \{\bigvee_{j=0}^{k_i} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee C_i \,|\, i \in B\} = \{\bigvee_{j=0}^{k_i} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee C_i \,|\, i \le n\} \subseteq clo^{\mathcal{BH}}(S)$; for all $\mathcal{S} \in \mathcal{S}el(\{\{j \,|\, j \le r-1\}_i \,|\, i \in A\} \cup \{\{j \,|\, j \le k_i\}_i \,|\, i \in B\}) \subseteq \mathcal{S}el(\{\{j \,|\, j \le k_i\}_i \,|\, i \le n\})$, there exists a contradiction of $\{\varepsilon_{\mathcal{S}(i)}^i \diamond_{\mathcal{S}(i)}^i \upsilon_{\mathcal{S}(i)}^i \,|\, i \le n\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $max(\{r-1\} \cup \{k_i \,|\, i \in B\}) = r - 1$; by induction hypothesis for $r-1$, there exist $I_A^* \subseteq A$, $I_B^* \subseteq B$, $\emptyset \ne I_A^* \cup I_B^* \subseteq \{i \,|\, i \le n\}$, $I_A^* \cap I_B^* \subseteq A \cap B = \emptyset$, and $\bigvee_{i\in I_A^*} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee \bigvee_{i\in I_B^*} C_i \in clo^{\mathcal{BH}}(S)$. We next prove the following statement:

For all $E \subseteq A$, there exist $I_E^* \subseteq A - E$ and $J_E^* \subseteq E \cup B$, $\emptyset \ne I_E^* \cup J_E^* \subseteq \{i \,|\, i \le n\}$, $I_E^* \cap J_E^* = \emptyset$, such that $\bigvee_{i\in I_E^*} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee \bigvee_{i\in J_E^*} C_i \in clo^{\mathcal{BH}}(S)$. (92)

$I_E^* \cap J_E^* \subseteq (A - E) \cap (E \cup B) = ((A - E) \cap E) \cup ((A - E) \cap B) \subseteq A \cap B = \emptyset$. We proceed by induction on $\|E\| \le \|A\| \le n + 1$.

Case 2.1 (the base case): $\|E\| = 0$. Then $E = \emptyset$. We put $I_\emptyset^* = I_A^* \subseteq A$ and $J_\emptyset^* = I_B^* \subseteq B$, $\emptyset \ne I_\emptyset^* \cup J_\emptyset^* \subseteq \{i \,|\, i \le n\}$, $I_\emptyset^* \cap J_\emptyset^* = \emptyset$. Hence, $\bigvee_{i\in I_\emptyset^*} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee \bigvee_{i\in J_\emptyset^*} C_i \in clo^{\mathcal{BH}}(S)$; (92) holds.

Case 2.2 (the induction case): $\|E\| > 0$. Then $E \ne \emptyset$, for all $e \in E \subseteq A$, by induction hypothesis for $E - \{e\} \subseteq E \subseteq A$, $\|E - \{e\}\| = \|E\| - 1$, there exist $I_{E-\{e\}}^* \subseteq A - (E - \{e\}) = (A - E) \cup \{e\}$, $J_{E-\{e\}}^* \subseteq (E - \{e\}) \cup B$, $\emptyset \ne I_{E-\{e\}}^* \cup J_{E-\{e\}}^* \subseteq \{i \,|\, i \le n\}$, $I_{E-\{e\}}^* \cap J_{E-\{e\}}^* = \emptyset$, and $\bigvee_{i\in I_{E-\{e\}}^*} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee \bigvee_{i\in J_{E-\{e\}}^*} C_i \in clo^{\mathcal{BH}}(S)$. We distinguish two cases.

Case 2.2.1: There exists $e^* \in E$ such that $I_{E-\{e^*\}}^* \subseteq A - E$. We put $I_E^* = I_{E-\{e^*\}}^* \subseteq A - E$ and $J_E^* = J_{E-\{e^*\}}^* \subseteq (E - \{e^*\}) \cup B \subseteq E \cup B$, $\emptyset \ne I_E^* \cup J_E^* \subseteq \{i \,|\, i \le n\}$, $I_E^* \cap J_E^* = \emptyset$. Hence, $\bigvee_{i\in I_E^*} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee \bigvee_{i\in J_E^*} C_i \in clo^{\mathcal{BH}}(S)$; (92) holds.

Case 2.2.2: For all $e \in E$, $e \in I_{E-\{e\}}^*$. Then, for all $i \in A - E$, $\bigvee_{j=0}^{r-1} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee \varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i \in clo^{\mathcal{BH}}(S)$; for all $e \in E$, $\bigvee_{i\in I_{E-\{e\}}^* - \{e\}} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee (\varepsilon_r^e \diamond_r^e \upsilon_r^e \vee C_e) \vee \bigvee_{i\in J_{E-\{e\}}^*} C_i = \bigvee_{i\in I_{E-\{e\}}^*} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee \bigvee_{i\in J_{E-\{e\}}^*} C_i \in clo^{\mathcal{BH}}(S)$; for all $i \in B$, $\bigvee_{j=0}^{k_i} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee C_i \in clo^{\mathcal{BH}}(S)$; $(A - E) \cup E \cup B = A \cup B = \{i \,|\, i \le n\}$, for all $\mathcal{S} \in \mathcal{S}el(\{\{j \,|\, j \le r-1\}_i \,|\, i \in A - E\} \cup \{\{r\}_e \,|\, e \in E\} \cup \{\{j \,|\, j \le k_i\}_i \,|\, i \in B\}) \subseteq \mathcal{S}el(\{\{j \,|\, j \le k_i\}_i \,|\, i \le n\})$, there exists a contradiction of $\{\varepsilon_{\mathcal{S}(i)}^i \diamond_{\mathcal{S}(i)}^i \upsilon_{\mathcal{S}(i)}^i \,|\, i \le n\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $max(\{r-1\} \cup \{0\} \cup \{k_i \,|\, i \in B\}) = r - 1$; by induction hypothesis for $r-1$, there exist $K_{A-E}^* \subseteq A - E$, $K_E^* \subseteq E$, $K_B^* \subseteq B$, $\emptyset \ne K_{A-E}^* \cup K_E^* \cup K_B^* \subseteq (A - E) \cup E \cup B = \{i \,|\, i \le n\}$, $K_{A-E}^* \cap K_E^* \subseteq (A - E) \cap E = \emptyset$, $K_{A-E}^* \cap K_B^* \subseteq (A - E) \cap B \subseteq A \cap B = \emptyset$, $K_E^* \cap K_B^* \subseteq E \cap B \subseteq A \cap B = \emptyset$, and $\bigvee_{i\in K_{A-E}^*} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee \bigvee_{e\in K_E^*} (\bigvee_{i\in I_{E-\{e\}}^* - \{e\}} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee C_e \vee \bigvee_{i\in J_{E-\{e\}}^*} C_i) \vee \bigvee_{i\in K_B^*} C_i \in clo^{\mathcal{BH}}(S)$. We put $I_E^* = K_{A-E}^* \cup \bigcup_{e\in K_E^*} (I_{E-\{e\}}^* - \{e\}) \subseteq (A - E) \cup \bigcup_{e\in K_E^*} (A - E) = (A - E)$ and $J_E^* = K_E^* \cup \bigcup_{e\in K_E^*} J_{E-\{e\}}^* \cup K_B^* \subseteq E \cup \bigcup_{e\in K_E^*} ((E - \{e\}) \cup B) \cup B \subseteq E \cup B$, $\emptyset \ne K_{A-E}^* \cup K_E^* \cup K_B^* \subseteq I_E^* \cup J_E^* \subseteq (A - E) \cup E \cup B = \{i \,|\, i \le n\}$, $I_E^* \cap J_E^* \subseteq (A - E) \cap (E \cup B) = \emptyset$. Hence, $\bigvee_{i\in I_E^*} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee \bigvee_{i\in J_E^*} C_i = \bigvee_{i\in K_{A-E}^*} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee \bigvee_{e\in K_E^*} (\bigvee_{i\in I_{E-\{e\}}^* - \{e\}} (\varepsilon_r^i \diamond_r^i \upsilon_r^i \vee C_i) \vee C_e \vee \bigvee_{i\in J_{E-\{e\}}^*} C_i) \vee \bigvee_{i\in K_B^*} C_i \in clo^{\mathcal{BH}}(S)$; (92) holds.

So, in both Cases 2.2.1 and 2.2.2, (92) holds; (92) holds. So, in both Cases 2.1 and 2.2, (92) holds. The induction is completed. Thus, (92) holds.

We conclude. By (92) for A, there exist $I_A^* \subseteq A - A = \emptyset$ and $J_A^* \subseteq A \cup B = \{i \,|\, i \le n\}$, $\emptyset \ne J_A^* \subseteq \{i \,|\, i \le n\}$, such that $\bigvee_{i\in J_A^*} C_i \in clo^{\mathcal{BH}}(S)$. We put $I^* = J_A^*$, $\emptyset \ne I^* \subseteq \{i \,|\, i \le n\}$. Then $\bigvee_{i\in I^*} C_i \in clo^{\mathcal{BH}}(S)$; the statement holds.

So, in both Cases 1 and 2, the statement holds. The induction is completed. The lemma is proved. □

12. Full proof of Lemma 8

Proof. We proceed by contradiction. Let, for all $\mathcal{S} \in \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota < \gamma\})$, there exist a contradiction of $\{\varepsilon^\iota_{\mathcal{S}(\iota)} \diamond^\iota_{\mathcal{S}(\iota)} \upsilon^\iota_{\mathcal{S}(\iota)} \,|\, \iota < \gamma\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. Since, a contradiction is a chain, finite sequence, for all $\mathcal{S} \in \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota < \gamma\})$, there exists the least $\kappa_{\mathcal{S}} < \gamma \leq \omega$ with respect to $\leq$ such that there exists a contradiction of $\{\varepsilon^\iota_{\mathcal{S}(\iota)} \diamond^\iota_{\mathcal{S}(\iota)} \upsilon^\iota_{\mathcal{S}(\iota)} \,|\, \iota \leq \kappa_{\mathcal{S}}\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. There exist $T_j = \{\mathcal{S}|_{\kappa_{\mathcal{S}}+1} \,|\, \mathcal{S} \in \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota < \gamma\}), \mathcal{S}(0) = j\}$, $j = 0, \dots, k_0$. Then, for all $j \leq k_0$, T_j is a finitely generated tree; T_j consists of the branches $\mathcal{S}|_{\kappa_{\mathcal{S}}+1} \in T_j$; j is the root of T_j if $T_j \neq \emptyset$; every vertex v_ι, $v_\iota \leq k_\iota$, $\iota < \gamma$, of T_j has at most the finite number $k_{\iota+1} + 1$ of children $v_{\iota+1}$, $v_{\iota+1} \leq k_{\iota+1}$, $\iota + 1 < \gamma$, in T_j; for all $\mathcal{S}|_{\kappa_{\mathcal{S}}+1} \in T_j$, $\mathcal{S}|_{\kappa_{\mathcal{S}}+1}$ is a finite mapping and a finite branch of T_j; by König's Lemma, T_j is a finite tree and a finite set; there exists $\eta_j = max\{\kappa_{\mathcal{S}} \,|\, \mathcal{S}|_{\kappa_{\mathcal{S}}+1} \in T_j\} < \gamma \leq \omega$; there exists $\eta = max\{\eta_j \,|\, j \leq k_0\} < \gamma \leq \omega$, and for all $\mathcal{S} \in \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota < \gamma\})$, there exists a contradiction of $\{\varepsilon^\iota_{\mathcal{S}(\iota)} \diamond^\iota_{\mathcal{S}(\iota)} \upsilon^\iota_{\mathcal{S}(\iota)} \,|\, \iota \leq \kappa_{\mathcal{S}} \leq \eta_{\mathcal{S}(0)} \leq \eta\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$; for all $\mathcal{S} \in \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota \leq \eta\}) \subseteq \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota < \gamma\})$, there exists a contradiction of $\{\varepsilon^\iota_{\mathcal{S}(\iota)} \diamond^\iota_{\mathcal{S}(\iota)} \upsilon^\iota_{\mathcal{S}(\iota)} \,|\, \iota \leq \eta\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$; by Lemma 7 for S, $\{\bigvee_{j=0}^{k_\iota} \varepsilon^\iota_j \diamond^\iota_j \upsilon^\iota_j \,|\, \iota \leq \eta\} \subseteq \{\bigvee_{j=0}^{k_\iota} \varepsilon^\iota_j \diamond^\iota_j \upsilon^\iota_j \,|\, \iota < \gamma\} = clo^{\mathcal{BH}}(S)$, $\square \in clo^{\mathcal{BH}}(S)$, which is a contradiction. So, there exists $\mathcal{S}^* \in \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota < \gamma\})$ such that there does not exist a contradiction of $\{\varepsilon^\iota_{\mathcal{S}^*(\iota)} \diamond^\iota_{\mathcal{S}^*(\iota)} \upsilon^\iota_{\mathcal{S}^*(\iota)} \,|\, \iota < \gamma\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. The lemma is proved. $\square$

13. Full proof of Theorem 9

PROOF. ($\Longrightarrow$) Let $\mathfrak{A}$ be a model of S for $\mathcal{L} \cup P$ and $C \in clo^{\mathcal{H}}(S) \subseteq OrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. Then there exists an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$ such that $\mathfrak{A}' \models C$. The proof is by complete induction on the length of a deduction of C from S by order hyperresolution. Let $\square \in clo^{\mathcal{H}}(S)$ and $\mathfrak{A}$ be a model of S for $\mathcal{L} \cup P$. Hence, there exists an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$ such that $\mathfrak{A}' \models \square$, which is a contradiction; S is unsatisfiable.

($\Longleftarrow$) Let $\square \notin clo^{\mathcal{H}}(S)$. Then, by Lemma 6 for S, $\square$, $\square \notin clo^{\mathcal{BH}}(S)$; we have $\mathcal{L}$, $\tilde{\mathbb{P}}$, $\tilde{\mathbb{W}}$ are countable, $P \subseteq \tilde{\mathbb{P}}$, $S \subseteq OrdCl_{\mathcal{L}\cup P}$, $clo^{\mathcal{BH}}(S) \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$; P, $\mathcal{L} \cup P$, $OrdCl_{\mathcal{L}\cup P}$, S, $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$, $GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $clo^{\mathcal{BH}}(S)$ are countable; there exists $\gamma_1 \leq \omega$ and $\square \notin clo^{\mathcal{BH}}(S) = \{\bigvee_{j=0}^{k_\iota} \varepsilon_j^\iota \diamond_j^\iota \upsilon_j^\iota \,|\, \iota < \gamma_1\}$; by Lemma 8 for S, there exists $\mathcal{S}^* \in \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota < \gamma_1\})$ and there does not exist a contradiction of $\{\varepsilon^\iota_{\mathcal{S}^*(\iota)} \diamond^\iota_{\mathcal{S}^*(\iota)} \upsilon^\iota_{\mathcal{S}^*(\iota)} \,|\, \iota < \gamma_1\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. We put $\mathbb{S} = \{\varepsilon^\iota_{\mathcal{S}^*(\iota)} \diamond^\iota_{\mathcal{S}^*(\iota)} \upsilon^\iota_{\mathcal{S}^*(\iota)} \,|\, \iota < \gamma_1\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. Then $ordtcons(S) \subseteq clo^{\mathcal{BH}}(S)$, $\mathbb{S} \supseteq ordtcons(S)$ is countable, unit, $(q)atoms(\mathbb{S}) \subseteq (q)atoms(clo^{\mathcal{BH}}(S))$; there does not exist a contradiction of $\mathbb{S}$. We have $\mathcal{L}$ contains a constant symbol. Hence, there exists $cn^* \in Func_{\mathcal{L}}$, $ar_{\mathcal{L}}(cn^*) = 0$. We put $\tilde{\mathbb{W}}^* = funcs(\mathbb{S}) \cap \tilde{\mathbb{W}} \subseteq \tilde{\mathbb{W}}$, $\tilde{\mathbb{W}}^* \cap (Func_{\mathcal{L}} \cup \{\tilde{f}_0\}) \subseteq \tilde{\mathbb{W}} \cap (Func_{\mathcal{L}} \cup \{\tilde{f}_0\}) = \emptyset$,

$$\mathcal{U}_{\mathfrak{A}} = GTerm_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}, cn^* \in \mathcal{U}_{\mathfrak{A}} \neq \emptyset,$$
$$\mathcal{B} = atoms(\mathbb{S}) \cup qatoms(\mathbb{S}) \subseteq GAtom_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P} \cup QAtom_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}.$$

We have $\mathbb{S}$ is countable. Then $tcons(S) = atoms(ordtcons(S)) \subseteq atoms(\mathbb{S}) \subseteq \mathcal{B}$, $\mathcal{B} = tcons(S) \cup (\mathcal{B} - tcons(S))$, $tcons(S) \cap (\mathcal{B} - tcons(S)) = \emptyset$, $atoms(\mathbb{S})$, $qatoms(\mathbb{S})$, $\mathcal{B}$, $tcons(S)$, $\mathcal{B} - tcons(S)$ are countable; there exist $\gamma_2 \leq \omega$ and a sequence $\delta_2 : \gamma_2 \longrightarrow \mathcal{B} - tcons(S)$ of $\mathcal{B} - tcons(S)$. Let $\varepsilon_1, \varepsilon_2 \in \mathcal{B}$. $\varepsilon_1 \triangleq \varepsilon_2$ iff there exists an equality chain $\varepsilon_1 \,\Xi\, \varepsilon_2$ of $\mathbb{S}$. Note that $\triangleq$ is a binary symmetric transitive relation on $\mathcal{B}$. $\varepsilon_1 \lhd \varepsilon_2$ iff there exists an increasing chain $\varepsilon_1 \,\Xi\, \varepsilon_2$ of $\mathbb{S}$. Note that $\lhd$ is a binary transitive relation on $\mathcal{B}$.

$$0 \not\triangleq 1, 1 \not\triangleq 0, 0 \lhd 1, 1 \ntriangleleft 0, \text{ for all } \varepsilon \in \mathcal{B}, \varepsilon \ntriangleleft 0, 1 \ntriangleleft \varepsilon, \varepsilon \ntriangleleft \varepsilon. \tag{93}$$

The proof is straightforward; we have that there does not exist a contradiction of $\mathbb{S}$. Note that $\lhd$ is also irreflexive and a partial strict order on $\mathcal{B}$.

Let $tcons(S) \subseteq X \subseteq \mathcal{B}$. A partial valuation $\mathcal{V}$ is a mapping $\mathcal{V} : X \longrightarrow [0, 1]$ such that $\mathcal{V}(0) = 0$, $\mathcal{V}(1) = 1$, for all $\bar{c} \in tcons(S) \cap \overline{C}_{\mathcal{L}}$, $\mathcal{V}(\bar{c}) = c$. We denote $dom(\mathcal{V}) = X$, $tcons(S) \subseteq dom(\mathcal{V}) \subseteq \mathcal{B}$. We define a partial valuation $\mathcal{V}_\alpha$ by recursion on $\alpha \leq \gamma_2$ as follows:

$$\mathcal{V}_0 = \{(0, 0), (1, 1)\} \cup \{(\bar{c}, c) \,|\, \bar{c} \in tcons(S) \cap \overline{C}_{\mathcal{L}}\};$$

$$\mathcal{V}_\alpha = \mathcal{V}_{\alpha-1} \cup \{(\delta_2(\alpha - 1), \lambda_{\alpha-1})\} \quad (1 \leq \alpha \leq \gamma_2 \textit{ is a successor ordinal}),$$

$$\mathbb{E}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, a \triangleq \delta_2(\alpha - 1), a \in dom(\mathcal{V}_{\alpha-1})\},$$
$$\mathbb{D}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, a \lhd \delta_2(\alpha - 1), a \in dom(\mathcal{V}_{\alpha-1})\},$$
$$\mathbb{U}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, \delta_2(\alpha - 1) \lhd a, a \in dom(\mathcal{V}_{\alpha-1})\},$$
$$\lambda_{\alpha-1} = \begin{cases} \dfrac{\bigvee \mathbb{D}_{\alpha-1} + \bigwedge \mathbb{U}_{\alpha-1}}{2} & \textit{if } \mathbb{E}_{\alpha-1} = \emptyset, \\ \bigvee \mathbb{E}_{\alpha-1} & \textit{else}; \end{cases}$$

$$\mathcal{V}_{\gamma_2} = \bigcup_{\alpha < \gamma_2} \mathcal{V}_\alpha \quad (\gamma_2 \textit{ is a limit ordinal}).$$

$$\text{For all } \alpha \leq \alpha' \leq \gamma_2, \mathcal{V}_\alpha \text{ is a partial valuation, } dom(\mathcal{V}_\alpha) = tcons(S) \cup \delta_2[\alpha], \mathcal{V}_\alpha \subseteq \mathcal{V}_{\alpha'}. \tag{94}$$

The proof is by induction on $\alpha \leq \gamma_2$.

At first, we prove the following statements:

$$\text{If } qatoms(S) = \emptyset, \text{ then } qatoms(clo^{\mathcal{BH}}(S)) = \emptyset. \tag{95}$$

Let $qatoms(S) = \emptyset$. Then, for all $C \in clo^{\mathcal{BH}}(S)$, $qatoms(C) = \emptyset$. The proof is by complete induction on the length of a deduction of C from S by basic order hyperresolution. Hence, $qatoms(clo^{\mathcal{BH}}(S)) = \bigcup_{C \in clo^{\mathcal{BH}}(S)} qatoms(C) = \emptyset$; (95) holds.

$$tcons(S) = tcons(clo^{\mathcal{BH}}(S)). \tag{96}$$

$ordtcons(S) \subseteq clo^{\mathcal{BH}}(S)$ and $tcons(S) = tcons(ordtcons(S)) \subseteq tcons(clo^{\mathcal{BH}}(S))$. For all $C \in clo^{\mathcal{BH}}(S)$, $tcons(C) \subseteq tcons(S)$. The proof is by complete induction on the length of a deduction of C from S by basic order hyperresolution. Then $tcons(clo^{\mathcal{BH}}(S)) = \bigcup_{C \in clo^{\mathcal{BH}}(S)} tcons(C) \subseteq tcons(S)$; (96) holds.

For all $a, b \in atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$, there exist a deduction $C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq GOrdCl_{\mathcal{L}_n}$, such that $a, b \in atoms(S_n) \cup qatoms(S_n)$. (97)

Let $a, b \in atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$. Then there exist $C_1 \in clo^{\mathcal{BH}}(S)$, $a \in atoms(C_1) \cup qatoms(C_1)$, a deduction $C_1^1, \dots, C_{n_1}^1$, $C_1 = C_{n_1}^1$, $n_1 \geq 1$, of C_1 from S by basic order hyperresolution, associated $\mathcal{L}_\sigma^1$, S_σ^1, $S_\sigma^1 \subseteq GOrdCl_{\mathcal{L}_\sigma^1}$, $\sigma = 0, \dots, n_1$, and $C_1 = C_{n_1}^1 \in S_{n_1}^1 \subseteq GOrdCl_{\mathcal{L}_{n_1}^1}$, $a \in atoms(C_1) \cup qatoms(C_1) \subseteq atoms(S_{n_1}^1) \cup qatoms(S_{n_1}^1)$; there exist $C_2 \in clo^{\mathcal{BH}}(S)$, $b \in atoms(C_2) \cup qatoms(C_2)$, a deduction $C_1^2, \dots, C_{n_2}^2$, $C_2 = C_{n_2}^2$, $n_2 \geq 1$, of C_2 from S by basic order hyperresolution, associated $\mathcal{L}_\sigma^2$, S_σ^2, $S_\sigma^2 \subseteq GOrdCl_{\mathcal{L}_\sigma^2}$, $\sigma = 0, \dots, n_2$, and $C_2 = C_{n_2}^2 \in S_{n_2}^2 \subseteq GOrdCl_{\mathcal{L}_{n_2}^2}$, $b \in atoms(C_2) \cup qatoms(C_2) \subseteq atoms(S_{n_2}^2) \cup qatoms(S_{n_2}^2)$. We next prove the following statement:

Let $A \subseteq atoms(S_{n_1}^1) \cup qatoms(S_{n_1}^1)$. For all $1 \leq \sigma \leq n_2$, there exist a deduction $\mathcal{D}_\sigma = C_1^1, \dots, C_{n_1}^1, D_1, \dots, D_\sigma$ of D_σ from S by basic order hyperresolution, associated $\mathcal{L}_{n_1+\sigma}$, $S_{n_1+\sigma}$, $D_\sigma \in S_{n_1+\sigma} \subseteq GOrdCl_{\mathcal{L}_{n_1+\sigma}}$, such that $A \subseteq atoms(S_{n_1+\sigma}) \cup qatoms(S_{n_1+\sigma})$, $\mathcal{L}_{n_1+\sigma}$ is an expansion of $\mathcal{L}_\sigma^2$, $atoms(S_\sigma^2) \cup qatoms(S_\sigma^2) \subseteq atoms(S_{n_1+\sigma}) \cup qatoms(S_{n_1+\sigma})$. (98)

We proceed by induction on $1 \leq \sigma \leq n_2$.

Case 1 (the base case): $\sigma = 1$. Then $\mathcal{L}_0^2 = \mathcal{L} \cup P$, $S_0^2 = \emptyset \subseteq GOrdCl_{\mathcal{L}_0^2}$, $(q)atoms(S_0^2) = \emptyset$, Rules (42)–(48) are not applicable to S_0^2, $C_1^2 \in ordtcons(S) \cup GInst_{\mathcal{L}_0^2}(S) = ordtcons(S) \cup GInst_{\mathcal{L} \cup P}(S) \subseteq GOrdCl_{\mathcal{L} \cup P}$, $\mathcal{L}_1^2 = \mathcal{L}_0^2 = \mathcal{L} \cup P$, $S_1^2 = \{C_1^2\} \subseteq GOrdCl_{\mathcal{L}_1^2} = GOrdCl_{\mathcal{L} \cup P}$. We put $\mathcal{L}_i = \mathcal{L}_i^1$, $i = 0, \dots, n_1$, $\mathcal{L}_{n_1+1} = \mathcal{L}_{n_1}$, $S_i = S_i^1$, $i = 0, \dots, n_1$, $S_i \subseteq GOrdCl_{\mathcal{L}_i}$, $D_1 = C_1^2 \in GOrdCl_{\mathcal{L} \cup P} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $D_1 \in ordtcons(S) \cup GInst_{\mathcal{L} \cup P}(S) \subseteq ordtcons(S) \cup GInst_{\mathcal{L}_{n_1}}(S)$, $\mathcal{D}_1 = C_1^1, \dots, C_{n_1}^1, D_1$, $Sn_1 + 1 = S_{n_1} \cup \{D_1\} \subseteq GOrdCl_{\mathcal{L}_{n_1}} \cup GOrdCl_{\mathcal{L} \cup P} = GOrdCl_{\mathcal{L}_{n_1}} = GOrdCl_{\mathcal{L}_{n_1+1}}$; $\mathcal{D}_1$ is a deduction of D_1 from S by basic order hyperresolution. Hence, $A \subseteq atoms(S_{n_1}^1) \cup qatoms(S_{n_1}^1) = atoms(S_{n_1}) \cup qatoms(S_{n_1}) \subseteq atoms(S_{n_1+1}) \cup qatoms(S_{n_1+1})$, $\mathcal{L}_{n_1+1}$ is an expansion of $\mathcal{L} \cup P = \mathcal{L}_1^2$, $atoms(S_1^2) \cup qatoms(S_1^2) = atoms(D_1) \cup qatoms(D_1) \subseteq atoms(S_{n_1+1}) \cup qatoms(S_{n_1+1})$; (98) holds.

Case 2 (the induction case): $1 < \sigma \leq n_2$. We get by induction hypothesis for $\sigma - 1$ that there exist a deduction $\mathcal{D}_{\sigma-1} = C_1^1, \dots, C_{n_1}^1, D_1, \dots, D_{\sigma-1}$ of $D_{\sigma-1}$ from S by basic order hyperresolution, associated $\mathcal{L}_{n_1+\sigma-1}$, $S_{n_1+\sigma-1}$, $D_{\sigma-1} \in S_{n_1+\sigma-1} \subseteq GOrdCl_{\mathcal{L}_{n_1+\sigma-1}}$, and $A \subseteq atoms(S_{n_1+\sigma-1}) \cup qatoms(S_{n_1+\sigma-1})$, $\mathcal{L}_{n_1+\sigma-1}$ is an expansion of $\mathcal{L}_{\sigma-1}^2$, $atoms(S_{\sigma-1}^2) \cup qatoms(S_{\sigma-1}^2) \subseteq atoms(S_{n_1+\sigma-1}) \cup qatoms(S_{n_1+\sigma-1})$. We distinguish seven cases for C_σ^2.

Case 2.1: $C_\sigma^2 \in ordtcons(S) \cup GInst_{\mathcal{L}_{\sigma-1}^2}(S)$. Then $\mathcal{L}_\sigma^2 = \mathcal{L}_{\sigma-1}^2$, $C_\sigma^2 \in ordtcons(S) \cup GInst_{\mathcal{L}_{\sigma-1}^2}(S) \subseteq ordtcons(S) \cup GInst_{\mathcal{L}_{n_1+\sigma-1}}(S)$, $S_\sigma^2 = S_{\sigma-1}^2 \cup \{C_\sigma^2\}$. We put $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$, $D_\sigma = C_\sigma^2 \in ordtcons(S) \cup GInst_{\mathcal{L}_{n_1+\sigma-1}}(S) \subseteq GOrdCl_{\mathcal{L}_{n_1+\sigma-1}} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, D_\sigma$, $S_{n_1+\sigma} = S_{n_1+\sigma-1} \cup \{D_\sigma\} \subseteq GOrdCl_{\mathcal{L}_{n_1+\sigma-1}} = GOrdCl_{\mathcal{L}_{n_1+\sigma}}$; $\mathcal{D}_\sigma$ is a deduction of D_σ from S by basic order hyperresolution. Hence, $A \subseteq atoms(S_{n_1+\sigma-1}) \cup qatoms(S_{n_1+\sigma-1}) \subseteq atoms(S_{n_1+\sigma}) \cup qatoms(S_{n_1+\sigma})$, $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$ is an expansion of $\mathcal{L}_{\sigma-1}^2 = \mathcal{L}_\sigma^2$, $atoms(S_\sigma^2) \cup qatoms(S_\sigma^2) = atoms(S_{\sigma-1}^2) \cup qatoms(S_{\sigma-1}^2) \cup atoms(C_\sigma^2) \cup qatoms(C_\sigma^2) \subseteq atoms(S_{n_1+\sigma-1}) \cup qatoms(S_{n_1+\sigma-1}) \cup atoms(D_\sigma) \cup qatoms(D_\sigma) = atoms(S_{n_1+\sigma}) \cup qatoms(S_{n_1+\sigma})$; (98) holds.

Case 2.2: There exist $1 \leq j_k^* \leq \sigma - 1$, $k = 0, \dots, m$, such that $C_\sigma^2 \in S_\sigma^2$ is a basic order hyperresolvent of $C_{j_1^*}^2, \dots, C_{j_m^*}^2 \in S_{\sigma-1}^2$ using Rule (42) with respect to $\mathcal{L}_{\sigma-1}^2$, $S_{\sigma-1}^2$. Then $\mathcal{L}_\sigma^2 = \mathcal{L}_{\sigma-1}^2$, $atoms(C_\sigma^2) \cup qatoms(C_\sigma^2) \subseteq \bigcup_{k=0}^m atoms(C_{j_k^*}^2) \cup qatoms(C_{j_k^*}^2) \subseteq atoms(S_{\sigma-1}^2) \cup qatoms(S_{\sigma-1}^2)$, $S_\sigma^2 = S_{\sigma-1}^2 \cup \{C_\sigma^2\}$. We put $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$, $D_\sigma = 0 \prec 1 \in ordtcons(S) \cup GInst_{\mathcal{L}_{n_1+\sigma-1}}(S) \subseteq GOrdCl_{\mathcal{L}_{n_1+\sigma-1}} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, D_\sigma$, $S_{n_1+\sigma} = S_{n_1+\sigma-1} \cup \{D_\sigma\} \subseteq GOrdCl_{\mathcal{L}_{n_1+\sigma-1}} = GOrdCl_{\mathcal{L}_{n_1+\sigma}}$; $\mathcal{D}_\sigma$ is a deduction of D_σ from S by basic order hyperresolution. Hence, $A \subseteq atoms(S_{n_1+\sigma-1}) \cup qatoms(S_{n_1+\sigma-1}) \subseteq atoms(S_{n_1+\sigma}) \cup qatoms(S_{n_1+\sigma})$, $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$ is an expansion of $\mathcal{L}_{\sigma-1}^2 = \mathcal{L}_\sigma^2$, $atoms(S_\sigma^2) \cup qatoms(S_\sigma^2) = atoms(S_{\sigma-1}^2) \cup qatoms(S_{\sigma-1}^2) \cup atoms(C_\sigma^2) \cup qatoms(C_\sigma^2) = atoms(S_{\sigma-1}^2) \cup qatoms(S_{\sigma-1}^2) \subseteq atoms(S_{n_1+\sigma-1}) \cup qatoms(S_{n_1+\sigma-1}) \subseteq atoms(S_{n_1+\sigma}) \cup qatoms(S_{n_1+\sigma})$; (98) holds.

Case 2.3: We get two cases for $qatoms(S)$.

Case 2.3.1: $qatoms(S) = \emptyset$. Then there exist $c, d \in atoms(S_{\sigma-1}^2)$, $c \in \overline{C}_{\mathcal{L}}$, $d \not\in Tcons_{\mathcal{L}}$, such that $C_\sigma^2 = c \prec d \vee c \eqcirc d \vee d \prec c \in S_\sigma^2$ is a basic order trichotomy resolvent of c and d using Rule (43) with respect to $\mathcal{L}_{\sigma-1}^2$, $S_{\sigma-1}^2$; $atoms(C_\sigma^2) \cup qatoms(C_\sigma^2) = \{c, d\} \subseteq atoms(S_{\sigma-1}^2) \cup qatoms(S_{\sigma-1}^2)$.

Case 2.3.2: $qatoms(S) \neq \emptyset$. Then there exist $c, d \in atoms(S_{\sigma-1}^2) - \{0, 1\}$, $\{c, d\} \not\subseteq Tcons_{\mathcal{L}}$, such that $C_\sigma^2 = c \prec d \vee c \eqcirc d \vee d \prec c \in S_\sigma^2$ is a basic order trichotomy resolvent of c and d using Rule (44) with respect to $\mathcal{L}_{\sigma-1}^2$, $S_{\sigma-1}^2$; $atoms(C_\sigma^2) \cup qatoms(C_\sigma^2) = \{c, d\} \subseteq atoms(S_{\sigma-1}^2) \cup qatoms(S_{\sigma-1}^2)$.

So, in both Cases 2.3.1 and 2.3.2, $atoms(C_\sigma^2) \cup qatoms(C_\sigma^2) \subseteq atoms(S_{\sigma-1}^2) \cup qatoms(S_{\sigma-1}^2)$; $atoms(C_\sigma^2) \cup qatoms(C_\sigma^2) \subseteq atoms(S_{\sigma-1}^2) \cup qatoms(S_{\sigma-1}^2)$. Then $\mathcal{L}_\sigma^2 = \mathcal{L}_{\sigma-1}^2$ and $S_\sigma^2 = S_{\sigma-1}^2 \cup \{C_\sigma^2\}$. We put $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$, $D_\sigma = 0 \prec 1 \in ordtcons(S) \cup GInst_{\mathcal{L}_{n_1+\sigma-1}}(S) \subseteq GOrdCl_{\mathcal{L}_{n_1+\sigma-1}} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, D_\sigma$, $S_{n_1+\sigma} = S_{n_1+\sigma-1} \cup \{D_\sigma\} \subseteq GOrdCl_{\mathcal{L}_{n_1+\sigma-1}} = GOrdCl_{\mathcal{L}_{n_1+\sigma}}$; $\mathcal{D}_\sigma$ is a deduction of D_σ from S by basic order hyperresolution. Hence, $A \subseteq atoms(S_{n_1+\sigma-1}) \cup qatoms(S_{n_1+\sigma-1}) \subseteq atoms(S_{n_1+\sigma}) \cup qatoms(S_{n_1+\sigma})$, $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$ is an expansion of $\mathcal{L}_{\sigma-1}^2 = \mathcal{L}_\sigma^2$, $atoms(S_\sigma^2) \cup qatoms(S_\sigma^2) = atoms(S_{\sigma-1}^2) \cup qatoms(S_{\sigma-1}^2) \cup atoms(C_\sigma^2) \cup qatoms(C_\sigma^2) = atoms(S_{\sigma-1}^2) \cup qatoms(S_{\sigma-1}^2) \subseteq atoms(S_{n_1+\sigma-1}) \cup qatoms(S_{n_1+\sigma-1}) \subseteq atoms(S_{n_1+\sigma}) \cup qatoms(S_{n_1+\sigma})$; (98) holds.

Case 2.4: There exist $\forall x\, c \in qatoms^{\forall}(S_{\sigma-1}^2)$, $t \in GTerm_{\mathcal{L}_{\sigma-1}^2}$, $\gamma = x/t \in Subst_{\mathcal{L}_{\sigma-1}^2}$, $dom(\gamma) = \{x\} = vars(c)$, such that $C_\sigma^2 = \forall x\, c \prec c\gamma \vee \forall x\, c \eqcirc c\gamma \in S_\sigma^2$ is a basic order $\forall$-quantification resolvent of $\forall x\, c$ using Rule (45) with respect to $\mathcal{L}_{\sigma-1}^2$, $S_{\sigma-1}^2$. Then $\mathcal{L}_\sigma^2 = \mathcal{L}_{\sigma-1}^2$, $S_\sigma^2 = S_{\sigma-1}^2 \cup \{C_\sigma^2\}$, $\forall x\, c \in qatoms^{\forall}(S_{\sigma-1}^2) \subseteq qatoms^{\forall}(S_{n_1+\sigma-1}) \subseteq QAtom_{\mathcal{L}_{n_1+\sigma-1}}$, $t \in GTerm_{\mathcal{L}_{\sigma-1}^2} \subseteq GTerm_{\mathcal{L}_{n_1+\sigma-1}}$, $\gamma \in Subst_{\mathcal{L}_{\sigma-1}^2} \subseteq Subst_{\mathcal{L}_{n_1+\sigma-1}}$, using Rule (45) with respect to $\mathcal{L}_{n_1+\sigma-1}$, $S_{n_1+\sigma-1}$, we derive $C_\sigma^2 = \forall x\, c \prec c\gamma \vee \forall x\, c \eqcirc c\gamma \in GOrdCl_{\mathcal{L}_{n_1+\sigma-1}}$. We put $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$, $D_\sigma = C_\sigma^2 \in GOrdCl_{\mathcal{L}_{n_1+\sigma-1}} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, D_\sigma$, $S_{n_1+\sigma} = S_{n_1+\sigma-1} \cup \{D_\sigma\} \subseteq GOrdCl_{\mathcal{L}_{n_1+\sigma-1}} = GOrdCl_{\mathcal{L}_{n_1+\sigma}}$; $\mathcal{D}_\sigma$ is a deduction of D_σ from S by basic order hyperresolution. Hence, $A \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \subseteq \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$, $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$ is an expansion of $\mathcal{L}^2_{\sigma-1} = \mathcal{L}^2_\sigma$, $\mathit{atoms}(S^2_\sigma) \cup \mathit{qatoms}(S^2_\sigma) = \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \cup \mathit{atoms}(C^2_\sigma) \cup \mathit{qatoms}(C^2_\sigma) \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \cup \mathit{atoms}(D_\sigma) \cup \mathit{qatoms}(D_\sigma) = \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$; (98) holds.

Case 2.5: There exist $\exists x\, c \in \mathit{qatoms}^\exists(S^2_{\sigma-1})$, $t \in \mathit{GTerm}_{\mathcal{L}^2_{\sigma-1}}$, $\gamma = x/t \in \mathit{Subst}_{\mathcal{L}^2_{\sigma-1}}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(c)$, such that $C^2_\sigma = c\gamma \prec \exists x\, c \vee c\gamma \eqcirc \exists x\, c \in S^2_\sigma$ is a basic order $\exists$-quantification resolvent of $\exists x\, c$ using Rule (46) with respect to $\mathcal{L}^2_{\sigma-1}$, $S^2_{\sigma-1}$. Then $\mathcal{L}^2_\sigma = \mathcal{L}^2_{\sigma-1}$, $S^2_\sigma = S^2_{\sigma-1} \cup \{C^2_\sigma\}$, $\exists x\, c \in \mathit{qatoms}^\exists(S^2_{\sigma-1}) \subseteq \mathit{qatoms}^\exists(S_{n_1+\sigma-1}) \subseteq \mathit{QAtom}_{\mathcal{L}_{n_1+\sigma-1}}$, $t \in \mathit{GTerm}_{\mathcal{L}^2_{\sigma-1}} \subseteq \mathit{GTerm}_{\mathcal{L}_{n_1+\sigma-1}}$, $\gamma \in \mathit{Subst}_{\mathcal{L}^2_{\sigma-1}} \subseteq \mathit{Subst}_{\mathcal{L}_{n_1+\sigma-1}}$, using Rule (46) with respect to $\mathcal{L}_{n_1+\sigma-1}$, $S_{n_1+\sigma-1}$, we derive $C^2_\sigma = c\gamma \prec \exists x\, c \vee c\gamma \eqcirc \exists x\, c \in \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1}}$. We put $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$, $D_\sigma = C^2_\sigma \in \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1}} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, D_\sigma$, $S_{n_1+\sigma} = S_{n_1+\sigma-1} \cup \{D_\sigma\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1}} = \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma}}$; $\mathcal{D}_\sigma$ is a deduction of D_σ from S by basic order hyperresolution. Hence, $A \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \subseteq \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$, $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$ is an expansion of $\mathcal{L}^2_{\sigma-1} = \mathcal{L}^2_\sigma$, $\mathit{atoms}(S^2_\sigma) \cup \mathit{qatoms}(S^2_\sigma) = \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \cup \mathit{atoms}(C^2_\sigma) \cup \mathit{qatoms}(C^2_\sigma) \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \cup \mathit{atoms}(D_\sigma) \cup \mathit{qatoms}(D_\sigma) = \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$; (98) holds.

Case 2.6: There exist $\forall x\, c \in \mathit{qatoms}^\forall(S^2_{\sigma-1})$, $d \in \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1})$, $\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}^2_{\sigma-1}}$, $\mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\forall x\, c), \mathit{freetermseq}(d)|$, $\gamma = x/\tilde{w}(\mathit{freetermseq}(\forall x\, c), \mathit{freetermseq}(d)) \in \mathit{Subst}_{\mathcal{L}^2_\sigma}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(a)$, such that $C^2_\sigma = c\gamma \prec d \vee d \eqcirc \forall x\, c \vee d \prec \forall x\, c \in S^2_\sigma$ is a basic order $\forall$-witnessing resolvent of $\forall x\, c$ and d using Rule (47) with respect to $\mathcal{L}^2_{\sigma-1}$, $S^2_{\sigma-1}$. Then $\mathcal{L}^2_\sigma = \mathcal{L}^2_{\sigma-1} \cup \{\tilde{w}\}$, $S^2_\sigma = S^2_{\sigma-1} \cup \{C^2_\sigma\}$, $\forall x\, c \in \mathit{qatoms}^\forall(S^2_{\sigma-1}) \subseteq \mathit{qatoms}^\forall(S_{n_1+\sigma-1}) \subseteq \mathit{QAtom}_{\mathcal{L}_{n_1+\sigma-1}}$, $d \in \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \subseteq \mathit{Atom}_{\mathcal{L}_{n_1+\sigma-1}} \cup \mathit{QAtom}_{\mathcal{L}_{n_1+\sigma-1}}$. We get two cases for $\tilde{w}$.

Case 2.6.1: $\tilde{w} \in \mathit{Func}_{\mathcal{L}_{n_1+\sigma-1}}$. Hence, $\mathcal{L}_{n_1+\sigma-1}$ is an expansion of $\mathcal{L}^2_{\sigma-1} \cup \{\tilde{w}\} = \mathcal{L}^2_\sigma$, $\gamma \in \mathit{Subst}_{\mathcal{L}^2_\sigma} \subseteq \mathit{Subst}_{\mathcal{L}_{n_1+\sigma-1}}$, $\tilde{w}(\mathit{freetermseq}(\forall x\, c), \mathit{freetermseq}(d)) \in \mathit{GTerm}_{\mathcal{L}^2_\sigma} \subseteq \mathit{GTerm}_{\mathcal{L}_{n_1+\sigma-1}}$, using Rule (45) with respect to $\mathcal{L}_{n_1+\sigma-1}$, $S_{n_1+\sigma-1}$, we derive $\forall x\, c \prec c\gamma \vee \forall x\, c \eqcirc c\gamma \in \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1}}$. We put $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$, $D_\sigma = \forall x\, c \prec c\gamma \vee \forall x\, c \eqcirc c\gamma \in \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1}} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, D_\sigma$, $S_{n_1+\sigma} = S_{n_1+\sigma-1} \cup \{D_\sigma\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1}} = \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma}}$; $\mathcal{D}_\sigma$ is a deduction of D_σ from S by basic order hyperresolution. Then $A \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \subseteq \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$, $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$ is an expansion of $\mathcal{L}^2_\sigma$, $\mathit{atoms}(S^2_\sigma) \cup \mathit{qatoms}(S^2_\sigma) = \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \cup \mathit{atoms}(C^2_\sigma) \cup \mathit{qatoms}(C^2_\sigma) = \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \cup \{\forall x\, c, c\gamma, d\} = \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \cup \{\forall x\, c, c\gamma\} \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \cup \mathit{atoms}(D_\sigma) \cup \mathit{qatoms}(D_\sigma) \subseteq \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$; (98) holds.

Case 2.6.2: $\tilde{w} \not\in \mathit{Func}_{\mathcal{L}_{n_1+\sigma-1}}$. Hence, $\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_{n_1+\sigma-1}}$, $\gamma \in \mathit{Subst}_{\mathcal{L}^2_\sigma} = \mathit{Subst}_{\mathcal{L}^2_{\sigma-1} \cup \{\tilde{w}\}} \subseteq \mathit{Subst}_{\mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}}$, using Rule (47) with respect to $\mathcal{L}_{n_1+\sigma-1}$, $S_{n_1+\sigma-1}$, we derive $C^2_\sigma = c\gamma \prec d \vee d \eqcirc \forall x\, c \vee d \prec \forall x\, c \in \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}}$. We put $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}$, $\gamma \in \mathit{Subst}_{\mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}} = \mathit{Subst}_{\mathcal{L}_{n_1+\sigma}}$, $D_\sigma = C^2_\sigma \in \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}} = \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma}} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, D_\sigma$, $S_{n_1+\sigma} = S_{n_1+\sigma-1} \cup \{D_\sigma\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1}} \cup \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma}} = \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma}}$; $\mathcal{D}_\sigma$ is a deduction of D_σ from S by basic order hyperresolution. Then $A \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \subseteq \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$, $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}$ is an expansion of $\mathcal{L}^2_{\sigma-1} \cup \{\tilde{w}\} = \mathcal{L}^2_\sigma$, $\mathit{atoms}(S^2_\sigma) \cup \mathit{qatoms}(S^2_\sigma) = \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \cup \mathit{atoms}(C^2_\sigma) \cup \mathit{qatoms}(C^2_\sigma) \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \cup \mathit{atoms}(D_\sigma) \cup \mathit{qatoms}(D_\sigma) = \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$; (98) holds.

So, in both Cases 2.6.1 and 2.6.2, (98) holds; (98) holds.

Case 2.7: There exist $\exists x\, c \in \mathit{qatoms}^\exists(S^2_{\sigma-1})$, $d \in \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1})$, $\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}^2_{\sigma-1}}$, $\mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\exists x\, c), \mathit{freetermseq}(d)|$, $\gamma = x/\tilde{w}(\mathit{freetermseq}(\exists x\, c), \mathit{freetermseq}(d)) \in \mathit{Subst}_{\mathcal{L}^2_\sigma}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(a)$, such that $C^2_\sigma = d \prec c\gamma \vee \exists x\, c \eqcirc d \vee \exists x\, c \prec d \in S^2_\sigma$ is a basic order $\exists$-witnessing resolvent of $\exists x\, c$ and d using Rule (48) with respect to $\mathcal{L}^2_{\sigma-1}$, $S^2_{\sigma-1}$. Then $\mathcal{L}^2_\sigma = \mathcal{L}^2_{\sigma-1} \cup \{\tilde{w}\}$, $S^2_\sigma = S^2_{\sigma-1} \cup \{C^2_\sigma\}$, $\exists x\, c \in \mathit{qatoms}^\exists(S^2_{\sigma-1}) \subseteq \mathit{qatoms}^\exists(S_{n_1+\sigma-1}) \subseteq \mathit{QAtom}_{\mathcal{L}_{n_1+\sigma-1}}$, $d \in \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \subseteq \mathit{Atom}_{\mathcal{L}_{n_1+\sigma-1}} \cup \mathit{QAtom}_{\mathcal{L}_{n_1+\sigma-1}}$. We get two cases for $\tilde{w}$.

Case 2.7.1: $\tilde{w} \in \mathit{Func}_{\mathcal{L}_{n_1+\sigma-1}}$. Hence, $\mathcal{L}_{n_1+\sigma-1}$ is an expansion of $\mathcal{L}^2_{\sigma-1} \cup \{\tilde{w}\} = \mathcal{L}^2_\sigma$, $\gamma \in \mathit{Subst}_{\mathcal{L}^2_\sigma} \subseteq \mathit{Subst}_{\mathcal{L}_{n_1+\sigma-1}}$, $\tilde{w}(\mathit{freetermseq}(\exists x\, c), \mathit{freetermseq}(d)) \in \mathit{GTerm}_{\mathcal{L}^2_\sigma} \subseteq \mathit{GTerm}_{\mathcal{L}_{n_1+\sigma-1}}$, using Rule (46) with respect to $\mathcal{L}_{n_1+\sigma-1}$, $S_{n_1+\sigma-1}$, we derive $c\gamma \prec \exists x\, c \vee c\gamma \eqcirc \exists x\, c \in \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1}}$. We put $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$, $D_\sigma = c\gamma \prec \exists x\, c \vee c\gamma \eqcirc \exists x\, c \in \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1}} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, D_\sigma$, $S_{n_1+\sigma} = S_{n_1+\sigma-1} \cup \{D_\sigma\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1}} = \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma}}$; $\mathcal{D}_\sigma$ is a deduction of D_σ from S by basic order hyperresolution. Then $A \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \subseteq \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$, $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1}$ is an expansion of $\mathcal{L}^2_\sigma$, $\mathit{atoms}(S^2_\sigma) \cup \mathit{qatoms}(S^2_\sigma) = \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \cup \mathit{atoms}(C^2_\sigma) \cup \mathit{qatoms}(C^2_\sigma) = \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \cup \{\exists x\, c, c\gamma, d\} = \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \cup \{\exists x\, c, c\gamma\} \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \cup \mathit{atoms}(D_\sigma) \cup \mathit{qatoms}(D_\sigma) \subseteq \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$; (98) holds.

Case 2.7.2: $\tilde{w} \not\in \mathit{Func}_{\mathcal{L}_{n_1+\sigma-1}}$. Hence, $\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_{n_1+\sigma-1}}$, $\gamma \in \mathit{Subst}_{\mathcal{L}^2_\sigma} = \mathit{Subst}_{\mathcal{L}^2_{\sigma-1} \cup \{\tilde{w}\}} \subseteq \mathit{Subst}_{\mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}}$, using Rule (48) with respect to $\mathcal{L}_{n_1+\sigma-1}$, $S_{n_1+\sigma-1}$, we derive $C^2_\sigma = d \prec c\gamma \vee \exists x\, c \eqcirc d \vee \exists x\, c \prec d \in \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}}$. We put $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}$, $\gamma \in \mathit{Subst}_{\mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}} = \mathit{Subst}_{\mathcal{L}_{n_1+\sigma}}$, $D_\sigma = C^2_\sigma \in \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}} = \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma}} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}_\sigma = \mathcal{D}_{\sigma-1}, D_\sigma$, $S_{n_1+\sigma} = S_{n_1+\sigma-1} \cup \{D_\sigma\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma-1}} \cup \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma}} = \mathit{GOrdCl}_{\mathcal{L}_{n_1+\sigma}}$; $\mathcal{D}_\sigma$ is a deduction of D_σ from S by basic order hyperresolution. Then $A \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \subseteq \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$, $\mathcal{L}_{n_1+\sigma} = \mathcal{L}_{n_1+\sigma-1} \cup \{\tilde{w}\}$ is an expansion of $\mathcal{L}^2_{\sigma-1} \cup \{\tilde{w}\} = \mathcal{L}^2_\sigma$, $\mathit{atoms}(S^2_\sigma) \cup \mathit{qatoms}(S^2_\sigma) = \mathit{atoms}(S^2_{\sigma-1}) \cup \mathit{qatoms}(S^2_{\sigma-1}) \cup \mathit{atoms}(C^2_\sigma) \cup \mathit{qatoms}(C^2_\sigma) \subseteq \mathit{atoms}(S_{n_1+\sigma-1}) \cup \mathit{qatoms}(S_{n_1+\sigma-1}) \cup \mathit{atoms}(D_\sigma) \cup \mathit{qatoms}(D_\sigma) = \mathit{atoms}(S_{n_1+\sigma}) \cup \mathit{qatoms}(S_{n_1+\sigma})$; (98) holds.

So, in both Cases 2.7.1 and 2.7.2, (98) holds; (98) holds.

So, in all Cases 2.1–2.7, (98) holds; (98) holds. So, in both Cases 1 and 2, (98) holds. The induction is completed. Thus, (98) holds.

Then, by (98) for $\{a\}$, n_2, there exist a deduction $\mathcal{D}_{n_2} = C^1_1, \dots, C^1_{n_1}, D_1, \dots, D_{n_2}$ of D_{n_2} from S by basic order hyperresolution, associated $\mathcal{L}_{n_1+n_2}$, $S_{n_1+n_2}$, $D_{n_2} \in S_{n_1+n_2} \subseteq \mathit{GOrdCl}_{\mathcal{L}_{n_1+n_2}}$, such that $a \in \mathit{atoms}(S_{n_1+n_2}) \cup \mathit{qatoms}(S_{n_1+n_2})$, $\mathcal{L}_{n_1+n_2}$ is an expansion of $\mathcal{L}^2_{n_2}$, $\mathit{atoms}(S^2_{n_2}) \cup \mathit{qatoms}(S^2_{n_2}) \subseteq \mathit{atoms}(S_{n_1+n_2}) \cup \mathit{qatoms}(S_{n_1+n_2})$. We put $n = n_1 + n_2$ and

$$C_i = \begin{cases} C^1_i & \mathit{if}\ 1 \le i \le n_1, \\ D_{i-n_1} & \mathit{if}\ n_1 + 1 \le i \le n_2, \end{cases} \qquad i = 1, \dots, n.$$

Hence, $C_1, \dots, C_n$, $n = n_1 + n_2 \ge 1 + 1 \ge 1$, is a deduction from S by basic order hyperresolution with associated $\mathcal{L}_n = \mathcal{L}_{n_1+n_2}$, $S_n = S_{n_1+n_2}$, $S_n = S_{n_1+n_2} \subseteq \mathit{GOrdCl}_{\mathcal{L}_{n_1+n_2}} = \mathit{GOrdCl}_{\mathcal{L}_n}$, such that $a \in \mathit{atoms}(S_{n_1+n_2}) \cup \mathit{qatoms}(S_{n_1+n_2}) = \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n)$, $b \in \mathit{atoms}(S^2_{n_2}) \cup \mathit{qatoms}(S^2_{n_2}) \subseteq \mathit{atoms}(S_{n_1+n_2}) \cup \mathit{qatoms}(S_{n_1+n_2}) = \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n)$; (97) holds.

For all $\emptyset \neq A \subseteq_{\mathcal{F}} atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$, there exist a deduction $C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq GOrdCl_{\mathcal{L}_n}$, such that $A \subseteq atoms(S_n) \cup qatoms(S_n)$. (99)

We proceed by induction on $1 \leq \|A\| < \omega$.

Case 1 (the base case): $\|A\| = 1$. Then there exist $\{a\} = A$, $a \in A \subseteq atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$, $C_n \in clo^{\mathcal{BH}}(S)$, $a \in atoms(C_n) \cup qatoms(C_n)$, a deduction $C_1, \dots, C_n$, $n \geq 1$, of C_n from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $C_n \in S_n \subseteq GOrdCl_{\mathcal{L}_n}$, and $A = \{a\} \subseteq atoms(C_n) \cup qatoms(C_n) \subseteq atoms(S_n) \cup qatoms(S_n)$; (99) holds.

Case 2 (the induction case): $1 < \|A\| < \omega$. Then there exist $A' \cup \{a\} = A$, $a \notin A'$, $1 \leq \|A'\| = \|A\| - 1 < \|A\| < \omega$, $\emptyset \neq A' \subseteq A \subseteq_{\mathcal{F}} atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$, $a \in A \subseteq atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$, $C_2 \in clo^{\mathcal{BH}}(S)$, $a \in atoms(C_2) \cup qatoms(C_2)$, a deduction $C_1^2, \dots, C_{n_2}^2$, $C_2 = C_{n_2}^2$, $n_2 \geq 1$, of C_2 from S by basic order hyperresolution, associated $\mathcal{L}_\sigma^2$, S_σ^2, $S_\sigma^2 \subseteq GOrdCl_{\mathcal{L}_\sigma^2}$, $\sigma = 0, \dots, n_2$, and $C_2 = C_{n_2}^2 \in S_{n_2}^2 \subseteq GOrdCl_{\mathcal{L}_{n_2}^2}$, $a \in atoms(C_2) \cup qatoms(C_2) \subseteq atoms(S_{n_2}^2) \cup qatoms(S_{n_2}^2)$. We get by induction hypothesis for A' that there exist a deduction $C_1^1, \dots, C_{n_1}^1$, $n_1 \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_{n_1}^1$, $S_{n_1}^1$, $S_{n_1}^1 \subseteq GOrdCl_{\mathcal{L}_{n_1}^1}$, and $A' \subseteq atoms(S_{n_1}^1) \cup qatoms(S_{n_1}^1)$. Then, by (98) for A', n_2, there exist a deduction $\mathcal{D}_{n_2} = C_1^1, \dots, C_{n_1}^1, D_1, \dots, D_{n_2}$ of D_{n_2} from S by basic order hyperresolution, associated $\mathcal{L}_{n_1+n_2}$, $S_{n_1+n_2}$, $D_{n_2} \in S_{n_1+n_2} \subseteq GOrdCl_{\mathcal{L}_{n_1+n_2}}$, such that $A' \subseteq atoms(S_{n_1+n_2}) \cup qatoms(S_{n_1+n_2})$, $\mathcal{L}_{n_1+n_2}$ is an expansion of $\mathcal{L}_{n_2}^2$, $atoms(S_{n_2}^2) \cup qatoms(S_{n_2}^2) \subseteq atoms(S_{n_1+n_2}) \cup qatoms(S_{n_1+n_2})$. We put $n = n_1 + n_2$ and

$$C_i = \begin{cases} C_i^1 & \textit{if } 1 \leq i \leq n_1, \\ D_{i-n_1} & \textit{if } n_1 + 1 \leq i \leq n_2, \end{cases} \qquad i = 1, \dots, n.$$

Hence, $C_1, \dots, C_n$, $n = n_1 + n_2 \geq 1 + 1 \geq 1$, is a deduction from S by basic order hyperresolution with associated $\mathcal{L}_n = \mathcal{L}_{n_1+n_2}$, $S_n = S_{n_1+n_2}$, $S_n = S_{n_1+n_2} \subseteq GOrdCl_{\mathcal{L}_{n_1+n_2}} = GOrdCl_{\mathcal{L}_n}$, such that $A' \subseteq atoms(S_{n_1+n_2}) \cup qatoms(S_{n_1+n_2}) = atoms(S_n) \cup qatoms(S_n)$, $a \in atoms(S_{n_2}^2) \cup qatoms(S_{n_2}^2) \subseteq atoms(S_{n_1+n_2}) \cup qatoms(S_{n_1+n_2}) = atoms(S_n) \cup qatoms(S_n)$, $A = A' \cup \{a\} \subseteq atoms(S_n) \cup qatoms(S_n)$; (99) holds.

So, in both Cases 1 and 2, (99) holds. The induction is completed. Thus, (99) holds.

For all $a \in tcons(S) \cap \overline{C}_{\mathcal{L}}$, $b \in \mathcal{B} - tcons(S)$, either $a \lhd b$ or $a \triangleq b$ or $b \lhd a$. (100)

Let $a \in tcons(S) \cap \overline{C}_{\mathcal{L}} \subseteq tcons(S) \subseteq \mathcal{B}$, $b \in \mathcal{B} - tcons(S) \subseteq \mathcal{B}$. Then $a, b \in \mathcal{B} \subseteq atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$, by (97) for a, b, there exist a deduction $\mathcal{D} = C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq GOrdCl_{\mathcal{L}_n}$, and $a, b \in atoms(S_n) \cup qatoms(S_n)$; $a \in atoms(S_n)$, $S_n \subseteq clo^{\mathcal{BH}}(S)$, $b \notin tcons(S) \overset{(96)}{=\!=} tcons(clo^{\mathcal{BH}}(S)) \supseteq tcons(S_n)$, $b \in (atoms(S_n) \cup qatoms(S_n)) - tcons(S_n) = (atoms(S_n) - tcons(S_n)) \cup qatoms(S_n) = (atoms(S_n) - Tcons_{\mathcal{L}}) \cup qatoms(S_n)$. We distinguish two cases for b.

Case 1: $b \in atoms(S_n) - Tcons_{\mathcal{L}}$. We get two cases for $qatoms(S)$.

Case 1.1: $qatoms(S) = \emptyset$. Then $a, b \in atoms(S_n) \subseteq GAtom_{\mathcal{L}_n}$, $a \in \overline{C}_{\mathcal{L}}$, $b \notin Tcons_{\mathcal{L}}$, using Rule (43) with respect to $\mathcal{L}_n$, S_n, we derive $a \prec b \vee a \eqcirc b \vee b \prec a \in GOrdCl_{\mathcal{L}_n}$.

Case 1.2: $qatoms(S) \neq \emptyset$. Then $a, b \in atoms(S_n) \subseteq GAtom_{\mathcal{L}_n}$, $a \in \overline{C}_{\mathcal{L}}$, $b \notin Tcons_{\mathcal{L}}$, $a, b \in atoms(S_n) - \{0, 1\}$, $\{a, b\} \not\subseteq Tcons_{\mathcal{L}}$, using Rule (44) with respect to $\mathcal{L}_n$, S_n, we derive $a \prec b \vee a \eqcirc b \vee b \prec a \in GOrdCl_{\mathcal{L}_n}$.

So, in both Cases 1.1 and 1.2, we derive $a \prec b \vee a \eqcirc b \vee b \prec a \in GOrdCl_{\mathcal{L}_n}$; we derive $a \prec b \vee a \eqcirc b \vee b \prec a \in GOrdCl_{\mathcal{L}_n}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n$, $C_{n+1} = a \prec b \vee a \eqcirc b \vee b \prec a \in GOrdCl_{\mathcal{L}_n} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq GOrdCl_{\mathcal{L}_n} = GOrdCl_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Hence, $C_{n+1} \in clo^{\mathcal{BH}}(S)$, $a \prec b \in \mathbb{S}$ or $a \eqcirc b \in \mathbb{S}$ or $b \prec a \in \mathbb{S}$; we have that there does not exist a contradiction of $\mathbb{S}$; either $a \prec b \in \mathbb{S}$ or $a \eqcirc b \in \mathbb{S}$ or $b \prec a \in \mathbb{S}$, either $a \lhd b$ or $a \triangleq b$ or $b \lhd a$; (100) holds.

Case 2: $b \in qatoms(S_n)$. We get two cases for b.

Case 2.1: $b = \forall x\, c$. Then $b \in qatoms^{\forall}(S_n) \subseteq QAtom_{\mathcal{L}_n}$, $a \in atoms(S_n) \subseteq GAtom_{\mathcal{L}_n}$, there exists $\tilde{w} \in \tilde{\mathbb{W}} - Func_{\mathcal{L}_n}$ and $ar(\tilde{w}) = |freetermseq(\forall x\, c), freetermseq(a)|$. We put $\gamma = x/\tilde{w}(freetermseq(\forall x\, c), freetermseq(a)) \in Subst_{\mathcal{L}_n \cup \{\tilde{w}\}}$, $dom(\gamma) = \{x\} = vars(c)$. Hence, using Rule (47) with respect to $\mathcal{L}_n$, S_n, we derive $c\gamma \prec a \vee a \eqcirc \forall x\, c \vee a \prec \forall x\, c \in GOrdCl_{\mathcal{L}_n \cup \{\tilde{w}\}}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{\tilde{w}\}$, $\gamma \in Subst_{\mathcal{L}_n \cup \{\tilde{w}\}} = Subst_{\mathcal{L}_{n+1}}$, $\tilde{w}(freetermseq(\forall x\, c), freetermseq(a)) \in GTerm_{\mathcal{L}_n \cup \{\tilde{w}\}} = GTerm_{\mathcal{L}_{n+1}}$, $C_{n+1} = c\gamma \prec a \vee a \eqcirc \forall x\, c \vee a \prec \forall x\, c \in GOrdCl_{\mathcal{L}_n \cup \{\tilde{w}\}} = GOrdCl_{\mathcal{L}_{n+1}} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq GOrdCl_{\mathcal{L}_n} \cup GOrdCl_{\mathcal{L}_{n+1}} = GOrdCl_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $\forall x\, c \in qatoms^{\forall}(C_{n+1}) \subseteq qatoms^{\forall}(S_{n+1}) \subseteq QAtom_{\mathcal{L}_{n+1}}$, using Rule (45) with respect to $\mathcal{L}_{n+1}$, S_{n+1}, we derive $\forall x\, c \prec c\gamma \vee \forall x\, c \eqcirc c\gamma \in GOrdCl_{\mathcal{L}_{n+1}}$. We put $\mathcal{L}_{n+2} = \mathcal{L}_{n+1}$, $C_{n+2} = \forall x\, c \prec c\gamma \vee \forall x\, c \eqcirc c\gamma \in GOrdCl_{\mathcal{L}_{n+1}} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}'' = \mathcal{D}', C_{n+2} = \mathcal{D}, C_{n+1}, C_{n+2}$, $S_{n+2} = S_{n+1} \cup \{C_{n+2}\} \subseteq GOrdCl_{\mathcal{L}_{n+1}} = GOrdCl_{\mathcal{L}_{n+2}}$; $\mathcal{D}''$ is a deduction of C_{n+2} from S by basic order hyperresolution. Hence, $C_{n+1}, C_{n+2} \in clo^{\mathcal{BH}}(S)$, $c\gamma \prec a \in \mathbb{S}$ or $a \eqcirc \forall x\, c \in \mathbb{S}$ or $a \prec \forall x\, c \in \mathbb{S}$, $\forall x\, c \prec c\gamma \in \mathbb{S}$ or $\forall x\, c \eqcirc c\gamma \in \mathbb{S}$; we have that there does not exist a contradiction of $\mathbb{S}$; either $\forall x\, c \prec c\gamma, c\gamma \prec a \in \mathbb{S}$ or $\forall x\, c \eqcirc c\gamma, c\gamma \prec a \in \mathbb{S}$ or $a \eqcirc \forall x\, c, \forall x\, c \prec c\gamma \in \mathbb{S}$ or $a \eqcirc \forall x\, c, \forall x\, c \eqcirc c\gamma \in \mathbb{S}$ or $a \prec \forall x\, c, \forall x\, c \prec c\gamma \in \mathbb{S}$ or $a \prec \forall x\, c, \forall x\, c \eqcirc c\gamma \in \mathbb{S}$, either $a \lhd \forall x\, c$ or $a \triangleq \forall x\, c$ or $\forall x\, c \lhd a$; (100) holds.

Case 2.2: $b = \exists x\, c$. Then $b \in qatoms^{\exists}(S_n) \subseteq QAtom_{\mathcal{L}_n}$, $a \in atoms(S_n) \subseteq GAtom_{\mathcal{L}_n}$, there exists $\tilde{w} \in \tilde{\mathbb{W}} - Func_{\mathcal{L}_n}$ and $ar(\tilde{w}) = |freetermseq(\exists x\, c), freetermseq(a)|$. We put $\gamma = x/\tilde{w}(freetermseq(\exists x\, c), freetermseq(a)) \in Subst_{\mathcal{L}_n \cup \{\tilde{w}\}}$, $dom(\gamma) = \{x\} = vars(c)$. Hence, using Rule (48) with respect to $\mathcal{L}_n$, S_n, we derive $a \prec c\gamma \vee \exists x\, c \eqcirc a \vee \exists x\, c \prec a \in GOrdCl_{\mathcal{L}_n \cup \{\tilde{w}\}}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{\tilde{w}\}$, $\gamma \in Subst_{\mathcal{L}_n \cup \{\tilde{w}\}} = Subst_{\mathcal{L}_{n+1}}$, $\tilde{w}(freetermseq(\exists x\, c), freetermseq(a)) \in GTerm_{\mathcal{L}_n \cup \{\tilde{w}\}} = GTerm_{\mathcal{L}_{n+1}}$, $C_{n+1} = a \prec c\gamma \vee \exists x\, c \eqcirc a \vee \exists x\, c \prec a \in GOrdCl_{\mathcal{L}_n \cup \{\tilde{w}\}} =$

$GOrdCl_{\mathcal{L}_{n+1}} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq GOrdCl_{\mathcal{L}_n} \cup GOrdCl_{\mathcal{L}_{n+1}} = GOrdCl_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $\exists x\, c \in qatoms^{\exists}(C_{n+1}) \subseteq qatoms^{\exists}(S_{n+1}) \subseteq QAtom_{\mathcal{L}_{n+1}}$, using Rule (46) with respect to $\mathcal{L}_{n+1}$, S_{n+1}, we derive $c\gamma \prec \exists x\, c \vee c\gamma \eqcirc \exists x\, c \in GOrdCl_{\mathcal{L}_{n+1}}$. We put $\mathcal{L}_{n+2} = \mathcal{L}_{n+1}$, $C_{n+2} = c\gamma \prec \exists x\, c \vee c\gamma \eqcirc \exists x\, c \in GOrdCl_{\mathcal{L}_{n+1}} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $\mathcal{D}'' = \mathcal{D}', C_{n+2} = \mathcal{D}, C_{n+1}, C_{n+2}$, $S_{n+2} = S_{n+1} \cup \{C_{n+2}\} \subseteq GOrdCl_{\mathcal{L}_{n+1}} = GOrdCl_{\mathcal{L}_{n+2}}$; $\mathcal{D}''$ is a deduction of C_{n+2} from S by basic order hyperresolution. Hence, $C_{n+1}, C_{n+2} \in clo^{\mathcal{BH}}(S)$, $a \prec c\gamma \in \mathbb{S}$ or $\exists x\, c \eqcirc a \in \mathbb{S}$ or $\exists x\, c \prec a \in \mathbb{S}$, $c\gamma \prec \exists x\, c \in \mathbb{S}$ or $c\gamma \eqcirc \exists x\, c \in \mathbb{S}$; we have that there does not exist a contradiction of $\mathbb{S}$; either $a \prec c\gamma, c\gamma \prec \exists x\, c \in \mathbb{S}$ or $a \prec c\gamma, c\gamma \eqcirc \exists x\, c \in \mathbb{S}$ or $c\gamma \prec \exists x\, c, \exists x\, c \eqcirc a \in \mathbb{S}$ or $c\gamma \eqcirc \exists x\, c, \exists x\, c \eqcirc a \in \mathbb{S}$ or $c\gamma \prec \exists x\, c, \exists x\, c \prec a \in \mathbb{S}$ or $c\gamma \eqcirc \exists x\, c, \exists x\, c \prec a \in \mathbb{S}$, either $a \lhd \exists x\, c$ or $a \triangleq \exists x\, c$ or $\exists x\, c \lhd a$; (100) holds.

So, in both Cases 2.1 and 2.2, (100) holds; (100) holds. Thus, in both Cases 1 and 2, (100) holds; (100) holds.

Let $qatoms(S) \neq \emptyset$. For all $a, b \in \mathcal{B} - \{0, 1\}$, either $a \lhd b$ or ($a = b$ or $a \triangleq b$) or $b \lhd a$. (101)

Let $a, b \in \mathcal{B} - \{0, 1\}$. Then $a, b \in \mathcal{B} - \{0, 1\} \subseteq \mathcal{B} \subseteq atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$, by (97) for a, b, there exist a deduction $\mathcal{D} = C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq GOrdCl_{\mathcal{L}_n}$, and $a, b \in atoms(S_n) \cup qatoms(S_n)$; $a, b \notin \{0, 1\}$, $a, b \in (atoms(S_n) \cup qatoms(S_n)) - \{0, 1\} = (atoms(S_n) - \{0, 1\}) \cup qatoms(S_n)$. We distinguish three cases for a, b.

Case 1: $a, b \in atoms(S_n) - \{0, 1\}$. We get two cases for $\{a, b\}$.

Case 1.1: $\{a, b\} \subseteq Tcons_{\mathcal{L}}$. Then $\{a, b\} \subseteq \mathcal{B} \cap Tcons_{\mathcal{L}} = atoms(\mathbb{S}) \cap Tcons_{\mathcal{L}} \subseteq atoms(clo^{\mathcal{BH}}(S)) \cap Tcons_{\mathcal{L}} \subseteq tcons(clo^{\mathcal{BH}}(S)) \overset{(96)}{=} tcons(S)$. We get two cases for a, b.

Case 1.1.1: $a = b$. Then $a \underset{(93)}{\ntriangleleft} a$; (101) holds.

Case 1.1.2: $a \neq b$. Then either $a \prec b \in ordtcons(S) \subseteq \mathbb{S}$ or $b \prec a \in ordtcons(S) \subseteq \mathbb{S}$, either $a \lhd b$ or $b \lhd a$; (101) holds.

Case 1.2: $\{a, b\} \nsubseteq Tcons_{\mathcal{L}}$. Then $a, b \in atoms(S_n) - \{0, 1\} \subseteq GAtom_{\mathcal{L}_n}$, using Rule (44) with respect to $\mathcal{L}_n$, S_n, we derive $a \prec b \vee a \eqcirc b \vee b \prec a \in GOrdCl_{\mathcal{L}_n}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n$, $C_{n+1} = a \prec b \vee a \eqcirc b \vee b \prec a \in GOrdCl_{\mathcal{L}_n} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq GOrdCl_{\mathcal{L}_n} = GOrdCl_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Hence, $C_{n+1} \in clo^{\mathcal{BH}}(S)$, $a \prec b \in \mathbb{S}$ or $a \eqcirc b \in \mathbb{S}$ or $b \prec a \in \mathbb{S}$; we have that there does not exist a contradiction of $\mathbb{S}$; either $a \prec b \in \mathbb{S}$ or $a \eqcirc b \in \mathbb{S}$ or $b \prec a \in \mathbb{S}$, either $a \lhd b$ or $a \triangleq b$ or $b \lhd a$; (101) holds.

So, in both Cases 1.1 and 1.2, (101) holds; (101) holds.

Case 2: $a \in qatoms(S_n)$, $b \in (atoms(S_n) - \{0, 1\}) \cup qatoms(S_n)$. We get two cases for a.

Case 2.1: $a = \forall x\, c$. Then $a \in qatoms^{\forall}(S_n) \subseteq QAtom_{\mathcal{L}_n}$, $b \in atoms(S_n) \cup qatoms(S_n) \subseteq GAtom_{\mathcal{L}_n} \cup QAtom_{\mathcal{L}_n}$, there exists $\tilde{w} \in \tilde{\mathbb{W}} - Func_{\mathcal{L}_n}$ and $ar(\tilde{w}) = |freetermseq(\forall x\, c), freetermseq(b)|$. We put $\gamma = x/\tilde{w}(freetermseq(\forall x\, c), freetermseq(b)) \in Subst_{\mathcal{L}_n \cup \{\tilde{w}\}}$, $dom(\gamma) = \{x\} = vars(c)$. Hence, using Rule (47) with respect to $\mathcal{L}_n$, S_n, we derive $c\gamma \prec b \vee b \eqcirc \forall x\, c \vee b \prec \forall x\, c \in GOrdCl_{\mathcal{L}_n \cup \{\tilde{w}\}}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{\tilde{w}\}$, $\gamma \in Subst_{\mathcal{L}_n \cup \{\tilde{w}\}} = Subst_{\mathcal{L}_{n+1}}$, $\tilde{w}(freetermseq(\forall x\, c), freetermseq(b)) \in GTerm_{\mathcal{L}_n \cup \{\tilde{w}\}} = GTerm_{\mathcal{L}_{n+1}}$, $C_{n+1} = c\gamma \prec b \vee b \eqcirc \forall x\, c \vee b \prec \forall x\, c \in GOrdCl_{\mathcal{L}_n \cup \{\tilde{w}\}} = GOrdCl_{\mathcal{L}_{n+1}} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq GOrdCl_{\mathcal{L}_n} \cup GOrdCl_{\mathcal{L}_{n+1}} = GOrdCl_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $\forall x\, c \in qatoms^{\forall}(C_{n+1}) \subseteq qatoms^{\forall}(S_{n+1}) \subseteq QAtom_{\mathcal{L}_{n+1}}$, using Rule (45) with respect to $\mathcal{L}_{n+1}$, S_{n+1}, we derive $\forall x\, c \prec c\gamma \vee \forall x\, c \eqcirc c\gamma \in GOrdCl_{\mathcal{L}_{n+1}}$. We put $\mathcal{L}_{n+2} = \mathcal{L}_{n+1}$, $C_{n+2} = \forall x\, c \prec c\gamma \vee \forall x\, c \eqcirc c\gamma \in GOrdCl_{\mathcal{L}_{n+1}} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $\mathcal{D}'' = \mathcal{D}', C_{n+2} = \mathcal{D}, C_{n+1}, C_{n+2}$, $S_{n+2} = S_{n+1} \cup \{C_{n+2}\} \subseteq GOrdCl_{\mathcal{L}_{n+1}} = GOrdCl_{\mathcal{L}_{n+2}}$; $\mathcal{D}''$ is a deduction of C_{n+2} from S by basic order hyperresolution. Hence, $C_{n+1}, C_{n+2} \in clo^{\mathcal{BH}}(S)$, $c\gamma \prec b \in \mathbb{S}$ or $b \eqcirc \forall x\, c \in \mathbb{S}$ or $b \prec \forall x\, c \in \mathbb{S}$, $\forall x\, c \prec c\gamma \in \mathbb{S}$ or $\forall x\, c \eqcirc c\gamma \in \mathbb{S}$; we have that there does not exist a contradiction of $\mathbb{S}$; either $\forall x\, c \prec c\gamma, c\gamma \prec b \in \mathbb{S}$ or $\forall x\, c \eqcirc c\gamma, c\gamma \prec b \in \mathbb{S}$ or $b \eqcirc \forall x\, c, \forall x\, c \prec c\gamma \in \mathbb{S}$ or $b \eqcirc \forall x\, c, \forall x\, c \eqcirc c\gamma \in \mathbb{S}$ or $b \prec \forall x\, c, \forall x\, c \prec c\gamma \in \mathbb{S}$ or $b \prec \forall x\, c, \forall x\, c \eqcirc c\gamma \in \mathbb{S}$, either $\forall x\, c \lhd b$ or $\forall x\, c \triangleq b$ or $b \lhd \forall x\, c$; (101) holds.

Case 2.2: $a = \exists x\, c$. Then $a \in qatoms^{\exists}(S_n) \subseteq QAtom_{\mathcal{L}_n}$, $b \in atoms(S_n) \cup qatoms(S_n) \subseteq GAtom_{\mathcal{L}_n} \cup QAtom_{\mathcal{L}_n}$, there exists $\tilde{w} \in \tilde{\mathbb{W}} - Func_{\mathcal{L}_n}$ and $ar(\tilde{w}) = |freetermseq(\exists x\, c), freetermseq(b)|$. We put $\gamma = x/\tilde{w}(freetermseq(\exists x\, c), freetermseq(b)) \in Subst_{\mathcal{L}_n \cup \{\tilde{w}\}}$, $dom(\gamma) = \{x\} = vars(c)$. Hence, using Rule (48) with respect to $\mathcal{L}_n$, S_n, we derive $b \prec c\gamma \vee \exists x\, c \eqcirc b \vee \exists x\, c \prec b \in GOrdCl_{\mathcal{L}_n \cup \{\tilde{w}\}}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{\tilde{w}\}$, $\gamma \in Subst_{\mathcal{L}_n \cup \{\tilde{w}\}} = Subst_{\mathcal{L}_{n+1}}$, $\tilde{w}(freetermseq(\exists x\, c), freetermseq(b)) \in GTerm_{\mathcal{L}_n \cup \{\tilde{w}\}} = GTerm_{\mathcal{L}_{n+1}}$, $C_{n+1} = b \prec c\gamma \vee \exists x\, c \eqcirc b \vee \exists x\, c \prec b \in GOrdCl_{\mathcal{L}_n \cup \{\tilde{w}\}} = GOrdCl_{\mathcal{L}_{n+1}} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq GOrdCl_{\mathcal{L}_n} \cup GOrdCl_{\mathcal{L}_{n+1}} = GOrdCl_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $\exists x\, c \in qatoms^{\exists}(C_{n+1}) \subseteq qatoms^{\exists}(S_{n+1}) \subseteq QAtom_{\mathcal{L}_{n+1}}$, using Rule (46) with respect to $\mathcal{L}_{n+1}$, S_{n+1}, we derive $c\gamma \prec \exists x\, c \vee c\gamma \eqcirc \exists x\, c \in GOrdCl_{\mathcal{L}_{n+1}}$. We put $\mathcal{L}_{n+2} = \mathcal{L}_{n+1}$, $C_{n+2} = c\gamma \prec \exists x\, c \vee c\gamma \eqcirc \exists x\, c \in GOrdCl_{\mathcal{L}_{n+1}} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $\mathcal{D}'' = \mathcal{D}', C_{n+2} = \mathcal{D}, C_{n+1}, C_{n+2}$, $S_{n+2} = S_{n+1} \cup \{C_{n+2}\} \subseteq GOrdCl_{\mathcal{L}_{n+1}} = GOrdCl_{\mathcal{L}_{n+2}}$; $\mathcal{D}''$ is a deduction of C_{n+2} from S by basic order hyperresolution. Hence, $C_{n+1}, C_{n+2} \in clo^{\mathcal{BH}}(S)$, $b \prec c\gamma \in \mathbb{S}$ or $\exists x\, c \eqcirc b \in \mathbb{S}$ or $\exists x\, c \prec b \in \mathbb{S}$, $c\gamma \prec \exists x\, c \in \mathbb{S}$ or $c\gamma \eqcirc \exists x\, c \in \mathbb{S}$; we have that there does not exist a contradiction of $\mathbb{S}$; either $b \prec c\gamma, c\gamma \prec \exists x\, c \in \mathbb{S}$ or $b \prec c\gamma, c\gamma \eqcirc \exists x\, c \in \mathbb{S}$ or $c\gamma \prec \exists x\, c, \exists x\, c \eqcirc b \in \mathbb{S}$ or $c\gamma \eqcirc \exists x\, c, \exists x\, c \eqcirc b \in \mathbb{S}$ or $c\gamma \prec \exists x\, c, \exists x\, c \prec b \in \mathbb{S}$ or $c\gamma \eqcirc \exists x\, c, \exists x\, c \prec b \in \mathbb{S}$, either $\exists x\, c \lhd b$ or $\exists x\, c \triangleq b$ or $b \lhd \exists x\, c$; (101) holds.

So, in both Cases 2.1 and 2.2, (101) holds; (101) holds.

Case 3: $a \in (atoms(S_n) - \{0, 1\}) \cup qatoms(S_n)$, $b \in qatoms(S_n)$. Analogous to Case 2; (101) holds.

Thus, in all Cases 1–3, (101) holds; (101) holds.

$$\begin{aligned}&\text{For all } \alpha \leq \gamma_2, \text{ for all } a, b \in \mathit{dom}(\mathcal{V}_\alpha), \\ &\quad \text{if } a \triangleq b, \text{ then } \mathcal{V}_\alpha(a) = \mathcal{V}_\alpha(b); \\ &\quad \text{if } a \lhd b, \text{ then } \mathcal{V}_\alpha(a) < \mathcal{V}_\alpha(b); \\ &\quad \text{if } \mathcal{V}_\alpha(a) = 0, \text{ then } a = \mathit{0} \text{ or } a \triangleq \mathit{0}; \\ &\quad \text{if } \mathcal{V}_\alpha(a) = 1, \text{ then } a = \mathit{1} \text{ or } a \triangleq \mathit{1}; \\ &\text{for all } \alpha < \gamma_2, \ \mathcal{V}_\alpha[\mathit{dom}(\mathcal{V}_\alpha)] \text{ is admissible with respect to suprema and infima.}\end{aligned} \tag{102}$$

We proceed by induction on $\alpha \leq \gamma_2$ using the assumption that $\overline{\mathit{tcons}(S)}$ is admissible with respect to suprema and infima.

Case 1 (the base case): $\alpha = 0$. Then $\mathcal{V}_0 = \{(\mathit{0}, 0), (\mathit{1}, 1)\} \cup \{(\bar{c}, c) \,|\, \bar{c} \in \mathit{tcons}(S) \cap \overline{C}_{\mathcal{L}}\}$. Let $a, b \in \mathit{dom}(\mathcal{V}_0) \overset{(94)}{=} \mathit{tcons}(S)$. Let $a \triangleq b$. We get two cases for a, b.

Case 1.1: $a = b$. Then $\mathcal{V}_0(a) = \mathcal{V}_0(b)$.

Case 1.2: $a \neq b$. Then either $a \prec b \in \mathit{ordtcons}(S) \subseteq \mathbb{S}$ or $b \prec a \in \mathit{ordtcons}(S) \subseteq \mathbb{S}$, either $a \lhd b$ or $b \lhd a$, for both the cases $a \lhd b$ and $b \lhd a$, $a \lhd a$; $a \lhd a$, which is a contradiction with $a \underset{(93)}{\ntriangleleft} a$.

So, in both Cases 1.1 and 1.2, the first point of (102) holds; the first point of (102) holds.

Let $a \lhd b$. We get two cases for a, b.

Case 1.3: $a = b$. Then $a \lhd a$, which is a contradiction with $a \underset{(93)}{\ntriangleleft} a$.

Case 1.4: $a \neq b$. We get two cases for a, b.

Case 1.4.1: $a \prec b \in \mathit{ordtcons}(S)$. Then $\mathcal{V}_0(a) < \mathcal{V}_0(b)$.

Case 1.4.2: $b \prec a \in \mathit{ordtcons}(S) \subseteq \mathbb{S}$. Then $b \lhd a$ and $a \lhd a$, which is a contradiction with $a \underset{(93)}{\ntriangleleft} a$.

So, in both Cases 1.3 and 1.4, the second point of (102) holds; the second point of (102) holds.

Hence, for all $a \in \mathit{dom}(\mathcal{V}_0)$, if $\mathcal{V}_0(a) = 0$, then $a = \mathit{0}$; if $\mathcal{V}_0(a) = 1$, then $a = \mathit{1}$; we have $\overline{\mathit{tcons}(S)}$ is admissible with respect to suprema and infima; $\mathcal{V}_0[\mathit{dom}(\mathcal{V}_0)] = \mathcal{V}_0[\mathit{tcons}(S)] = \overline{\mathit{tcons}(S)}$ is admissible with respect to suprema and infima; (102) holds.

Case 2 (the induction case): $0 < \alpha \leq \gamma_2$ is a successor ordinal. Then $0 \leq \alpha - 1 < \alpha \leq \gamma_2$, $\mathcal{V}_\alpha = \mathcal{V}_{\alpha-1} \cup \{(\delta_2(\alpha-1), \lambda_{\alpha-1})\}$, $\mathit{dom}(\mathcal{V}_\alpha) = \mathit{dom}(\mathcal{V}_{\alpha-1}) \cup \{\delta_2(\alpha-1)\}$, $\delta_2(\alpha-1) \not\in \mathit{tcons}(S) \cup \delta_2[\alpha-1] \overset{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha-1})$. At first, we prove the following statements:

$$\text{If } \mathbb{E}_{\alpha-1} \neq \emptyset, \text{ then there exists } a^* \in \mathit{dom}(\mathcal{V}_{\alpha-1}) \text{ such that } a^* \triangleq \delta_2(\alpha-1), \ \mathcal{V}_{\alpha-1}(a^*) \in \mathbb{E}_{\alpha-1}, \ \mathcal{V}_{\alpha-1}(a^*) = \mathcal{V}_\alpha(\delta_2(\alpha-1)). \tag{103}$$

Let $\mathbb{E}_{\alpha-1} \neq \emptyset$. Then $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigvee \mathbb{E}_{\alpha-1}$, there exists $a^* \in \mathit{dom}(\mathcal{V}_{\alpha-1})$ and $a^* \triangleq \delta_2(\alpha-1)$, $\mathcal{V}_{\alpha-1}(a^*) \in \mathbb{E}_{\alpha-1}$; for all $c \in \mathbb{E}_{\alpha-1}$, there exists $\varepsilon^* \in \mathit{dom}(\mathcal{V}_{\alpha-1})$ and $\varepsilon^* \triangleq \delta_2(\alpha-1)$, $\mathcal{V}_{\alpha-1}(\varepsilon^*) = c \in \mathbb{E}_{\alpha-1}$, $\varepsilon^* \triangleq a^*$, by induction hypothesis for $\alpha - 1$, $c = \mathcal{V}_{\alpha-1}(\varepsilon^*) = \mathcal{V}_{\alpha-1}(a^*)$; $\mathcal{V}_{\alpha-1}(a^*) = \bigvee \mathbb{E}_{\alpha-1} = \mathcal{V}_\alpha(\delta_2(\alpha-1))$; (103) holds.

$$\text{If } \mathbb{E}_{\alpha-1} = \emptyset, \text{ then } \bigvee \mathbb{D}_{\alpha-1} < \mathcal{V}_\alpha(\delta_2(\alpha-1)) < \bigwedge \mathbb{U}_{\alpha-1}. \tag{104}$$

Let $\mathbb{E}_{\alpha-1} = \emptyset$. Then $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \dfrac{\bigvee \mathbb{D}_{\alpha-1} + \bigwedge \mathbb{U}_{\alpha-1}}{2}$. We get four cases for $\mathbb{D}_{\alpha-1}$, $\mathbb{U}_{\alpha-1}$.

Case (a): $\mathbb{D}_{\alpha-1} = \emptyset$, $\mathbb{U}_{\alpha-1} = \emptyset$. Then $\bigvee \mathbb{D}_{\alpha-1} = 0 < \mathcal{V}_\alpha(\delta_2(\alpha-1)) = \dfrac{0+1}{2} = \dfrac{1}{2} < 1 = \bigwedge \mathbb{U}_{\alpha-1}$; (104) holds.

Case (b): $\mathbb{D}_{\alpha-1} = \emptyset$, $\mathbb{U}_{\alpha-1} \neq \emptyset$. Then $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \dfrac{0 + \bigwedge \mathbb{U}_{\alpha-1}}{2} = \dfrac{\bigwedge \mathbb{U}_{\alpha-1}}{2}$. We get two cases for $\bigwedge \mathbb{U}_{\alpha-1}$.

Case (b.1): $\bigwedge \mathbb{U}_{\alpha-1} = 0$. Then $\mathit{0} \in \mathit{tcons}(S) \subseteq \mathit{dom}(\mathcal{V}_{\alpha-1})$, $\emptyset \neq \{\mathcal{V}_{\alpha-1}(\mathit{0})\}, \mathbb{U}_{\alpha-1} \subseteq \mathcal{V}_{\alpha-1}[\mathit{dom}(\mathcal{V}_{\alpha-1})]$, $\bigvee\{\mathcal{V}_{\alpha-1}(\mathit{0})\} = \bigvee\{0\} = 0 = \bigwedge \mathbb{U}_{\alpha-1}$, by induction hypothesis for $\alpha - 1 < \alpha \leq \gamma_2$, $\mathcal{V}_{\alpha-1}[\mathit{dom}(\mathcal{V}_{\alpha-1})]$ is admissible with respect to suprema and infima, $0 = \bigwedge \mathbb{U}_{\alpha-1} \in \mathbb{U}_{\alpha-1}$; there exists $\varepsilon^* \in \mathit{dom}(\mathcal{V}_{\alpha-1})$ and $\delta_2(\alpha-1) \lhd \varepsilon^*$, $\mathcal{V}_{\alpha-1}(\varepsilon^*) = 0$, by induction hypothesis for $\alpha - 1$, $\varepsilon^* = \mathit{0}$ or $\varepsilon^* \triangleq \mathit{0}$, for both the cases $\varepsilon^* = \mathit{0}$ and $\varepsilon^* \triangleq \mathit{0}$, $\delta_2(\alpha-1) \lhd \mathit{0}$; $\delta_2(\alpha-1) \lhd \mathit{0}$, which is a contradiction with $\delta_2(\alpha-1) \underset{(93)}{\ntriangleleft} \mathit{0}$; (104) holds.

Case (b.2): $\bigwedge \mathbb{U}_{\alpha-1} > 0$. Then $\bigvee \mathbb{D}_{\alpha-1} = 0 < \mathcal{V}_\alpha(\delta_2(\alpha-1)) = \dfrac{\bigwedge \mathbb{U}_{\alpha-1}}{2} < \bigwedge \mathbb{U}_{\alpha-1}$; (104) holds.

Case (c): $\mathbb{D}_{\alpha-1} \neq \emptyset$, $\mathbb{U}_{\alpha-1} = \emptyset$. Then $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \dfrac{\bigvee \mathbb{D}_{\alpha-1} + 1}{2} = \dfrac{\bigvee \mathbb{D}_{\alpha-1}}{2} + \dfrac{1}{2}$. We get two cases for $\bigvee \mathbb{D}_{\alpha-1}$.

Case (c.1): $\bigvee \mathbb{D}_{\alpha-1} = 1$. Then $\mathit{1} \in tcons(S) \subseteq dom(\mathcal{V}_{\alpha-1})$, $\emptyset \neq \mathbb{D}_{\alpha-1}, \{\mathcal{V}_{\alpha-1}(\mathit{1})\} \subseteq \mathcal{V}_{\alpha-1}[dom(\mathcal{V}_{\alpha-1})]$, $\bigvee \mathbb{D}_{\alpha-1} = 1 = \bigwedge\{1\} = \bigwedge\{\mathcal{V}_{\alpha-1}(\mathit{1})\}$, by induction hypothesis for $\alpha - 1 < \alpha \leq \gamma_2$, $\mathcal{V}_{\alpha-1}[dom(\mathcal{V}_{\alpha-1})]$ is admissible with respect to suprema and infima, $1 = \bigvee \mathbb{D}_{\alpha-1} \in \mathbb{D}_{\alpha-1}$; there exists $\varepsilon^* \in dom(\mathcal{V}_{\alpha-1})$ and $\varepsilon^* \lhd \delta_2(\alpha-1)$, $\mathcal{V}_{\alpha-1}(\varepsilon^*) = 1$, by induction hypothesis for $\alpha - 1$, $\varepsilon^* = \mathit{1}$ or $\varepsilon^* \triangleq \mathit{1}$, for both the cases $\varepsilon^* = \mathit{1}$ and $\varepsilon^* \triangleq \mathit{1}$, $\mathit{1} \lhd \delta_2(\alpha-1)$; $\mathit{1} \lhd \delta_2(\alpha-1)$, which is a contradiction with $\mathit{1} \underset{(93)}{\not\lhd} \delta_2(\alpha-1)$; (104) holds.

Case (c.2): $\bigvee \mathbb{D}_{\alpha-1} < 1$. Then $\bigvee \mathbb{D}_{\alpha-1} < \mathcal{V}_\alpha(\delta_2(\alpha-1)) = \dfrac{\bigvee \mathbb{D}_{\alpha-1}}{2} + \dfrac{1}{2} < 1 = \bigwedge \mathbb{U}_{\alpha-1}$; (104) holds.

Case (d): $\mathbb{D}_{\alpha-1} \neq \emptyset$, $\mathbb{U}_{\alpha-1} \neq \emptyset$. Then, for all $d \in \mathbb{D}_{\alpha-1}$, $u \in \mathbb{U}_{\alpha-1}$, there exist $\varepsilon^*_d, \varepsilon^*_u \in dom(\mathcal{V}_{\alpha-1})$ and $\varepsilon^*_d \lhd \delta_2(\alpha-1)$, $\delta_2(\alpha-1) \lhd \varepsilon^*_u$, $\mathcal{V}_{\alpha-1}(\varepsilon^*_d) = d$, $\mathcal{V}_{\alpha-1}(\varepsilon^*_u) = u$, $\varepsilon^*_d \lhd \varepsilon^*_u$, by induction hypothesis for $\alpha - 1$, $d = \mathcal{V}_{\alpha-1}(\varepsilon^*_d) < \mathcal{V}_{\alpha-1}(\varepsilon^*_u) = u$; $\bigvee \mathbb{D}_{\alpha-1} \leq \bigwedge \mathbb{U}_{\alpha-1}$; $\emptyset \neq \mathbb{D}_{\alpha-1}, \mathbb{U}_{\alpha-1} \subseteq \mathcal{V}_{\alpha-1}[dom(\mathcal{V}_{\alpha-1})]$. We get two cases for $\bigvee \mathbb{D}_{\alpha-1}$, $\bigwedge \mathbb{U}_{\alpha-1}$.

Case (d.1): $\bigvee \mathbb{D}_{\alpha-1} = \bigwedge \mathbb{U}_{\alpha-1}$. Then, by induction hypothesis for $\alpha - 1 < \alpha \leq \gamma_2$, $\mathcal{V}_{\alpha-1}[dom(\mathcal{V}_{\alpha-1})]$ is admissible with respect to suprema and infima, $\bigvee \mathbb{D}_{\alpha-1} \in \mathbb{D}_{\alpha-1}$, $\bigwedge \mathbb{U}_{\alpha-1} \in \mathbb{U}_{\alpha-1}$; there exist $\varepsilon^*, \varepsilon^{**} \in dom(\mathcal{V}_{\alpha-1})$ and $\varepsilon^* \lhd \delta_2(\alpha-1)$, $\delta_2(\alpha-1) \lhd \varepsilon^{**}$, $\mathcal{V}_{\alpha-1}(\varepsilon^*) = \bigvee \mathbb{D}_{\alpha-1}$, $\mathcal{V}_{\alpha-1}(\varepsilon^{**}) = \bigwedge \mathbb{U}_{\alpha-1}$, $\varepsilon^* \lhd \varepsilon^{**}$, by induction hypothesis for $\alpha - 1$, $\bigvee \mathbb{D}_{\alpha-1} = \mathcal{V}_{\alpha-1}(\varepsilon^*) < \mathcal{V}_{\alpha-1}(\varepsilon^{**}) = \bigwedge \mathbb{U}_{\alpha-1}$, which is a contradiction; (104) holds.

Case (d.2): $\bigvee \mathbb{D}_{\alpha-1} < \bigwedge \mathbb{U}_{\alpha-1}$. Then $\bigvee \mathbb{D}_{\alpha-1} < \mathcal{V}_\alpha(\delta_2(\alpha-1)) = \dfrac{\bigvee \mathbb{D}_{\alpha-1} + \bigwedge \mathbb{U}_{\alpha-1}}{2} < \bigwedge \mathbb{U}_{\alpha-1}$; (104) holds.

So, in all Cases (a)–(d), (104) holds; (104) holds.

Let $a, b \in dom(\mathcal{V}_\alpha)$. We distinguish two cases for $\{a, b\}$.

Case 2.1: $\{a, b\} \subseteq dom(\mathcal{V}_{\alpha-1})$. Then, by induction hypothesis for $\alpha - 1$,

$$\begin{aligned}
&\text{if } a \triangleq b, \text{ then } \mathcal{V}_\alpha(a) = \mathcal{V}_{\alpha-1}(a) = \mathcal{V}_{\alpha-1}(b) = \mathcal{V}_\alpha(b); \\
&\text{if } a \lhd b, \text{ then } \mathcal{V}_\alpha(a) = \mathcal{V}_{\alpha-1}(a) < \mathcal{V}_{\alpha-1}(b) = \mathcal{V}_\alpha(b); \\
&\text{if } \mathcal{V}_\alpha(a) = \mathcal{V}_{\alpha-1}(a) = 0, \text{ then } a = \mathit{0} \text{ or } a \triangleq \mathit{0}; \\
&\text{if } \mathcal{V}_\alpha(a) = \mathcal{V}_{\alpha-1}(a) = 1, \text{ then } a = \mathit{1} \text{ or } a \triangleq \mathit{1};
\end{aligned}$$

the first four points of (102) hold.

Case 2.2: $\{a, b\} \not\subseteq dom(\mathcal{V}_{\alpha-1})$. We distinguish three cases for a, b.

Case 2.2.1: $a = \delta_2(\alpha-1)$, $b \in dom(\mathcal{V}_{\alpha-1})$. Then $b \neq \delta_2(\alpha-1)$. We get two cases for $\mathbb{E}_{\alpha-1}$.

Case 2.2.1.1: $\mathbb{E}_{\alpha-1} = \emptyset$. Then $b \not\triangleq \delta_2(\alpha-1)$, $0 \leq \bigvee \mathbb{D}_{\alpha-1} \underset{(104)}{<} \mathcal{V}_\alpha(\delta_2(\alpha-1)) \underset{(104)}{<} \bigwedge \mathbb{U}_{\alpha-1} \leq 1$; the third and fourth point of (102) holds trivially. We get two cases for b.

Case 2.2.1.1.1: $\delta_2(\alpha-1) \lhd b$. Then $\mathcal{V}_{\alpha-1}(b) \in \mathbb{U}_{\alpha-1}$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) \underset{(104)}{<} \bigwedge \mathbb{U}_{\alpha-1} \leq \mathcal{V}_{\alpha-1}(b) = \mathcal{V}_\alpha(b)$.

Case 2.2.1.1.2: $b \lhd \delta_2(\alpha-1)$. Then $\mathcal{V}_{\alpha-1}(b) \in \mathbb{D}_{\alpha-1}$, $\mathcal{V}_\alpha(b) = \mathcal{V}_{\alpha-1}(b) \leq \bigvee \mathbb{D}_{\alpha-1} \underset{(104)}{<} \mathcal{V}_\alpha(\delta_2(\alpha-1))$.

So, the first four points of (102) hold.

Case 2.2.1.2: $\mathbb{E}_{\alpha-1} \neq \emptyset$. Then, by (103), there exists $a^* \in dom(\mathcal{V}_{\alpha-1})$ and $a^* \triangleq \delta_2(\alpha-1)$, $\mathcal{V}_{\alpha-1}(a^*) = \mathcal{V}_\alpha(\delta_2(\alpha-1))$; if $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \mathcal{V}_{\alpha-1}(a^*) = 0$, then, by induction hypothesis for $\alpha - 1$, $a^* = \mathit{0}$ or $a^* \triangleq \mathit{0}$, for both the cases $a^* = \mathit{0}$ and $a^* \triangleq \mathit{0}$, $\delta_2(\alpha-1) \triangleq \mathit{0}$; $\delta_2(\alpha-1) \triangleq \mathit{0}$; if $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \mathcal{V}_{\alpha-1}(a^*) = 1$, then, by induction hypothesis for $\alpha - 1$, $a^* = \mathit{1}$ or $a^* \triangleq \mathit{1}$, for both the cases $a^* = \mathit{1}$ and $a^* \triangleq \mathit{1}$, $\delta_2(\alpha-1) \triangleq \mathit{1}$; $\delta_2(\alpha-1) \triangleq \mathit{1}$; the third and fourth point of (102) holds. We get three cases for b.

Case 2.2.1.2.1: $\delta_2(\alpha-1) \triangleq b$. Then $a^* \triangleq b$, by induction hypothesis for $\alpha - 1$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \mathcal{V}_{\alpha-1}(a^*) = \mathcal{V}_{\alpha-1}(b) = \mathcal{V}_\alpha(b)$.

Case 2.2.1.2.2: $\delta_2(\alpha-1) \lhd b$. Then $a^* \lhd b$, by induction hypothesis for $\alpha - 1$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \mathcal{V}_{\alpha-1}(a^*) < \mathcal{V}_{\alpha-1}(b) = \mathcal{V}_\alpha(b)$.

Case 2.2.1.2.3: $b \lhd \delta_2(\alpha-1)$. Then $b \lhd a^*$, by induction hypothesis for $\alpha - 1$, $\mathcal{V}_\alpha(b) = \mathcal{V}_{\alpha-1}(b) < \mathcal{V}_{\alpha-1}(a^*) = \mathcal{V}_\alpha(\delta_2(\alpha-1))$.

So, the first four points of (102) hold.

Case 2.2.2: $a \in dom(\mathcal{V}_{\alpha-1})$, $b = \delta_2(\alpha-1)$. Analogous to Case 2.2.1; the first four points of (102) hold.

Case 2.2.3: $a = b = \delta_2(\alpha-1)$. We get two cases for $\mathbb{E}_{\alpha-1}$.

Case 2.2.3.1: $\mathbb{E}_{\alpha-1} = \emptyset$. Then $0 \leq \bigvee \mathbb{D}_{\alpha-1} \underset{(104)}{<} \mathcal{V}_\alpha(\delta_2(\alpha-1)) \underset{(104)}{<} \bigwedge \mathbb{U}_{\alpha-1} \leq 1$; the third and fourth point of (102) holds trivially. We get two cases for $\delta_2(\alpha-1)$.

Case 2.2.3.1.1: $\delta_2(\alpha-1) \triangleq \delta_2(\alpha-1)$. Then $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \mathcal{V}_\alpha(\delta_2(\alpha-1))$.

Case 2.2.3.1.2: $\delta_2(\alpha-1) \lhd \delta_2(\alpha-1)$. This is a contradiction with $\delta_2(\alpha-1) \underset{(93)}{\not\lhd} \delta_2(\alpha-1)$.

So, the first four points of (102) hold.

Case 2.2.3.2: $\mathbb{E}_{\alpha-1} \neq \emptyset$. Then, by (103), there exists $a^* \in dom(\mathcal{V}_{\alpha-1})$ and $a^* \triangleq \delta_2(\alpha-1)$, $\mathcal{V}_{\alpha-1}(a^*) = \mathcal{V}_\alpha(\delta_2(\alpha-1))$; if $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \mathcal{V}_{\alpha-1}(a^*) = 0$, then, by induction hypothesis for $\alpha - 1$, $a^* = \mathit{0}$ or $a^* \triangleq \mathit{0}$, for both the cases $a^* = \mathit{0}$ and $a^* \triangleq \mathit{0}$, $\delta_2(\alpha-1) \triangleq \mathit{0}$; $\delta_2(\alpha-1) \triangleq \mathit{0}$; if $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \mathcal{V}_{\alpha-1}(a^*) = 1$, then, by induction hypothesis for $\alpha - 1$, $a^* = \mathit{1}$ or $a^* \triangleq \mathit{1}$, for both the cases $a^* = \mathit{1}$ and $a^* \triangleq \mathit{1}$, $\delta_2(\alpha-1) \triangleq \mathit{1}$; $\delta_2(\alpha-1) \triangleq \mathit{1}$; the third and fourth point of (102) holds. We get two cases for $\delta_2(\alpha-1)$.

Case 2.2.3.2.1: $\delta_2(\alpha-1) \triangleq \delta_2(\alpha-1)$. The same as Case 2.2.3.1.1.

Case 2.2.3.2.2: $\delta_2(\alpha-1) \lhd \delta_2(\alpha-1)$. The same as Case 2.2.3.1.2.

So, the first four points of (102) hold.

So, in all Cases 2.2.1–2.2.3, the first four points of (102) hold; the first four points of (102) hold. Thus, in both Cases 2.1 and 2.2, the first four points of (102) hold; the first four points of (102) hold.

Case 2.3: It remains to prove the fifth point of (102). We distinguish two cases for $\mathbb{E}_{\alpha-1}$.

Case 2.3.1: $\mathbb{E}_{\alpha-1} = \emptyset$. Then $\bigvee \mathbb{D}_{\alpha-1} \underset{(104)}{<} \mathcal{V}_\alpha(\delta_2(\alpha-1)) \underset{(104)}{<} \bigwedge \mathbb{U}_{\alpha-1}$. Let $\emptyset \neq Y_1, Y_2 \subseteq \mathcal{V}_\alpha[\mathit{dom}(\mathcal{V}_\alpha)] = \mathcal{V}_\alpha[\mathit{dom}(\mathcal{V}_{\alpha-1}) \cup \{\delta_2(\alpha-1)\}] = \mathcal{V}_{\alpha-1}[\mathit{dom}(\mathcal{V}_{\alpha-1})] \cup \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\}$ and $\bigvee Y_1 = \bigwedge Y_2$. We get three cases for $\mathcal{V}_\alpha(\delta_2(\alpha-1))$, $\bigvee Y_1 = \bigwedge Y_2$.

Case 2.3.1.1: $\mathcal{V}_\alpha(\delta_2(\alpha-1)) < \bigvee Y_1 = \bigwedge Y_2$. Then $\mathcal{V}_\alpha(\delta_2(\alpha-1)) \notin Y_2$, $\emptyset \neq Y_1 \neq \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\}$, $\emptyset \neq Y_1 - \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\}, Y_2 \subseteq \mathcal{V}_{\alpha-1}[\mathit{dom}(\mathcal{V}_{\alpha-1})]$, $\bigvee(Y_1 - \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\}) = \bigvee Y_1 = \bigwedge Y_2$, by induction hypothesis for $\alpha-1 < \alpha \leq \gamma_2$, $\mathcal{V}_{\alpha-1}[\mathit{dom}(\mathcal{V}_{\alpha-1})]$ is admissible with respect to suprema and infima, $\bigvee Y_1 = \bigvee(Y_1 - \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\}) \in Y_1 - \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\} \subseteq Y_1$, $\bigwedge Y_2 \in Y_2$; $\mathcal{V}_\alpha[\mathit{dom}(\mathcal{V}_\alpha)]$ is admissible with respect to suprema and infima; the fifth point of (102) holds.

Case 2.3.1.2: $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigvee Y_1 = \bigwedge Y_2$. Then there exist $Y_1^*, Y_2^* \subseteq \mathit{dom}(\mathcal{V}_\alpha) \overset{(94)}{=} \mathit{tcons}(S) \cup \delta_2[\alpha-1] \cup \{\delta_2(\alpha-1)\}$ and $\mathcal{V}_\alpha[Y_1^*] = Y_1$, $\mathcal{V}_\alpha[Y_2^*] = Y_2$, $\alpha-1 < \alpha \leq \gamma_2 \leq \omega$, $\delta_2[\alpha-1] \underset{(94)}{\subseteq_{\mathcal{F}}} \mathit{dom}(\mathcal{V}_{\alpha-1})$; $\mathit{tcons}(S)$, $\delta_2[\alpha-1]$, $\{\delta_2(\alpha-1)\}$ are pairwise disjoint; for both i, $Y_i^* \cap \delta_2[\alpha-1] \subseteq \delta_2[\alpha-1] \subseteq_{\mathcal{F}} \mathit{dom}(\mathcal{V}_{\alpha-1})$, $Y_i^* = (Y_i^* \cap \mathit{tcons}(S)) \cup (Y_i^* \cap \delta_2[\alpha-1]) \cup (Y_i^* \cap \{\delta_2(\alpha-1)\})$, $Y_i = \mathcal{V}_\alpha[Y_i^*] = \mathcal{V}_\alpha[(Y_i^* \cap \mathit{tcons}(S)) \cup (Y_i^* \cap \delta_2[\alpha-1]) \cup (Y_i^* \cap \{\delta_2(\alpha-1)\})] = \mathcal{V}_\alpha[Y_i^* \cap \mathit{tcons}(S)] \cup \mathcal{V}_\alpha[Y_i^* \cap \delta_2[\alpha-1]] \cup \mathcal{V}_\alpha[Y_i^* \cap \{\delta_2(\alpha-1)\}]$, $Y_i^* - \{\delta_2(\alpha-1)\} = ((Y_i^* \cap \mathit{tcons}(S)) \cup (Y_i^* \cap \delta_2[\alpha-1]) \cup (Y_i^* \cap \{\delta_2(\alpha-1)\})) - \{\delta_2(\alpha-1)\} = ((Y_i^* \cap \mathit{tcons}(S)) - \{\delta_2(\alpha-1)\}) \cup ((Y_i^* \cap \delta_2[\alpha-1]) - \{\delta_2(\alpha-1)\}) \cup ((Y_i^* \cap \{\delta_2(\alpha-1)\}) - \{\delta_2(\alpha-1)\}) = (Y_i^* \cap \mathit{tcons}(S)) \cup (Y_i^* \cap \delta_2[\alpha-1]) \subseteq \mathit{tcons}(S) \cup \delta_2[\alpha-1] \overset{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha-1})$, $\mathcal{V}_\alpha[Y_i^* - \{\delta_2(\alpha-1)\}] = \mathcal{V}_{\alpha-1}[Y_i^* - \{\delta_2(\alpha-1)\}] = \mathcal{V}_{\alpha-1}[(Y_i^* \cap \mathit{tcons}(S)) \cup (Y_i^* \cap \delta_2[\alpha-1])] = \mathcal{V}_{\alpha-1}[Y_i^* \cap \mathit{tcons}(S)] \cup \mathcal{V}_{\alpha-1}[Y_i^* \cap \delta_2[\alpha-1]]$, $\mathcal{V}_{\alpha-1}[Y_i^* \cap \delta_2[\alpha-1]] \subseteq_{\mathcal{F}} [0,1]$. We get two cases for $\mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)]$, $\mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]]$.

Case 2.3.1.2.1: $\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] < \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]]$. Then $\bigvee(\mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] \cup \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]]) = (\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)]) \vee (\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]]) = \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]]$, $0 \leq \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] < \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]]$, $\bigvee \emptyset = 0$, $\mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]] \neq \emptyset$, $Y_1^* \cap \delta_2[\alpha-1] \neq \emptyset$, there exists $y^* \in Y_1^* \cap \delta_2[\alpha-1]$ and $\mathcal{V}_{\alpha-1}(y^*) \in \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]] \subseteq \mathcal{V}_\alpha[Y_1^* - \{\delta_2(\alpha-1)\}] \subseteq \mathcal{V}_\alpha[Y_1^*] = Y_1$, $\mathcal{V}_{\alpha-1}(y^*) = \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]] \leq \bigvee Y_1 = \mathcal{V}_\alpha(\delta_2(\alpha-1))$. We get two cases for $\mathcal{V}_{\alpha-1}(y^*)$, $\mathcal{V}_\alpha(\delta_2(\alpha-1))$.

Case 2.3.1.2.1.1: $\mathcal{V}_{\alpha-1}(y^*) < \mathcal{V}_\alpha(\delta_2(\alpha-1))$. Then $\bigvee \mathcal{V}_\alpha[Y_1^* - \{\delta_2(\alpha-1)\}] = \bigvee(\mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] \cup \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]]) = \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]] = \mathcal{V}_{\alpha-1}(y^*) < \mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigvee Y_1 = \bigvee \mathcal{V}_\alpha[Y_1^*]$, $Y_1^* - \{\delta_2(\alpha-1)\} \subset Y_1^*$, $\delta_2(\alpha-1) \in Y_1^*$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) \in \mathcal{V}_\alpha[Y_1^*] = Y_1$.

Case 2.3.1.2.1.2: $\mathcal{V}_{\alpha-1}(y^*) = \mathcal{V}_\alpha(\delta_2(\alpha-1))$. Then $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \mathcal{V}_{\alpha-1}(y^*) \in Y_1$.

Case 2.3.1.2.2: $\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] \geq \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]]$. Then $\bigvee(\mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] \cup \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]]) = (\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)]) \vee (\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]]) = \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)]$, $\mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] \subseteq \mathcal{V}_\alpha[Y_1^* - \{\delta_2(\alpha-1)\}] \subseteq \mathcal{V}_\alpha[Y_1^*] = Y_1$, $\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] \leq \bigvee Y_1 = \mathcal{V}_\alpha(\delta_2(\alpha-1))$. We get two cases for $\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)]$, $\mathcal{V}_\alpha(\delta_2(\alpha-1))$.

Case 2.3.1.2.2.1: $\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] < \mathcal{V}_\alpha(\delta_2(\alpha-1))$. Then $\bigvee \mathcal{V}_\alpha[Y_1^* - \{\delta_2(\alpha-1)\}] = \bigvee(\mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] \cup \mathcal{V}_{\alpha-1}[Y_1^* \cap \delta_2[\alpha-1]]) = \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] < \mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigvee Y_1 = \bigvee \mathcal{V}_\alpha[Y_1^*]$, $Y_1^* - \{\delta_2(\alpha-1)\} \subset Y_1^*$, $\delta_2(\alpha-1) \in Y_1^*$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) \in \mathcal{V}_\alpha[Y_1^*] = Y_1$.

Case 2.3.1.2.2.2: $\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] = \mathcal{V}_\alpha(\delta_2(\alpha-1))$. We get two cases for $\mathit{1}$, $Y_1^* \cap \mathit{tcons}(S)$.

Case 2.3.1.2.2.2.1: $\mathit{1} \in Y_1^* \cap \mathit{tcons}(S)$. Then $\mathcal{V}_{\alpha-1}(\mathit{1}) \in \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] \subseteq \mathcal{V}_\alpha[Y_1^* - \{\delta_2(\alpha-1)\}] \subseteq Y_1$, $1 = \mathcal{V}_{\alpha-1}(\mathit{1}) = \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)]$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] = \mathcal{V}_{\alpha-1}(\mathit{1}) \in Y_1$.

Case 2.3.1.2.2.2.2: $\mathit{1} \notin Y_1^* \cap \mathit{tcons}(S)$. Then $\mathcal{V}_{\alpha-1}(\mathit{0}) = 0$, $Y_1^* \cap \mathit{tcons}(S) \subseteq ((Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}) \cup \{\mathit{0}\}$, $\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] \leq \bigvee \mathcal{V}_{\alpha-1}[((Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}) \cup \{\mathit{0}\}] = \bigvee(\mathcal{V}_{\alpha-1}[(Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}] \cup \mathcal{V}_{\alpha-1}[\{\mathit{0}\}]) = (\bigvee \mathcal{V}_{\alpha-1}[(Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}]) \vee (\bigvee \mathcal{V}_{\alpha-1}[\{\mathit{0}\}]) = \bigvee \mathcal{V}_{\alpha-1}[(Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}]$, $(Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\} \subseteq Y_1^* \cap \mathit{tcons}(S)$, $\bigvee \mathcal{V}_{\alpha-1}[(Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}] \leq \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)]$, $\bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] = \bigvee \mathcal{V}_{\alpha-1}[(Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}]$, for all $y \in (Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\} \subseteq \mathit{tcons}(S) \cap \overline{C}_{\mathcal{L}}$, by (100) for y, $\delta_2(\alpha-1)$, either $y \lhd \delta_2(\alpha-1)$ or $y \triangleq \delta_2(\alpha-1)$ or $\delta_2(\alpha-1) \lhd y$. We get two cases for y, $\delta_2(\alpha-1)$.

Case 2.3.1.2.2.2.2.1: $y \triangleq \delta_2(\alpha-1)$. Then $y \in (Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\} \subseteq Y_1^* \cap \mathit{tcons}(S) \subseteq \mathit{dom}(\mathcal{V}_{\alpha-1})$, $\mathcal{V}_{\alpha-1}(y) \in \mathbb{E}_{\alpha-1} = \emptyset$, which is a contradiction.

Case 2.3.1.2.2.2.2.2: $\delta_2(\alpha-1) \lhd y$. Then $\mathcal{V}_{\alpha-1}(y) \leq \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] = \mathcal{V}_\alpha(\delta_2(\alpha-1))$, $y \in \mathit{dom}(\mathcal{V}_{\alpha-1})$, $\mathcal{V}_{\alpha-1}(y) \in \mathbb{U}_{\alpha-1}$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) \underset{(104)}{<} \bigwedge \mathbb{U}_{\alpha-1} \leq \mathcal{V}_{\alpha-1}(y)$, which is a contradiction.

Hence, for all $y \in (Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}$, $y \lhd \delta_2(\alpha-1)$, $y \in \mathit{dom}(\mathcal{V}_{\alpha-1})$, $\mathcal{V}_{\alpha-1}(y) \in \mathbb{D}_{\alpha-1}$; $\mathcal{V}_{\alpha-1}[(Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}] \subseteq \mathbb{D}_{\alpha-1}$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigvee \mathcal{V}_{\alpha-1}[Y_1^* \cap \mathit{tcons}(S)] = \bigvee \mathcal{V}_{\alpha-1}[(Y_1^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}] \leq \bigvee \mathbb{D}_{\alpha-1}$, $\bigvee \mathbb{D}_{\alpha-1} \underset{(104)}{<} \mathcal{V}_\alpha(\delta_2(\alpha-1))$, which is a contradiction.

We get two cases for $\mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)]$, $\mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]]$.

Case 2.3.1.2.3: $\bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)] \leq \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]]$. Then $\bigwedge(\mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)] \cup \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]]) = (\bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)]) \wedge (\bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]]) = \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)]$, $\mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)] \subseteq \mathcal{V}_\alpha[Y_2^* - \{\delta_2(\alpha-1)\}] \subseteq \mathcal{V}_\alpha[Y_2^*] = Y_2$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigwedge Y_2 \leq \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)]$. We get two cases for $\mathcal{V}_\alpha(\delta_2(\alpha-1))$, $\bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)]$.

Case 2.3.1.2.3.1: $\mathcal{V}_\alpha(\delta_2(\alpha-1)) < \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)]$. Then $\bigwedge \mathcal{V}_\alpha[Y_2^* - \{\delta_2(\alpha-1)\}] = \bigwedge(\mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)] \cup \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]]) = \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)] > \mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigwedge Y_2 = \bigwedge \mathcal{V}_\alpha[Y_2^*]$, $Y_2^* - \{\delta_2(\alpha-1)\} \subset Y_2^*$, $\delta_2(\alpha-1) \in Y_2^*$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) \in \mathcal{V}_\alpha[Y_2^*] = Y_2$.

Case 2.3.1.2.3.2: $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)]$. We get two cases for $\mathit{0}$, $Y_2^* \cap \mathit{tcons}(S)$.

Case 2.3.1.2.3.2.1: $\mathit{0} \in Y_2^* \cap \mathit{tcons}(S)$. Then $\mathcal{V}_{\alpha-1}(\mathit{0}) \in \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)] \subseteq \mathcal{V}_\alpha[Y_2^* - \{\delta_2(\alpha-1)\}] \subseteq Y_2$, $0 = \mathcal{V}_{\alpha-1}(\mathit{0}) = \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)]$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)] = \mathcal{V}_{\alpha-1}(\mathit{0}) \in Y_2$.

Case 2.3.1.2.3.2.2: $\mathit{0} \notin Y_2^* \cap \mathit{tcons}(S)$. Then $\mathcal{V}_{\alpha-1}(\mathit{1}) = 1$, $Y_2^* \cap \mathit{tcons}(S) \subseteq ((Y_2^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}) \cup \{\mathit{1}\}$, $\bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \mathit{tcons}(S)] \geq \bigwedge \mathcal{V}_{\alpha-1}[((Y_2^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}) \cup \{\mathit{1}\}] = \bigwedge(\mathcal{V}_{\alpha-1}[(Y_2^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}] \cup \mathcal{V}_{\alpha-1}[\{\mathit{1}\}]) = (\bigwedge \mathcal{V}_{\alpha-1}[(Y_2^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}]) \wedge (\bigwedge \mathcal{V}_{\alpha-1}[\{\mathit{1}\}]) = \bigwedge \mathcal{V}_{\alpha-1}[(Y_2^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\}]$, $(Y_2^* \cap \mathit{tcons}(S)) - \{\mathit{0}, \mathit{1}\} \subseteq Y_2^* \cap \mathit{tcons}(S)$, $\bigwedge \mathcal{V}_{\alpha-1}[(Y_2^* \cap$

$tcons(S)) - \{0, 1\}] \geq \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap tcons(S)]$, $\bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap tcons(S)] = \bigwedge \mathcal{V}_{\alpha-1}[(Y_2^* \cap tcons(S)) - \{0, 1\}]$, for all $y \in (Y_2^* \cap tcons(S)) - \{0, 1\} \subseteq tcons(S) \cap \overline{C}_{\mathcal{L}}$, by (100) for y, $\delta_2(\alpha-1)$, either $y \lhd \delta_2(\alpha-1)$ or $y \triangleq \delta_2(\alpha-1)$ or $\delta_2(\alpha-1) \lhd y$. We get two cases for y, $\delta_2(\alpha-1)$.

Case 2.3.1.2.3.2.2.1: $y \lhd \delta_2(\alpha-1)$. Then $y \in (Y_2^* \cap tcons(S)) - \{0, 1\} \subseteq Y_2^* \cap tcons(S) \subseteq dom(\mathcal{V}_{\alpha-1})$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap tcons(S)] \leq \mathcal{V}_{\alpha-1}(y)$, $\mathcal{V}_{\alpha-1}(y) \in \mathbb{D}_{\alpha-1}$, $\mathcal{V}_{\alpha-1}(y) \leq \bigvee \mathbb{D}_{\alpha-1} \underset{(104)}{<} \mathcal{V}_\alpha(\delta_2(\alpha-1))$, which is a contradiction.

Case 2.3.1.2.3.2.2.2: $y \triangleq \delta_2(\alpha-1)$. Then $y \in dom(\mathcal{V}_{\alpha-1})$, $\mathcal{V}_{\alpha-1}(y) \in \mathbb{E}_{\alpha-1} = \emptyset$, which is a contradiction.

Hence, for all $y \in (Y_2^* \cap tcons(S)) - \{0, 1\}$, $\delta_2(\alpha-1) \lhd y$, $y \in dom(\mathcal{V}_{\alpha-1})$, $\mathcal{V}_{\alpha-1}(y) \in \mathbb{U}_{\alpha-1}$; $\mathcal{V}_{\alpha-1}[(Y_2^* \cap tcons(S)) - \{0, 1\}] \subseteq \mathbb{U}_{\alpha-1}$, $\bigwedge \mathbb{U}_{\alpha-1} \leq \bigwedge \mathcal{V}_{\alpha-1}[(Y_2^* \cap tcons(S)) - \{0, 1\}] = \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap tcons(S)] = \mathcal{V}_\alpha(\delta_2(\alpha-1))$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) \underset{(104)}{<} \bigwedge \mathbb{U}_{\alpha-1}$, which is a contradiction.

Case 2.3.1.2.4: $\bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap tcons(S)] > \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]]$. Then $\bigwedge(\mathcal{V}_{\alpha-1}[Y_2^* \cap tcons(S)] \cup \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]]) = (\bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap tcons(S)]) \wedge (\bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]]) = \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]]$, $\bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]] < \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap tcons(S)] \leq 1$, $\bigwedge \emptyset = 1$, $\mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]] \neq \emptyset$, $Y_2^* \cap \delta_2[\alpha-1] \neq \emptyset$, there exists $y^* \in Y_2^* \cap \delta_2[\alpha-1]$ and $\mathcal{V}_{\alpha-1}(y^*) \in \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]] \subseteq \mathcal{V}_\alpha[Y_2^* - \{\delta_2(\alpha-1)\}] \subseteq \mathcal{V}_\alpha[Y_2^*] = Y_2$, $\mathcal{V}_{\alpha-1}(y^*) = \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]] \geq \bigwedge Y_2 = \mathcal{V}_\alpha(\delta_2(\alpha-1))$. We get two cases for $\mathcal{V}_\alpha(\delta_2(\alpha-1))$, $\mathcal{V}_{\alpha-1}(y^*)$.

Case 2.3.1.2.4.1: $\mathcal{V}_\alpha(\delta_2(\alpha-1)) < \mathcal{V}_{\alpha-1}(y^*)$. Then $\bigwedge \mathcal{V}_\alpha[Y_2^* - \{\delta_2(\alpha-1)\}] = \bigwedge(\mathcal{V}_{\alpha-1}[Y_2^* \cap tcons(S)] \cup \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]]) = \bigwedge \mathcal{V}_{\alpha-1}[Y_2^* \cap \delta_2[\alpha-1]] = \mathcal{V}_{\alpha-1}(y^*) > \mathcal{V}_\alpha(\delta_2(\alpha-1)) = \bigwedge Y_2 = \bigwedge \mathcal{V}_\alpha[Y_2^*]$, $Y_2^* - \{\delta_2(\alpha-1)\} \subset Y_2^*$, $\delta_2(\alpha-1) \in Y_2^*$, $\mathcal{V}_\alpha(\delta_2(\alpha-1)) \in \mathcal{V}_\alpha[Y_2^*] = Y_2$.

Case 2.3.1.2.4.2: $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \mathcal{V}_{\alpha-1}(y^*)$. Then $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \mathcal{V}_{\alpha-1}(y^*) \in Y_2$.

So, $\mathcal{V}_\alpha[dom(\mathcal{V}_\alpha)]$ is admissible with respect to suprema and infima; the fifth point of (102) holds.

Case 2.3.1.3: $\mathcal{V}_\alpha(\delta_2(\alpha-1)) > \bigvee Y_1 = \bigwedge Y_2$. Then $\mathcal{V}_\alpha(\delta_2(\alpha-1)) \notin Y_1$, $\emptyset \neq Y_2 \neq \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\}$, $\emptyset \neq Y_1, Y_2 - \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\} \subseteq \mathcal{V}_{\alpha-1}[dom(\mathcal{V}_{\alpha-1})]$, $\bigvee Y_1 = \bigwedge Y_2 = \bigwedge(Y_2 - \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\})$; we have $\mathcal{V}_{\alpha-1}[dom(\mathcal{V}_{\alpha-1})]$ is admissible with respect to suprema and infima; $\bigvee Y_1 \in Y_1$, $\bigwedge Y_2 = \bigwedge(Y_2 - \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\}) \in Y_2 - \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\} \subseteq Y_2$; $\mathcal{V}_\alpha[dom(\mathcal{V}_\alpha)]$ is admissible with respect to suprema and infima; the fifth point of (102) holds.

So, in all Cases 2.3.1.1–2.3.1.3, the fifth point of (102) holds; the fifth point of (102) holds.

Case 2.3.2: $\mathbb{E}_{\alpha-1} \neq \emptyset$. Then, by (103), there exists $a^* \in dom(\mathcal{V}_{\alpha-1})$ and $a^* \triangleq \delta_2(\alpha-1)$, $\mathcal{V}_{\alpha-1}(a^*) = \mathcal{V}_\alpha(\delta_2(\alpha-1))$; we have $\mathcal{V}_{\alpha-1}[dom(\mathcal{V}_{\alpha-1})]$ is admissible with respect to suprema and infima; $\mathcal{V}_\alpha(\delta_2(\alpha-1)) = \mathcal{V}_{\alpha-1}(a^*) \in \mathcal{V}_{\alpha-1}[dom(\mathcal{V}_{\alpha-1})]$, $\mathcal{V}_\alpha[dom(\mathcal{V}_\alpha)] = \mathcal{V}_{\alpha-1}[dom(\mathcal{V}_{\alpha-1})] \cup \{\mathcal{V}_\alpha(\delta_2(\alpha-1))\} = \mathcal{V}_{\alpha-1}[dom(\mathcal{V}_{\alpha-1})]$ is admissible with respect to suprema and infima; the fifth point of (102) holds.

So, in both Cases 2.3.1 and 2.3.2, the fifth point of (102) holds; the fifth point of (102) holds. Thus, (102) holds.

Case 3 (the induction case): $\alpha = \gamma_2$ is a limit ordinal. Then $\mathcal{V}_{\gamma_2} = \bigcup_{\alpha<\gamma_2} \mathcal{V}_\alpha$, $dom(\mathcal{V}_{\gamma_2}) = \bigcup_{\alpha<\gamma_2} dom(\mathcal{V}_\alpha)$. Let $a, b \in dom(\mathcal{V}_{\gamma_2})$. Then there exist $\alpha_a, \alpha_b < \gamma_2$ and $a \in dom(\mathcal{V}_{\alpha_a})$, $b \in dom(\mathcal{V}_{\alpha_b})$. We put $\beta = max(\alpha_a, \alpha_b) < \gamma_2$. Hence, $\alpha_a \leq \beta$, $\alpha_b \leq \beta$, $a \in dom(\mathcal{V}_{\alpha_a}) \underset{(94)}{\subseteq} dom(\mathcal{V}_\beta)$, $b \in dom(\mathcal{V}_{\alpha_b}) \underset{(94)}{\subseteq} dom(\mathcal{V}_\beta)$, by induction hypothesis for β,

$$\begin{aligned}
&\text{if } a \triangleq b, \text{ then } \mathcal{V}_{\gamma_2}(a) = \mathcal{V}_\beta(a) = \mathcal{V}_\beta(b) = \mathcal{V}_{\gamma_2}(b);\\
&\text{if } a \lhd b, \text{ then } \mathcal{V}_{\gamma_2}(a) = \mathcal{V}_\beta(a) < \mathcal{V}_\beta(b) = \mathcal{V}_{\gamma_2}(b);\\
&\text{if } \mathcal{V}_{\gamma_2}(a) = \mathcal{V}_\beta(a) = 0, \text{ then } a = \mathit{0} \text{ or } a \triangleq \mathit{0};\\
&\text{if } \mathcal{V}_{\gamma_2}(a) = \mathcal{V}_\beta(a) = 1, \text{ then } a = \mathit{1} \text{ or } a \triangleq \mathit{1};
\end{aligned}$$

the first four points of (102) hold; the fifth point of (102) holds trivially; (102) holds.

So, in all Cases 1–3, (102) holds. The induction is completed. Thus, (102) holds.

We put $\mathcal{V} = \mathcal{V}_{\gamma_2}$, $dom(\mathcal{V}) \overset{(94)}{=\!=} tcons(S) \cup \delta[\gamma_2] = tcons(S) \cup (\mathcal{B} - tcons(S)) = \mathcal{B}$. We next prove the following statements:

$$\begin{aligned}
&\text{For all } a, b \in \mathcal{B},\\
&\quad \text{if } a \triangleq b, \text{ then } \mathcal{V}(a) = \mathcal{V}(b);\\
&\quad \text{if } a \lhd b, \text{ then } \mathcal{V}(a) < \mathcal{V}(b).
\end{aligned} \tag{105}$$

By (102) for γ_2, for all $a, b \in dom(\mathcal{V}) = \mathcal{B}$, if $a \triangleq b$, then $\mathcal{V}(a) = \mathcal{V}(b)$; if $a \lhd b$, then $\mathcal{V}(a) < \mathcal{V}(b)$; (105) holds.

$$\text{For all } Qx\, a \in qatoms(clo^{\mathcal{BH}}(S)) \text{ and } u \in \mathcal{U}_{\mathfrak{A}},\ a(x/u) \in atoms(clo^{\mathcal{BH}}(S)). \tag{106}$$

Let $Qx\, a \in qatoms(clo^{\mathcal{BH}}(S))$ and $u \in \mathcal{U}_{\mathfrak{A}}$. Then $u \in \mathcal{U}_{\mathfrak{A}} = GTerm_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, there exists $\{\tilde{w}_j \,|\, 1 \leq j \leq m\} = funcs(u) \cap \tilde{\mathbb{W}}^* \subseteq_{\mathcal{F}} \tilde{\mathbb{W}}^* = funcs(\mathbb{S}) \cap \tilde{\mathbb{W}}$; for all $1 \leq j \leq m$, there exists $a_j \in atoms(\mathbb{S}) \cup qatoms(\mathbb{S}) \subseteq atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$ and $\tilde{w}_j \in funcs(a_j)$; $\emptyset \neq \{Qx\, a\} \cup \{a_j \,|\, 1 \leq j \leq m\} \subseteq_{\mathcal{F}} atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$; by (99) for $\{Qx\, a\} \cup \{a_j \,|\, 1 \leq j \leq m\}$, there exist a deduction $\mathcal{D} = C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq GOrdCl_{\mathcal{L}_n}$, and $\{Qx\, a\} \cup \{a_j \,|\, 1 \leq j \leq m\} \subseteq atoms(S_n) \cup qatoms(S_n)$, $funcs(u) \subseteq Func_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $Func_{\mathcal{L}_n} \supseteq \bigcup_{j=1}^m funcs(a_j) \supseteq \{\tilde{w}_j \,|\, 1 \leq j \leq m\} = funcs(u) \cap \tilde{\mathbb{W}}^*$, $Func_{\mathcal{L}_n} \supseteq Func_{\mathcal{L} \cup P} \supseteq funcs(u) \cap Func_{\mathcal{L} \cup P}$, $Func_{\mathcal{L}_n} \supseteq (funcs(u) \cap \tilde{\mathbb{W}}^*) \cup (funcs(u) \cap Func_{\mathcal{L} \cup P}) = funcs(u) \cap (\tilde{\mathbb{W}}^* \cup Func_{\mathcal{L} \cup P}) = funcs(u) \cap Func_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P} = funcs(u)$, $u \in GTerm_{\mathcal{L}_n}$. We distinguish two cases for Q.

Case 1: $Q = \forall$. Then $\forall x\, a \in \mathit{qatoms}^{\forall}(S_n) \subseteq \mathit{QAtom}_{\mathcal{L}_n}$. We put $\gamma = x/u \in \mathit{Subst}_{\mathcal{L}_n}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(a)$. Hence, $a\gamma = a(x/u)$, using Rule (45) with respect to $\mathcal{L}_n$, S_n, we derive $\forall x\, a \prec a\gamma \vee \forall x\, a \eqcirc a\gamma = \forall x\, a \prec a(x/u) \vee \forall x\, a \eqcirc a(x/u) \in \mathit{GOrdCl}_{\mathcal{L}_n}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n$, $C_{n+1} = \forall x\, a \prec a(x/u) \vee \forall x\, a \eqcirc a(x/u) \in \mathit{GOrdCl}_{\mathcal{L}_n}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $C_{n+1} \in \mathit{GOrdCl}_{\mathcal{L}_n} \subseteq \mathit{GOrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $C_{n+1} \in \mathit{clo}^{\mathcal{BH}}(S)$, $a(x/u) \in \mathit{atoms}(C_{n+1}) \subseteq \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S))$; (106) holds.

Case 2: $Q = \exists$. Then $\exists x\, a \in \mathit{qatoms}^{\exists}(S_n) \subseteq \mathit{QAtom}_{\mathcal{L}_n}$. We put $\gamma = x/u \in \mathit{Subst}_{\mathcal{L}_n}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(a)$. Hence, $a\gamma = a(x/u)$, using Rule (46) with respect to $\mathcal{L}_n$, S_n, we derive $a\gamma \prec \exists x\, a \vee a\gamma \eqcirc \exists x\, a = a(x/u) \prec \exists x\, a \vee a(x/u) \eqcirc \exists x\, a \in \mathit{GOrdCl}_{\mathcal{L}_n}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n$, $C_{n+1} = a(x/u) \prec \exists x\, a \vee a(x/u) \eqcirc \exists x\, a \in \mathit{GOrdCl}_{\mathcal{L}_n}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $C_{n+1} \in \mathit{GOrdCl}_{\mathcal{L}_n} \subseteq \mathit{GOrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $C_{n+1} \in \mathit{clo}^{\mathcal{BH}}(S)$, $a(x/u) \in \mathit{atoms}(C_{n+1}) \subseteq \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S))$; (106) holds.

So, in both Cases 1 and 2, (106) holds; (106) holds.

$$\begin{aligned} &\text{For all } a \in \mathcal{B}, \\ &\quad \text{if } a = \forall x b, \text{ then } \mathcal{V}(a) = \textstyle\bigwedge_{u\in\mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u)); \\ &\quad \text{if } a = \exists x b, \text{ then } \mathcal{V}(a) = \textstyle\bigvee_{u\in\mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u)). \end{aligned} \tag{107}$$

Let $a = \forall x\, b \in \mathcal{B}$. Then $\forall x\, b \in \mathcal{B} - \mathit{tcons}(S)$, there exists $\alpha^* < \gamma_2$ and $\delta_2(\alpha^*) = \forall x\, b$; $\forall x\, b \in \mathit{qatoms}(\mathbb{S}) \subseteq \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$. Let $u \in \mathcal{U}_{\mathfrak{A}}$. Hence, by (106) for $\forall x\, b$, u, $b(x/u) \in \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S))$, by (97) for $\forall x\, b$, $b(x/u)$, there exist a deduction $\mathcal{D} = C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq \mathit{GOrdCl}_{\mathcal{L}_n}$, and $\forall x\, b, b(x/u) \in \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n)$, $\forall x\, b \in \mathit{qatoms}^{\forall}(S_n) \subseteq \mathit{QAtom}_{\mathcal{L}_n}$, $b(x/u) \in \mathit{atoms}(S_n) \subseteq \mathit{GAtom}_{\mathcal{L}_n}$, $u \in \mathit{GTerm}_{\mathcal{L}_n}$. We put $\gamma = x/u \in \mathit{Subst}_{\mathcal{L}_n}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(b)$. Then $b\gamma = b(x/u)$, using Rule (45) with respect to $\mathcal{L}_n$, S_n, we derive $\forall x\, b \prec b\gamma \vee \forall x\, b \eqcirc b\gamma = \forall x\, b \prec b(x/u) \vee \forall x\, b \eqcirc b(x/u) \in \mathit{GOrdCl}_{\mathcal{L}_n}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n$, $C_{n+1} = \forall x\, b \prec b(x/u) \vee \forall x\, b \eqcirc b(x/u) \in \mathit{GOrdCl}_{\mathcal{L}_n} \subseteq \mathit{GOrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Hence, $C_{n+1} \in \mathit{clo}^{\mathcal{BH}}(S)$, $\forall x\, b \prec b(x/u) \in \mathbb{S}$ or $\forall x\, b \eqcirc b(x/u) \in \mathbb{S}$, for both the cases $\forall x\, b \prec b(x/u) \in \mathbb{S}$ and $\forall x\, b \eqcirc b(x/u) \in \mathbb{S}$, $b(x/u) \in \mathit{atoms}(\mathbb{S})$; $b(x/u) \in \mathit{atoms}(\mathbb{S}) \subseteq \mathcal{B}$; we have that there does not exist a contradiction of $\mathbb{S}$; either $\forall x\, b \prec b(x/u) \in \mathbb{S}$ or $\forall x\, b \eqcirc b(x/u) \in \mathbb{S}$, either $\forall x\, b \lhd b(x/u)$ or $\forall x\, b \triangleq b(x/u)$, by (105) for $\forall x\, b$, $b(x/u)$, either $\mathcal{V}(\forall x\, b) < \mathcal{V}(b(x/u))$ or $\mathcal{V}(\forall x\, b) = \mathcal{V}(b(x/u))$; $\mathcal{V}(\forall x\, b) \leq \bigwedge_{u\in\mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u))$. We distinguish two cases.

Case 1: There exists $u^* \in \mathcal{U}_{\mathfrak{A}}$ such that $b(x/u^*) \in \mathcal{B}$ and $\forall x\, b \triangleq b(x/u^*)$. Then, by (105) for $\forall x\, b$, $b(x/u^*)$, $\mathcal{V}(\forall x\, b) = \mathcal{V}(b(x/u^*))$, $\bigwedge_{u\in\mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u)) \leq \mathcal{V}(b(x/u^*)) = \mathcal{V}(\forall x\, b)$, $\mathcal{V}(\forall x\, b) = \bigwedge_{u\in\mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u))$.

Case 2: For all $u \in \mathcal{U}_{\mathfrak{A}}$, $b(x/u) \in \mathcal{B}$ and $\forall x\, b \lhd b(x/u)$. At first, we prove the following statements:

$$\begin{aligned} &\text{For all } \alpha < \gamma_2 \text{ and } \forall x\, b \lhd \delta_2(\alpha), \text{ there exists } \tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha))) \in \mathcal{U}_{\mathfrak{A}} \text{ such that} \\ &\quad b(x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \in \mathcal{B}, \\ &\quad \forall x\, b \lhd b(x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))), \\ &\quad b(x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \lhd \delta_2(\alpha). \end{aligned} \tag{108}$$

Let $\alpha < \gamma_2$ and $\forall x\, b \lhd \delta_2(\alpha)$. Then $\delta_2(\alpha) \in \mathcal{B} \subseteq \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S)) \cup \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$, by (97) for $\forall x\, b$, $\delta_2(\alpha)$, there exist a deduction $C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq \mathit{GOrdCl}_{\mathcal{L}_n}$, and $\forall x\, b, \delta_2(\alpha) \in \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n)$; $\forall x\, b \in \mathit{qatoms}^{\forall}(S_n) \subseteq \mathit{QAtom}_{\mathcal{L}_n}$, $\delta_2(\alpha) \in \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n) \subseteq \mathit{GAtom}_{\mathcal{L}_n} \cup \mathit{QAtom}_{\mathcal{L}_n}$, there exists $\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_n}$ and $\mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha))|$. We put $\gamma = x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha))) \in \mathit{Subst}_{\mathcal{L}_n \cup \{\tilde{w}\}}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(b)$. Hence, $\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha))) \in \mathit{GTerm}_{\mathcal{L}_n \cup \{\tilde{w}\}}$, $b\gamma = b(x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha))))$, using Rule (47) with respect to $\mathcal{L}_n$, S_n, we derive $b\gamma \prec \delta_2(\alpha) \vee \delta_2(\alpha) \eqcirc \forall x\, b \vee \delta_2(\alpha) \prec \forall x\, b = b(x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \prec \delta_2(\alpha) \vee \delta_2(\alpha) \eqcirc \forall x\, b \vee \delta_2(\alpha) \prec \forall x\, b \in \mathit{GOrdCl}_{\mathcal{L}_n \cup \{\tilde{w}\}}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{\tilde{w}\}$, $\gamma \in \mathit{Subst}_{\mathcal{L}_n \cup \{\tilde{w}\}} = \mathit{Subst}_{\mathcal{L}_{n+1}}$, $\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha))) \in \mathit{GTerm}_{\mathcal{L}_n \cup \{\tilde{w}\}} = \mathit{GTerm}_{\mathcal{L}_{n+1}} \subseteq \mathit{GTerm}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $C_{n+1} = b(x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \prec \delta_2(\alpha) \vee \delta_2(\alpha) \eqcirc \forall x\, b \vee \delta_2(\alpha) \prec \forall x\, b \in \mathit{GOrdCl}_{\mathcal{L}_n \cup \{\tilde{w}\}} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}} \subseteq \mathit{GOrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} \cup \mathit{GOrdCl}_{\mathcal{L}_{n+1}} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $C_{n+1} \in \mathit{clo}^{\mathcal{BH}}(S)$, $b(x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \prec \delta_2(\alpha) \in \mathbb{S}$ or $\delta_2(\alpha) \eqcirc \forall x\, b \in \mathbb{S}$ or $\delta_2(\alpha) \prec \forall x\, b \in \mathbb{S}$; we have that $\forall x\, b \lhd \delta_2(\alpha)$, there does not exist a contradiction of $\mathbb{S}$; $b(x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \prec \delta_2(\alpha) \in \mathbb{S}$, $\mathit{funcs}(\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \subseteq \mathit{Func}_{\mathcal{L}} \cup \tilde{\mathbb{W}}$, $\mathit{funcs}(\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \cap \tilde{\mathbb{W}} \subseteq \mathit{funcs}(\mathbb{S}) \cap \tilde{\mathbb{W}} = \tilde{\mathbb{W}}^*$, $\mathit{funcs}(\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \subseteq \mathit{Func}_{\mathcal{L}} \cup \tilde{\mathbb{W}}^*$, $\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha))) \in \mathit{GTerm}_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P} = \mathcal{U}_{\mathfrak{A}}$,

$$\begin{aligned} &b(x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \in \mathcal{B}, \\ &\forall x\, b \lhd b(x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))), \\ &b(x/\tilde{w}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \lhd \delta_2(\alpha); \end{aligned}$$

(108) holds.

Let there exist $c^* \in \mathit{tcons}(S)$ and $\forall x\, b \lhd c^*$. There exists $\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)) \in \mathcal{U}_{\mathfrak{A}}$ such that $b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \in \mathcal{B}$, for all $c \in \mathit{tcons}(S)$ and $\forall x\, b \lhd c$, (109)

$$\forall x\, b \lhd b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))),$$
$$b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \lhd c.$$

We have $\forall x\, b \in \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$. Then there exists $C \in \mathit{clo}^{\mathcal{BH}}(S)$ and $\forall x\, b \in \mathit{qatoms}(C)$; there exists a deduction $\mathcal{D} = C_1, \dots, C_n = C$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $C = C_n \in S_n \subseteq \mathit{GOrdCl}_{\mathcal{L}_n}$, and $\forall x\, b \in \mathit{qatoms}^{\forall}(C) \subseteq \mathit{qatoms}^{\forall}(S_n) \subseteq \mathit{QAtom}_{\mathcal{L}_n}$; there exists $\tilde{w}^* \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_n}$ and $\mathit{ar}(\tilde{w}^*) = |\mathit{freetermseq}(\forall x\, b)|$. Let $c \in \mathit{tcons}(S)$ and $\forall x\, b \lhd c$. We have that there does not exist a contradiction of $\mathbb{S}$. Hence, $c \neq \mathit{0}$ and $\mathit{0} \lhd c \in \mathit{ordtcons}(S) \subseteq \mathit{ordtcons}(S) \cup \mathit{GInst}_{\mathcal{L}_n}(S)$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n$, $C_{n+1} = \mathit{0} \lhd c \in \mathit{ordtcons}(S) \cup \mathit{GInst}_{\mathcal{L}_n}(S) \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $\forall x\, b \in \mathit{qatoms}^{\forall}(S_n) \subseteq \mathit{qatoms}^{\forall}(S_{n+1}) \subseteq \mathit{QAtom}_{\mathcal{L}_{n+1}}$, $\tilde{w}^* \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_n} = \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_{n+1}}$, $\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)) = \tilde{w}^*(\mathit{freetermseq}(\forall x\, b), \ell) = \tilde{w}^*(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(c))$, $c \in \mathit{atoms}(C_{n+1}) \subseteq \mathit{atoms}(S_{n+1}) \subseteq \mathit{GAtom}_{\mathcal{L}_{n+1}}$. We put $\gamma = x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(c)) = x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)) \in \mathit{Subst}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(b)$. Hence, $\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)) \in \mathit{GTerm}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}}$, $b\gamma = b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)))$, using Rule (47) with respect to $\mathcal{L}_{n+1}$, S_{n+1}, we derive $b\gamma \prec c \vee c \eqcirc \forall x\, b \vee c \prec \forall x\, b = b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \prec c \vee c \eqcirc \forall x\, b \vee c \prec \forall x\, b \in \mathit{GOrdCl}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}}$. We put $\mathcal{L}_{n+2} = \mathcal{L}_{n+1} \cup \{\tilde{w}^*\}$, $\gamma \in \mathit{Subst}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}} = \mathit{Subst}_{\mathcal{L}_{n+2}}$, $\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)) \in \mathit{GTerm}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}} = \mathit{GTerm}_{\mathcal{L}_{n+2}} \subseteq \mathit{GTerm}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $C_{n+2} = b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \prec c \vee c \eqcirc \forall x\, b \vee c \prec \forall x\, b \in \mathit{GOrdCl}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}} = \mathit{GOrdCl}_{\mathcal{L}_{n+2}} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}'' = \mathcal{D}', C_{n+2}$, $S_{n+2} = S_{n+1} \cup \{C_{n+2}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_{n+1}} \cup \mathit{GOrdCl}_{\mathcal{L}_{n+2}} = \mathit{GOrdCl}_{\mathcal{L}_{n+2}}$; $\mathcal{D}''$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $C_{n+2} \in \mathit{clo}^{\mathcal{BH}}(S)$, $b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \prec c \in \mathbb{S}$ or $c \eqcirc \forall x\, b \in \mathbb{S}$ or $c \prec \forall x\, b \in \mathbb{S}$; we have that $\forall x\, b \lhd c$, there does not exist a contradiction of $\mathbb{S}$; $b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \prec c \in \mathbb{S}$, $\mathit{funcs}(\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \subseteq \mathit{Func}_{\mathcal{L}} \cup \tilde{\mathbb{W}}$, $\mathit{funcs}(\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \cap \tilde{\mathbb{W}} \subseteq \mathit{funcs}(\mathbb{S}) \cap \tilde{\mathbb{W}} = \tilde{\mathbb{W}}^*$, $\mathit{funcs}(\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \subseteq \mathit{Func}_{\mathcal{L}} \cup \tilde{\mathbb{W}}^*$, $\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)) \in \mathit{GTerm}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P} = \mathcal{U}_{\mathfrak{A}}$,

$$b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \in \mathcal{B},$$
$$\forall x\, b \lhd b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))),$$
$$b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \lhd c;$$

for $c = c^*$, $\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)) \in \mathcal{U}_{\mathfrak{A}}$, $b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \in \mathcal{B}$, for all $c \in \mathit{tcons}(S)$ and $\forall x\, b \lhd c$, $\forall x\, b \lhd b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)))$, $b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \lhd c$; (109) holds.

We have $\alpha^* < \gamma_2$, $\forall x\, b = \delta_2(\alpha^*)$, for all $u \in \mathcal{U}_{\mathfrak{A}} \neq \emptyset$, $b(x/u) \in \mathcal{B}$, $\forall x\, b \lhd b(x/u)$. Then there exist $u^* \in \mathcal{U}_{\mathfrak{A}} \neq \emptyset$ and $b(x/u^*) \in \mathcal{B}$, $\forall x\, b \lhd b(x/u^*)$; $b(x/u^*) \in \mathcal{B} - \mathit{tcons}(S)$, there exists $\beta^* < \gamma_2$ and $\delta_2(\beta^*) = b(x/u^*)$, $\forall x\, b \lhd b(x/u^*) = \delta_2(\beta^*)$. We get two cases.

Case 2.1: There exists $c^* \in \mathit{tcons}(S)$ and $\forall x\, b \lhd c^*$. Then, by (109), there exists $\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)) \in \mathcal{U}_{\mathfrak{A}}$ and $b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \in \mathcal{B}$, for all $c \in \mathit{tcons}(S)$ and $\forall x\, b \lhd c$, $\forall x\, b \lhd b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)))$, $b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \lhd c$; $b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b))) \in \mathcal{B} - \mathit{tcons}(S)$, there exists $\beta^{**} < \gamma_2$ and $\delta_2(\beta^{**}) = b(x/\tilde{w}^*(\mathit{freetermseq}(\forall x\, b)))$, for all $c \in \mathit{tcons}(S)$ and $\forall x\, b \lhd c$, $\forall x\, b \lhd \delta_2(\beta^{**})$, $\delta_2(\beta^{**}) \lhd c$. We put $\kappa = \mathit{max}(\alpha^*, \beta^*, \beta^{**}) < \gamma_2 \leq \omega$ and $M = \{\delta_2(\alpha) \,|\, \alpha \leq \kappa < \gamma_2 \leq \omega, \forall x\, b \lhd \delta_2(\alpha)\} \subseteq_{\mathcal{F}} \mathcal{B} - \mathit{tcons}(S)$. Hence, $\beta^*, \beta^{**} \leq \kappa$ and $\delta_2(\beta^*), \delta_2(\beta^{**}) \in M \neq \emptyset$; we have $\mathit{qatoms}(S) \neq \emptyset$; by (101), for all $c, d \in \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd d$ or ($c = d$ or $c \triangleq d$) or $d \lhd c$, there exists $\alpha_0^* \leq \kappa < \gamma_2$ and $\delta_2(\alpha_0^*) \in M$ is a minimal element of M with respect to $\lhd$; for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\} \subseteq \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or $\forall x\, b \lhd c$, either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$, $\forall x\, b \lhd \delta_2(\alpha_0^*)$; we have that there does not exist a contradiction of $\mathbb{S}$; either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or $\forall x\, b \lhd c$, $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$. Let $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\} = (\mathit{tcons}(S) - \{\mathit{0}, \mathit{1}\}) \cup \delta_2[\kappa+1]$, $\forall x\, b \lhd c$, $c \lhd \delta_2(\alpha_0^*)$. We get two cases for c.

Case 2.1.1: $c \in \mathit{tcons}(S) - \{\mathit{0}, \mathit{1}\}$. Then $\delta_2(\beta^{**}) \lhd c$, $\delta_2(\beta^{**}) \lhd \delta_2(\alpha_0^*)$; we have $\delta_2(\beta^{**}) \in M$; which is a contradiction that $\delta_2(\alpha_0^*)$ is a minimal element of M with respect to $\lhd$.

Case 2.1.2: $c \in \delta_2[\kappa + 1]$. Then there exists $\alpha_c \leq \kappa$ and $\delta_2(\alpha_c) = c$, $\forall x\, b \lhd c = \delta_2(\alpha_c)$, $c = \delta_2(\alpha_c) \in M$; we have $c \lhd \delta_2(\alpha_0^*)$; which is a contradiction that $\delta_2(\alpha_0^*)$ is a minimal element of M with respect to $\lhd$.

Hence, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa + 1]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$; $\forall x\, b \lhd \delta_2(\alpha_0^*)$, by (105) for $\forall x\, b$, $\delta_2(\alpha_0^*)$, $\mathcal{V}(\delta_2(\alpha_0^*)) > \mathcal{V}(\forall x\, b)$. We put $D = \mathcal{V}(\delta_2(\alpha_0^*)) - \mathcal{V}(\forall x\, b)$, $0 < D \leq 1$. Then, by (108) for α_0^*, there exists $\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))) \in \mathcal{U}_{\mathfrak{A}}$ and

$$b(x/\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \in \mathcal{B},$$
$$\forall x\, b \lhd b(x/\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))),$$
$$b(x/\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \lhd \delta_2(\alpha_0^*);$$

$b(x/\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \in \mathcal{B} - \mathit{tcons}(S)$, there exists $\kappa < \alpha_0^{**} < \gamma_2$ and $\delta_2(\alpha_0^{**}) = b(x/\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))))$, $\forall x\, b \lhd \delta_2(\alpha_0^{**})$, $\delta_2(\alpha_0^{**}) \lhd \delta_2(\alpha_0^*)$.

Case 2.2: There does not exist $c^* \in \mathit{tcons}(S)$ and $\forall x\, b \lhd c^*$. We put $\kappa = \mathit{max}(\alpha^*, \beta^*) < \gamma_2 \leq \omega$ and $M = \{\delta_2(\alpha) \,|\, \alpha \leq \kappa < \gamma_2 \leq \omega, \forall x\, b \lhd \delta_2(\alpha)\} \subseteq_{\mathcal{F}} \mathcal{B} - \mathit{tcons}(S)$. Hence, $\beta^* \leq \kappa$ and $\delta_2(\beta^*) \in M \neq \emptyset$; we have, for all $c, d \in \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd d$ or ($c = d$ or $c \triangleq d$) or $d \lhd c$; there exists $\alpha_0^* \leq \kappa < \gamma_2$ and $\delta_2(\alpha_0^*) \in M$ is a minimal element of M with respect to $\lhd$; for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\} \subseteq \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or $\forall x\, b \lhd c$, either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$, $\forall x\, b \lhd \delta_2(\alpha_0^*)$; we have that there does not exist a contradiction of $\mathbb{S}$; either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or $\forall x\, b \lhd c$, $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$. Let $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\} = (\mathit{tcons}(S) - \{\mathit{0}, \mathit{1}\}) \cup \delta_2[\kappa+1]$, $\forall x\, b \lhd c$, $c \lhd \delta_2(\alpha_0^*)$. We get two cases for c.

Case 2.2.1: $c \in \mathit{tcons}(S) - \{\mathit{0}, \mathit{1}\}$. Then $c \in \mathit{tcons}(S) - \{\mathit{0}, \mathit{1}\} \subseteq \mathit{tcons}(S)$; we have $\forall x\, b \lhd c$; which is a contradiction that there does not exist $c^* \in \mathit{tcons}(S)$ and $\forall x\, b \lhd c^*$.

Case 2.2.2: $c \in \delta_2[\kappa + 1]$. Then there exists $\alpha_c \leq \kappa$ and $c = \delta_2(\alpha_c)$, $\forall x\, b \lhd c = \delta_2(\alpha_c)$, $c = \delta_2(\alpha_c) \in M$; we have $c \lhd \delta_2(\alpha_0^*)$; which is a contradiction that $\delta_2(\alpha_0^*)$ is a minimal element of M with respect to $\lhd$.

Hence, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$; $\forall x\, b \lhd \delta_2(\alpha_0^*)$, by (105) for $\forall x\, b$, $\delta_2(\alpha_0^*)$, $\mathcal{V}(\delta_2(\alpha_0^*)) > \mathcal{V}(\forall x\, b)$. We put $D = \mathcal{V}(\delta_2(\alpha_0^*)) - \mathcal{V}(\forall x\, b)$, $0 < D \leq 1$. Then, by (108) for α_0^*, there exists $\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))) \in \mathcal{U}_{\mathfrak{A}}$ and

$$\begin{gathered} b(x/\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \in \mathcal{B}, \\ \forall x\, b \lhd b(x/\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))), \\ b(x/\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \lhd \delta_2(\alpha_0^*); \end{gathered}$$

$b(x/\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \in \mathcal{B} - \mathit{tcons}(S)$, there exists $\kappa < \alpha_0^{**} < \gamma_2$ and $\delta_2(\alpha_0^{**}) = b(x/\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))))$, $\forall x\, b \lhd \delta_2(\alpha_0^{**})$, $\delta_2(\alpha_0^{**}) \lhd \delta_2(\alpha_0^*)$.

So, in both Cases 2.1 and 2.2, there exist $\alpha^*, \alpha_0^* \leq \kappa < \alpha_0^{**} < \gamma_2$, $\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))) \in \mathcal{U}_{\mathfrak{A}}$, $0 < D = \mathcal{V}(\delta_2(\alpha_0^*)) - \mathcal{V}(\forall x\, b) \leq 1$ such that $\delta_2(\alpha^*) = \forall x\, b \lhd \delta_2(\alpha_0^*)$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa + 1]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$, $\delta_2(\alpha_0^{**}) = b(x/\tilde{w}_0(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))))$, $\forall x\, b \lhd \delta_2(\alpha_0^{**})$, $\delta_2(\alpha_0^{**}) \lhd \delta_2(\alpha_0^*)$.

We next prove the following statement:

For all $n \geq 1$, there exist $\kappa, \alpha_{n-1}^* < \alpha_n^* < \alpha_n^{**} < \gamma_2$, $\tilde{w}_n(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_n^*))) \in \mathcal{U}_{\mathfrak{A}}$ such that (110)

$$\begin{aligned} &\forall x\, b \lhd \delta_2(\alpha_n^*), \delta_2(\alpha_n^*) \lhd \delta_2(\alpha_{n-1}^*), \\ &\text{for all } c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_n^*]) - \{\mathit{0}, \mathit{1}\}, \\ &\quad \text{either } c \lhd \forall x\, b \text{ or } (c = \forall x\, b \text{ or } c \triangleq \forall x\, b) \text{ or } (c = \delta_2(\alpha_{n-1}^*) \text{ or } c \triangleq \delta_2(\alpha_{n-1}^*)) \text{ or } \delta_2(\alpha_{n-1}^*) \lhd c, \\ &\delta_2(\alpha_n^{**}) = b(x/\tilde{w}_n(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_n^*)))), \\ &\forall x\, b \lhd \delta_2(\alpha_n^{**}), \delta_2(\alpha_n^{**}) \lhd \delta_2(\alpha_n^*), \\ &\mathcal{V}(\delta_2(\alpha_n^*)) = \mathcal{V}(\forall x\, b) + \frac{D}{2^n}, \mathcal{V}(\forall x\, b) < \mathcal{V}(\delta_2(\alpha_n^{**})) < \mathcal{V}(\delta_2(\alpha_n^*)). \end{aligned}$$

We proceed by induction on $1 \leq n$.

Case 2.3 (the base case): $n = 1$. We have $\alpha^*, \alpha_0^* \leq \kappa < \alpha_0^{**} < \gamma_2$, $\delta_2(\alpha^*) = \forall x\, b \lhd \delta_2(\alpha_0^*)$, $\forall x\, b \lhd \delta_2(\alpha_0^{**})$, $\delta_2(\alpha_0^{**}) \lhd \delta_2(\alpha_0^*)$, $0 < D = \mathcal{V}(\delta_2(\alpha_0^*)) - \mathcal{V}(\forall x\, b) \leq 1$, for all $c, d \in \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd d$ or ($c = d$ or $c \triangleq d$) or $d \lhd c$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$. We put $M_1 = \{\alpha \,|\, \kappa < \alpha \leq \alpha_0^{**} < \gamma_2 \leq \omega, \forall x\, b \lhd \delta_2(\alpha), \delta_2(\alpha) \lhd \delta_2(\alpha_0^*)\} \subseteq_{\mathcal{F}} \gamma_2$. Then $\alpha_0^{**} \in M_1 \neq \emptyset$, there exists $\kappa < \alpha_1^* \leq \alpha_0^{**} < \gamma_2$ and $\alpha_1^* = \mathit{min}\, M_1$, $\alpha_1^* \in M_1$, $\forall x\, b \lhd \delta_2(\alpha_1^*)$, $\delta_2(\alpha_1^*) \lhd \delta_2(\alpha_0^*)$, for all $\kappa < \alpha < \alpha_1^* < \gamma_2$, either $\delta_2(\alpha) \lhd \forall x\, b$ or ($\delta_2(\alpha) = \forall x\, b$ or $\delta_2(\alpha) \triangleq \forall x\, b$) or $\forall x\, b \lhd \delta_2(\alpha)$, either $\delta_2(\alpha) \lhd \delta_2(\alpha_0^*)$ or ($\delta_2(\alpha) = \delta_2(\alpha_0^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha)$; we have that $\forall x\, b \lhd \delta_2(\alpha_0^*)$, there does not exist a contradiction of $\mathbb{S}$; either $\delta_2(\alpha) \lhd \forall x\, b$ or ($\delta_2(\alpha) = \forall x\, b$ or $\delta_2(\alpha) \triangleq \forall x\, b$) or $\forall x\, b \lhd \delta_2(\alpha)$, $\delta_2(\alpha) \lhd \delta_2(\alpha_0^*)$ or ($\delta_2(\alpha) = \delta_2(\alpha_0^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha)$. Let $\kappa < \alpha < \alpha_1^* \leq \alpha_0^{**} < \gamma_2$, $\forall x\, b \lhd \delta_2(\alpha)$, $\delta_2(\alpha) \lhd \delta_2(\alpha_0^*)$. Then $\alpha \in M_1$, $\alpha < \alpha_1^* = \mathit{min}\, M_1$, which is a contradiction. Hence, for all $\kappa < \alpha < \alpha_1^*$, either $\delta_2(\alpha) \lhd \forall x\, b$ or ($\delta_2(\alpha) = \forall x\, b$ or $\delta_2(\alpha) \triangleq \forall x\, b$) or ($\delta_2(\alpha) = \delta_2(\alpha_0^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha)$, for all $c \in ((\mathit{tcons}(S) \cup \delta_2[\kappa + 1]) - \{\mathit{0}, \mathit{1}\}) \cup \{\delta_2(\alpha) \,|\, \kappa < \alpha < \alpha_1^*\} = (\mathit{tcons}(S) \cup \delta_2[\alpha_1^*]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$; we have $\alpha_1^* < \gamma_2$; $\alpha_1^* + 1 \leq \gamma_2$ is a successor ordinal; we have that $\forall x\, b \lhd \delta_2(\alpha_1^*)$, $\delta_2(\alpha_1^*) \lhd \delta_2(\alpha_0^*)$, there does not exist a contradiction of $\mathbb{S}$; $\mathit{0} \not\triangleq \delta_2(\alpha_1^*)$, $\mathit{1} \not\triangleq \delta_2(\alpha_1^*)$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_1^*]) - \{\mathit{0}, \mathit{1}\}$, $c \not\triangleq \delta_2(\alpha_1^*)$, for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_1^*] \overset{(94)}{=\!=} \mathit{dom}(\mathcal{V}_{\alpha_1^*})$, $c \not\triangleq \delta_2(\alpha_1^*)$, $\mathbb{E}_{\alpha_1^*} = \emptyset$; $\alpha^*, \alpha_0^* \leq \kappa < \alpha_1^*$, $\delta_2(\alpha^*), \delta_2(\alpha_0^*) \in \mathit{tcons}(S) \cup \delta_2[\alpha_1^*] \overset{(94)}{=\!=} \mathit{dom}(\mathcal{V}_{\alpha_1^*})$, $\forall x\, b = \delta_2(\alpha^*) \in \mathit{dom}(\mathcal{V}_{\alpha_1^*})$, $\mathcal{V}_{\alpha_1^*}(\mathit{0}) = 0 \leq \mathcal{V}_{\alpha_1^*}(\forall x\, b)$, $\mathit{1} \not\lhd \delta_2(\alpha_1^*)$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_1^*]) - \{\mathit{0}, \mathit{1}\}$ and $c \lhd \delta_2(\alpha_1^*)$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$), for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_1^*] \overset{(94)}{=\!=} \mathit{dom}(\mathcal{V}_{\alpha_1^*})$ and $c \lhd \delta_2(\alpha_1^*)$, $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or $c = \mathit{0}$, by (102) for α_1^*, c, $\forall x\, b$, either $\mathcal{V}_{\alpha_1^*}(c) < \mathcal{V}_{\alpha_1^*}(\forall x\, b)$ or $\mathcal{V}_{\alpha_1^*}(c) = \mathcal{V}_{\alpha_1^*}(\forall x\, b)$; $\mathcal{V}_{\alpha_1^*}(c) \leq \mathcal{V}_{\alpha_1^*}(\forall x\, b)$; $\mathcal{V}_{\alpha_1^*}(\forall x\, b) \in \mathbb{D}_{\alpha_1^*}$, $\bigvee \mathbb{D}_{\alpha_1^*} = \mathcal{V}_{\alpha_1^*}(\forall x\, b)$; $\mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*)) \leq 1 = \mathcal{V}_{\alpha_1^*}(\mathit{1})$, $\delta_2(\alpha_1^*) \not\lhd \mathit{0}$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_1^*]) - \{\mathit{0}, \mathit{1}\}$ and $\delta_2(\alpha_1^*) \lhd c$, either ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$, for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_1^*] \overset{(94)}{=\!=} \mathit{dom}(\mathcal{V}_{\alpha_1^*})$ and $\delta_2(\alpha_1^*) \lhd c$, ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$ or $c = \mathit{1}$, by (102) for α_1^*, $\delta_2(\alpha_0^*)$, c, either $\mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*)) = \mathcal{V}_{\alpha_1^*}(c)$ or $\mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*)) < \mathcal{V}_{\alpha_1^*}(c)$; $\mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*)) \leq \mathcal{V}_{\alpha_1^*}(c)$; $\mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*)) \in \mathbb{U}_{\alpha_1^*}$, $\bigwedge \mathbb{U}_{\alpha_1^*} = \mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*))$; we have $\mathbb{E}_{\alpha_1^*} = \emptyset$; $\mathcal{V}(\delta_2(\alpha_1^*)) \overset{(94)}{=\!=} \mathcal{V}_{\alpha_1^*+1}(\delta_2(\alpha_1^*)) = \dfrac{\bigvee \mathbb{D}_{\alpha_1^*} + \bigwedge \mathbb{U}_{\alpha_1^*}}{2} = \dfrac{\mathcal{V}_{\alpha_1^*}(\forall x\, b) + \mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*))}{2} \overset{(94)}{=\!=} \dfrac{\mathcal{V}(\forall x\, b) + \mathcal{V}(\delta_2(\alpha_0^*))}{2} = \mathcal{V}(\forall x\, b) + \dfrac{\mathcal{V}(\delta_2(\alpha_0^*)) - \mathcal{V}(\forall x\, b)}{2} = \mathcal{V}(\forall x\, b) + \dfrac{D}{2}$; we have $\forall x\, b \lhd \delta_2(\alpha_1^*)$; by (108) for α_1^*, there exists $\tilde{w}_1(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*))) \in \mathcal{U}_{\mathfrak{A}}$ and

$$\begin{gathered} b(x/\tilde{w}_1(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*)))) \in \mathcal{B}, \\ \forall x\, b \lhd b(x/\tilde{w}_1(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*)))), \\ b(x/\tilde{w}_1(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*)))) \lhd \delta_2(\alpha_1^*); \end{gathered}$$

$b(x/\tilde{w}_1(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*)))) \in \mathcal{B} - \mathit{tcons}(S)$, $\delta_2(\alpha_1^*) \not\lhd \delta_2(\alpha_1^*)$, there exists $\kappa, \alpha_0^* < \alpha_1^* < \alpha_1^{**} < \gamma_2$ and $\delta_2(\alpha_1^{**}) = b(x/\tilde{w}_1(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*))))$, $\forall x\, b \lhd \delta_2(\alpha_1^{**})$, $\delta_2(\alpha_1^{**}) \lhd \delta_2(\alpha_1^*)$, by (105) for $\forall x\, b$, $\delta_2(\alpha_1^{**})$, $\delta_2(\alpha_1^*)$, $\mathcal{V}(\forall x\, b) < \mathcal{V}(\delta_2(\alpha_1^{**}))$, $\mathcal{V}(\delta_2(\alpha_1^{**})) < \mathcal{V}(\delta_2(\alpha_1^*))$; (110) holds.

Case 2.4 (the induction case): $n > 1$. We have $\alpha^* \leq \kappa < \gamma_2$, $\forall x\, b = \delta_2(\alpha^*)$, for all $c, d \in \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd d$ or ($c = d$ or $c \triangleq d$) or $d \lhd c$. We get by induction hypothesis for $n-1$ that there exist $\kappa, \alpha_{n-2}^* < \alpha_{n-1}^* < \alpha_{n-1}^{**} < \gamma_2$, $\tilde{w}_{n-1}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_{n-1}^*))) \in \mathcal{U}_{\mathfrak{A}}$ satisfying $\forall x\, b \lhd \delta_2(\alpha_{n-1}^*)$, $\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha_{n-2}^*)$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_{n-1}^*]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or ($c = \delta_2(\alpha_{n-2}^*)$ or $c \triangleq \delta_2(\alpha_{n-2}^*)$) or $\delta_2(\alpha_{n-2}^*) \lhd c$, $\delta_2(\alpha_{n-1}^{**}) = b(x/\tilde{w}_{n-1}(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_{n-1}^*))))$, $\forall x\, b \lhd \delta_2(\alpha_{n-1}^{**})$, $\delta_2(\alpha_{n-1}^{**}) \lhd \delta_2(\alpha_{n-1}^*)$, $\mathcal{V}(\delta_2(\alpha_{n-1}^*)) = \mathcal{V}(\forall x\, b) + \dfrac{D}{2^{n-1}}$. We put $M_n = \{\alpha \mid \alpha_{n-1}^* < \alpha \leq \alpha_{n-1}^{**} < \gamma_2 \leq \omega, \forall x\, b \lhd \delta_2(\alpha), \delta_2(\alpha) \lhd \delta_2(\alpha_{n-1}^*)\} \subseteq_{\mathcal{F}} \gamma_2$. Then $\alpha_{n-1}^{**} \in M_n \neq \emptyset$, there exists $\alpha_{n-1}^* < \alpha_n^* \leq \alpha_{n-1}^{**} < \gamma_2$ and $\alpha_n^* = \mathit{min}\, M_n$, $\alpha_n^* \in M_n$, $\forall x\, b \lhd \delta_2(\alpha_n^*)$, $\delta_2(\alpha_n^*) \lhd \delta_2(\alpha_{n-1}^*)$, for all $\alpha_{n-1}^* \leq \alpha < \alpha_n^* < \gamma_2$, either $\delta_2(\alpha) \lhd \forall x\, b$ or ($\delta_2(\alpha) = \forall x\, b$ or $\delta_2(\alpha) \triangleq \forall x\, b$) or $\forall x\, b \lhd \delta_2(\alpha)$, either $\delta_2(\alpha) \lhd \delta_2(\alpha_{n-1}^*)$ or ($\delta_2(\alpha) = \delta_2(\alpha_{n-1}^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_{n-1}^*)$) or $\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha)$; we have that $\forall x\, b \lhd \delta_2(\alpha_{n-1}^*)$, there does not exist a contradiction of $\mathbb{S}$; either $\delta_2(\alpha) \lhd \forall x\, b$ or ($\delta_2(\alpha) = \forall x\, b$ or $\delta_2(\alpha) \triangleq \forall x\, b$) or $\forall x\, b \lhd \delta_2(\alpha)$, $\delta_2(\alpha) \lhd \delta_2(\alpha_{n-1}^*)$ or ($\delta_2(\alpha) = \delta_2(\alpha_{n-1}^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_{n-1}^*)$) or $\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha)$. Let $\alpha_{n-1}^* < \alpha < \alpha_n^* \leq \alpha_{n-1}^{**} < \gamma_2$, $\forall x\, b \lhd \delta_2(\alpha)$, $\delta_2(\alpha) \lhd \delta_2(\alpha_{n-1}^*)$. Then $\alpha \in M_n$, $\alpha < \alpha_n^* = \mathit{min}\, M_n$, which is a contradiction. Hence, for all $\alpha_{n-1}^* \leq \alpha < \alpha_n^*$, either $\delta_2(\alpha) \lhd \forall x\, b$ or ($\delta_2(\alpha) = \forall x\, b$ or $\delta_2(\alpha) \triangleq \forall x\, b$) or ($\delta_2(\alpha) = \delta_2(\alpha_{n-1}^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_{n-1}^*)$) or $\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha)$; we have $\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha_{n-2}^*)$; for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_{n-1}^*]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or ($c = \delta_2(\alpha_{n-1}^*)$ or $c \triangleq \delta_2(\alpha_{n-1}^*)$) or $\delta_2(\alpha_{n-1}^*) \lhd c$, for all $c \in ((\mathit{tcons}(S) \cup \delta_2[\alpha_{n-1}^*]) - \{\mathit{0}, \mathit{1}\}) \cup \{\delta_2(\alpha) \mid \alpha_{n-1}^* \leq \alpha < \alpha_n^*\} = (\mathit{tcons}(S) \cup \delta_2[\alpha_n^*]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or ($c = \delta_2(\alpha_{n-1}^*)$ or $c \triangleq \delta_2(\alpha_{n-1}^*)$) or $\delta_2(\alpha_{n-1}^*) \lhd c$; we have $\alpha_n^* < \gamma_2$; $\alpha_n^* + 1 \leq \gamma_2$ is a successor ordinal; we have that $\forall x\, b \lhd \delta_2(\alpha_n^*)$, $\delta_2(\alpha_n^*) \lhd \delta_2(\alpha_{n-1}^*)$, there does not exist a contradiction of $\mathbb{S}$; $\mathit{0} \not\triangleq \delta_2(\alpha_n^*)$, $\mathit{1} \not\triangleq \delta_2(\alpha_n^*)$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_n^*]) - \{\mathit{0}, \mathit{1}\}$, $c \not\triangleq \delta_2(\alpha_n^*)$, for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_n^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_n^*})$, $c \not\triangleq \delta_2(\alpha_n^*)$, $\mathbb{E}_{\alpha_n^*} = \emptyset$; $\alpha^* \leq \kappa < \alpha_{n-1}^* < \alpha_n^*$, $\delta_2(\alpha^*), \delta_2(\alpha_{n-1}^*) \in \mathit{tcons}(S) \cup \delta_2[\alpha_n^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_n^*})$, $\forall x\, b = \delta_2(\alpha^*) \in \mathit{dom}(\mathcal{V}_{\alpha_n^*})$, $\mathcal{V}_{\alpha_n^*}(\mathit{0}) = 0 \leq \mathcal{V}_{\alpha_n^*}(\forall x\, b)$, $\mathit{1} \not\lhd \delta_2(\alpha_n^*)$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_n^*]) - \{\mathit{0}, \mathit{1}\}$ and $c \lhd \delta_2(\alpha_n^*)$, either $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$), for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_n^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_n^*})$ and $c \lhd \delta_2(\alpha_n^*)$, $c \lhd \forall x\, b$ or ($c = \forall x\, b$ or $c \triangleq \forall x\, b$) or $c = \mathit{0}$, by (102) for α_n^*, c, $\forall x\, b$, either $\mathcal{V}_{\alpha_n^*}(c) < \mathcal{V}_{\alpha_n^*}(\forall x\, b)$ or $\mathcal{V}_{\alpha_n^*}(c) = \mathcal{V}_{\alpha_n^*}(\forall x\, b)$; $\mathcal{V}_{\alpha_n^*}(c) \leq \mathcal{V}_{\alpha_n^*}(\forall x\, b)$; $\mathcal{V}_{\alpha_n^*}(\forall x\, b) \in \mathbb{D}_{\alpha_n^*}$, $\bigvee \mathbb{D}_{\alpha_n^*} = \mathcal{V}_{\alpha_n^*}(\forall x\, b)$; $\mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*)) \leq 1 = \mathcal{V}_{\alpha_n^*}(\mathit{1})$, $\delta_2(\alpha_n^*) \not\lhd \mathit{0}$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_n^*]) - \{\mathit{0}, \mathit{1}\}$ and $\delta_2(\alpha_n^*) \lhd c$, either ($c = \delta_2(\alpha_{n-1}^*)$ or $c \triangleq \delta_2(\alpha_{n-1}^*)$) or $\delta_2(\alpha_{n-1}^*) \lhd c$, for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_n^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_n^*})$ and $\delta_2(\alpha_n^*) \lhd c$, ($c = \delta_2(\alpha_{n-1}^*)$ or $c \triangleq \delta_2(\alpha_{n-1}^*)$) or $\delta_2(\alpha_{n-1}^*) \lhd c$ or $c = \mathit{1}$, by (102) for α_n^*, $\delta_2(\alpha_{n-1}^*)$, c, either $\mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*)) = \mathcal{V}_{\alpha_n^*}(c)$ or $\mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*)) < \mathcal{V}_{\alpha_n^*}(c)$; $\mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*)) \leq \mathcal{V}_{\alpha_n^*}(c)$; $\mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*)) \in \mathbb{U}_{\alpha_n^*}$, $\bigwedge \mathbb{U}_{\alpha_n^*} = \mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*))$; we have $\mathbb{E}_{\alpha_n^*} = \emptyset$; $\mathcal{V}(\delta_2(\alpha_n^*)) \stackrel{(94)}{=} \mathcal{V}_{\alpha_n^*+1}(\delta_2(\alpha_n^*)) = \dfrac{\bigvee \mathbb{D}_{\alpha_n^*} + \bigwedge \mathbb{U}_{\alpha_n^*}}{2} = \dfrac{\mathcal{V}_{\alpha_n^*}(\forall x\, b) + \mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*))}{2} \stackrel{(94)}{=} \dfrac{\mathcal{V}(\forall x\, b) + \mathcal{V}(\delta_2(\alpha_{n-1}^*))}{2} = \mathcal{V}(\forall x\, b) + \dfrac{\mathcal{V}(\delta_2(\alpha_{n-1}^*)) - \mathcal{V}(\forall x\, b)}{2} = \mathcal{V}(\forall x\, b) + \dfrac{\mathcal{V}(\forall x\, b) + \dfrac{D}{2^{n-1}} - \mathcal{V}(\forall x\, b)}{2} = \mathcal{V}(\forall x\, b) + \dfrac{D}{2^n}$; we have $\forall x\, b \lhd \delta_2(\alpha_n^*)$; by (108) for α_n^*, there exists $\tilde{w}_n(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_n^*))) \in \mathcal{U}_{\mathfrak{A}}$ and

$$b(x/\tilde{w}_n(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_n^*)))) \in \mathcal{B},$$
$$\forall x\, b \lhd b(x/\tilde{w}_n(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_n^*)))),$$
$$b(x/\tilde{w}_n(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_n^*)))) \lhd \delta_2(\alpha_n^*);$$

$b(x/\tilde{w}_n(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_n^*)))) \in \mathcal{B} - \mathit{tcons}(S)$, $\delta_2(\alpha_n^*) \not\lhd \delta_2(\alpha_n^*)$, there exists $\kappa, \alpha_{n-1}^* < \alpha_n^* < \alpha_n^{**} < \gamma_2$ and $\delta_2(\alpha_n^{**}) = b(x/\tilde{w}_n(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_n^*))))$, $\forall x\, b \lhd \delta_2(\alpha_n^{**})$, $\delta_2(\alpha_n^{**}) \lhd \delta_2(\alpha_n^*)$, by (105) for $\forall x\, b$, $\delta_2(\alpha_n^{**})$, $\delta_2(\alpha_n^*)$, $\mathcal{V}(\forall x\, b) < \mathcal{V}(\delta_2(\alpha_n^{**}))$, $\mathcal{V}(\delta_2(\alpha_n^{**})) < \mathcal{V}(\delta_2(\alpha_n^*))$; (110) holds.

So, in both Cases 2.3 and 2.4, (110) holds. The induction is completed. Thus, (110) holds.

Then, by (110),

$$\bigwedge_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u)) \leq \bigwedge_{n \geq 1} \mathcal{V}(b(x/\tilde{w}_n(\mathit{freetermseq}(\forall x\, b), \mathit{freetermseq}(\delta_2(\alpha_n^*))))) =$$
$$\bigwedge_{n \geq 1} \mathcal{V}(\delta_2(\alpha_n^{**})) \leq \bigwedge_{n \geq 1} \mathcal{V}(\delta_2(\alpha_n^*)) = \bigwedge_{n \geq 1} \left(\mathcal{V}(\forall x\, b) + \frac{D}{2^n}\right) = \mathcal{V}(\forall x\, b) + \bigwedge_{n \geq 1} \frac{D}{2^n} = \mathcal{V}(\forall x\, b),$$

$\mathcal{V}(\forall x\, b) = \bigwedge_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u))$.

So, in both Cases 1 and 2, $\mathcal{V}(\forall x\, b) = \bigwedge_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u))$; $\mathcal{V}(\forall x\, b) = \bigwedge_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u))$.

Let $a = \exists x\, b \in \mathcal{B}$. Then $\exists x\, b \in \mathcal{B} - \mathit{tcons}(S)$, there exists $\alpha^* < \gamma_2$ and $\delta_2(\alpha^*) = \exists x\, b$; $\exists x\, b \in \mathit{qatoms}(\mathbb{S}) \subseteq \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$. Let $u \in \mathcal{U}_{\mathfrak{A}}$. Hence, by (106) for $\exists x\, b$, u, $b(x/u) \in \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S))$, by (97) for $\exists x\, b$, $b(x/u)$, there exist a deduction $\mathcal{D} = C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq \mathit{GOrdCl}_{\mathcal{L}_n}$, and $\exists x\, b, b(x/u) \in \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n)$, $\exists x\, b \in \mathit{qatoms}^{\exists}(S_n) \subseteq \mathit{QAtom}_{\mathcal{L}_n}$, $b(x/u) \in \mathit{atoms}(S_n) \subseteq \mathit{GAtom}_{\mathcal{L}_n}$, $u \in \mathit{GTerm}_{\mathcal{L}_n}$. We put $\gamma = x/u \in \mathit{Subst}_{\mathcal{L}_n}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(b)$. Then $b\gamma = b(x/u)$, using Rule (46) with respect to $\mathcal{L}_n$, S_n, we derive $b\gamma \prec \exists x\, b \vee b\gamma \eqcirc \exists x\, b = b(x/u) \prec \exists x\, b \vee b(x/u) \eqcirc \exists x\, b \in \mathit{GOrdCl}_{\mathcal{L}_n}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n$, $C_{n+1} = b(x/u) \prec \exists x\, b \vee b(x/u) \eqcirc \exists x\, b \in \mathit{GOrdCl}_{\mathcal{L}_n} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Hence, $C_{n+1} \in \mathit{clo}^{\mathcal{BH}}(S)$, $b(x/u) \prec \exists x\, b \in \mathbb{S}$ or $b(x/u) \eqcirc \exists x\, b \in \mathbb{S}$, for both the cases $b(x/u) \prec \exists x\, b \in \mathbb{S}$ and $b(x/u) \eqcirc \exists x\, b \in \mathbb{S}$, $b(x/u) \in \mathit{atoms}(\mathbb{S})$; $b(x/u) \in \mathit{atoms}(\mathbb{S}) \subseteq \mathcal{B}$; we have that there does not exist a contradiction of $\mathbb{S}$; either $b(x/u) \prec \exists x\, b \in \mathbb{S}$ or $b(x/u) \eqcirc \exists x\, b \in \mathbb{S}$, either $b(x/u) \lhd \exists x\, b$ or $b(x/u) \triangleq \exists x\, b$, by (105) for $b(x/u)$, $\exists x\, b$, either $\mathcal{V}(b(x/u)) < \mathcal{V}(\exists x\, b)$ or $\mathcal{V}(b(x/u)) = \mathcal{V}(\exists x\, b)$; $\bigvee_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u)) \leq \mathcal{V}(\exists x\, b)$. We distinguish two cases.

Case 1: There exists $u^* \in \mathcal{U}_{\mathfrak{A}}$ such that $b(x/u^*) \in \mathcal{B}$ and $b(x/u^*) \triangleq \exists x\, b$. Then, by (105) for $b(x/u^*)$, $\exists x\, b$, $\mathcal{V}(b(x/u^*)) = \mathcal{V}(\exists x\, b)$, $\mathcal{V}(\exists x\, b) = \mathcal{V}(b(x/u^*)) \leq \bigvee_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u))$, $\mathcal{V}(\exists x\, b) = \bigvee_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u))$.

Case 2: For all $u \in \mathcal{U}_{\mathfrak{A}}$, $b(x/u) \in \mathcal{B}$ and $b(x/u) \lhd \exists x\, b$. At first, we prove the following statements:

For all $\alpha < \gamma_2$ and $\delta_2(\alpha) \lhd \exists x\, b$, there exists $\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha))) \in \mathcal{U}_{\mathfrak{A}}$ such that (111)

$$\begin{gathered} b(x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \in \mathcal{B}, \\ \delta_2(\alpha) \lhd b(x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))), \\ b(x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \lhd \exists x\, b. \end{gathered}$$

Let $\alpha < \gamma_2$ and $\delta_2(\alpha) \lhd \exists x\, b$. Then $\delta_2(\alpha) \in \mathcal{B} \subseteq \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S)) \cup \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$, by (97) for $\exists x\, b$, $\delta_2(\alpha)$, there exist a deduction $C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq \mathit{GOrdCl}_{\mathcal{L}_n}$, and $\exists x\, b, \delta_2(\alpha) \in \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n)$; $\exists x\, b \in \mathit{qatoms}^{\exists}(S_n) \subseteq \mathit{QAtom}_{\mathcal{L}_n}$, $\delta_2(\alpha) \in \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n) \subseteq \mathit{GAtom}_{\mathcal{L}_n} \cup \mathit{QAtom}_{\mathcal{L}_n}$, there exists $\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_n}$ and $\mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha))|$. We put $\gamma = x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha))) \in \mathit{Subst}_{\mathcal{L}_n \cup \{\tilde{w}\}}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(b)$. Hence, $\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha))) \in \mathit{GTerm}_{\mathcal{L}_n \cup \{\tilde{w}\}}$, $b\gamma = b(x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha))))$, using Rule (48) with respect to $\mathcal{L}_n$, S_n, we derive $\delta_2(\alpha) \prec b\gamma \vee \exists x\, b \eqcirc \delta_2(\alpha) \vee \exists x\, b \prec \delta_2(\alpha) = \delta_2(\alpha) \prec b(x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \vee \exists x\, b \eqcirc \delta_2(\alpha) \vee \exists x\, b \prec \delta_2(\alpha) \in \mathit{GOrdCl}_{\mathcal{L}_n \cup \{\tilde{w}\}}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{\tilde{w}\}$, $\gamma \in \mathit{Subst}_{\mathcal{L}_n \cup \{\tilde{w}\}} = \mathit{Subst}_{\mathcal{L}_{n+1}}$, $\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha))) \in \mathit{GTerm}_{\mathcal{L}_n \cup \{\tilde{w}\}} = \mathit{GTerm}_{\mathcal{L}_{n+1}} \subseteq \mathit{GTerm}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $C_{n+1} = \delta_2(\alpha) \prec b(x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \vee \exists x\, b \eqcirc \delta_2(\alpha) \vee \exists x\, b \prec \delta_2(\alpha) \in \mathit{GOrdCl}_{\mathcal{L}_n \cup \{\tilde{w}\}} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} \cup \mathit{GOrdCl}_{\mathcal{L}_{n+1}} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $C_{n+1} \in \mathit{clo}^{\mathcal{BH}}(S)$, $\delta_2(\alpha) \prec b(x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta(\alpha)))) \in \mathbb{S}$ or $\exists x\, b \eqcirc \delta_2(\alpha) \in \mathbb{S}$ or $\exists x\, b \prec \delta_2(\alpha) \in \mathbb{S}$; we have that $\delta_2(\alpha) \lhd \exists x\, b$, there does not exist a contradiction of $\mathbb{S}$; $\delta_2(\alpha) \prec b(x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \in \mathbb{S}$, $\mathit{funcs}(\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \subseteq \mathit{Func}_{\mathcal{L}} \cup \tilde{\mathbb{W}}$, $\mathit{funcs}(\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \cap \tilde{\mathbb{W}} \subseteq \mathit{funcs}(\mathbb{S}) \cap \tilde{\mathbb{W}} = \tilde{\mathbb{W}}^*$, $\mathit{funcs}(\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \subseteq \mathit{Func}_{\mathcal{L}} \cup \tilde{\mathbb{W}}^*$, $\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha))) \in \mathit{GTerm}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P} = \mathcal{U}_{\mathfrak{A}}$,

$$\begin{gathered} b(x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \in \mathcal{B}, \\ \delta_2(\alpha) \lhd b(x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))), \\ b(x/\tilde{w}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha)))) \lhd \exists x\, b; \end{gathered}$$

(111) holds.

Let there exist $c^* \in \mathit{tcons}(S)$ and $c^* \lhd \exists x\, b$. There exists $\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)) \in \mathcal{U}_{\mathfrak{A}}$ such that $b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \in \mathcal{B}$, for all $c \in \mathit{tcons}(S)$ and $c \lhd \exists x\, b$, (112)

$$\begin{gathered} c \lhd b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))), \\ b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \lhd \exists x\, b. \end{gathered}$$

We have $\exists x\, b \in \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$. Then there exists $C \in \mathit{clo}^{\mathcal{BH}}(S)$ and $\exists x\, b \in \mathit{qatoms}(C)$; there exists a deduction $\mathcal{D} = C_1, \dots, C_n = C$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $C = C_n \in S_n \subseteq \mathit{GOrdCl}_{\mathcal{L}_n}$, and $\exists x\, b \in \mathit{qatoms}^{\exists}(C) \subseteq \mathit{qatoms}^{\exists}(S_n) \subseteq \mathit{QAtom}_{\mathcal{L}_n}$; there exists $\tilde{w}^* \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_n}$ and $\mathit{ar}(\tilde{w}^*) = |\mathit{freetermseq}(\exists x\, b)|$. Let $c \in \mathit{tcons}(S)$ and $c \lhd \exists x\, b$. We have that there does not exist a contradiction of $\mathbb{S}$. Hence, $c \neq \mathit{1}$ and $c \lhd \mathit{1} \in \mathit{ordtcons}(S) \subseteq \mathit{ordtcons}(S) \cup \mathit{GInst}_{\mathcal{L}_n}(S)$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n$, $C_{n+1} = c \lhd \mathit{1} \in \mathit{ordtcons}(S) \cup \mathit{GInst}_{\mathcal{L}_n}(S) \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $\exists x\, b \in \mathit{qatoms}^{\exists}(S_n) \subseteq \mathit{qatoms}^{\exists}(S_{n+1}) \subseteq \mathit{QAtom}_{\mathcal{L}_{n+1}}$, $\tilde{w}^* \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_n} = \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_{n+1}}$, $\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)) = \tilde{w}^*(\mathit{freetermseq}(\exists x\, b), \ell) = \tilde{w}^*(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(c))$, $c \in \mathit{atoms}(C_{n+1}) \subseteq \mathit{atoms}(S_{n+1}) \subseteq \mathit{GAtom}_{\mathcal{L}_{n+1}}$. We put $\gamma = x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(c)) = x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)) \in \mathit{Subst}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(b)$. Hence, $\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)) \in \mathit{GTerm}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}}$, $b\gamma = b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)))$, using Rule (48) with respect to $\mathcal{L}_{n+1}$, S_{n+1}, we derive $c \prec b\gamma \vee \exists x\, b \eqcirc c \vee \exists x\, b \prec c = c \prec b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \vee \exists x\, b \eqcirc c \vee \exists x\, b \prec c \in \mathit{GOrdCl}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}}$. We put $\mathcal{L}_{n+2} = \mathcal{L}_{n+1} \cup \{\tilde{w}^*\}$, $\gamma \in \mathit{Subst}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}} = \mathit{Subst}_{\mathcal{L}_{n+2}}$, $\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)) \in \mathit{GTerm}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}} = \mathit{GTerm}_{\mathcal{L}_{n+2}} \subseteq \mathit{GTerm}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $C_{n+2} = c \prec b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \vee \exists x\, b \eqcirc c \vee \exists x\, b \prec c \in \mathit{GOrdCl}_{\mathcal{L}_{n+1} \cup \{\tilde{w}^*\}} = \mathit{GOrdCl}_{\mathcal{L}_{n+2}} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}'' = \mathcal{D}', C_{n+2}$, $S_{n+2} = S_{n+1} \cup \{C_{n+2}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_{n+1}} \cup \mathit{GOrdCl}_{\mathcal{L}_{n+2}} = \mathit{GOrdCl}_{\mathcal{L}_{n+2}}$; $\mathcal{D}''$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $C_{n+2} \in \mathit{clo}^{\mathcal{BH}}(S)$, $c \prec b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \in \mathbb{S}$ or $\exists x\, b \eqcirc c \in \mathbb{S}$ or $\exists x\, b \prec c \in \mathbb{S}$; we have that $c \lhd \exists x\, b$, there does not exist a contradiction of $\mathbb{S}$; $c \prec b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \in \mathbb{S}$, $\mathit{funcs}(\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \subseteq \mathit{Func}_{\mathcal{L}} \cup \tilde{\mathbb{W}}$, $\mathit{funcs}(\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \cap \tilde{\mathbb{W}} \subseteq \mathit{funcs}(\mathbb{S}) \cap \tilde{\mathbb{W}} = \tilde{\mathbb{W}}^*$, $\mathit{funcs}(\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \subseteq \mathit{Func}_{\mathcal{L}} \cup \tilde{\mathbb{W}}^*$, $\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)) \in \mathit{GTerm}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P} = \mathcal{U}_{\mathfrak{A}}$,

$$\begin{gathered} b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \in \mathcal{B}, \\ c \lhd b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))), \\ b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \lhd \exists x\, b; \end{gathered}$$

for $c = c^*$, $\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)) \in \mathcal{U}_{\mathfrak{A}}$, $b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \in \mathcal{B}$, for all $c \in \mathit{tcons}(S)$ and $c \lhd \exists x\, b$, $c \lhd b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)))$, $b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \lhd \exists x\, b$; (112) holds.

We have $\alpha^* < \gamma_2$, $\exists x\, b = \delta_2(\alpha^*)$, for all $u \in \mathcal{U}_{\mathfrak{A}} \neq \emptyset$, $b(x/u) \in \mathcal{B}$, $b(x/u) \lhd \exists x\, b$. Then there exist $u^* \in \mathcal{U}_{\mathfrak{A}} \neq \emptyset$ and $b(x/u^*) \in \mathcal{B}$, $b(x/u^*) \lhd \exists x\, b$; $b(x/u^*) \in \mathcal{B} - \mathit{tcons}(S)$, there exists $\beta^* < \gamma_2$ and $\delta_2(\beta^*) = b(x/u^*)$, $\delta_2(\beta^*) = b(x/u^*) \lhd \exists x\, b$. We get two cases.

Case 2.1: There exists $c^* \in \mathit{tcons}(S)$ and $c^* \lhd \exists x\, b$. Then, by (112), there exists $\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)) \in \mathcal{U}_{\mathfrak{A}}$ and $b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \in \mathcal{B}$, for all $c \in \mathit{tcons}(S)$ and $c \lhd \exists x\, b$, $c \lhd b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)))$, $b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \lhd \exists x\, b$; $b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b))) \in \mathcal{B} - \mathit{tcons}(S)$, there exists $\beta^{**} < \gamma_2$ and $\delta_2(\beta^{**}) = b(x/\tilde{w}^*(\mathit{freetermseq}(\exists x\, b)))$, for all $c \in \mathit{tcons}(S)$ and $c \lhd \exists x\, b$, $c \lhd \delta_2(\beta^{**})$, $\delta_2(\beta^{**}) \lhd \exists x\, b$. We put $\kappa = \mathit{max}(\alpha^*, \beta^*, \beta^{**}) < \gamma_2 \leq \omega$ and $M = \{\delta_2(\alpha) \mid \alpha \leq \kappa < \gamma_2 \leq \omega, \delta_2(\alpha) \lhd \exists x\, b\} \subseteq_{\mathcal{F}} \mathcal{B} - \mathit{tcons}(S)$. Hence, $\beta^*, \beta^{**} \leq \kappa$ and $\delta_2(\beta^*), \delta_2(\beta^{**}) \in M \neq \emptyset$; we have $\mathit{qatoms}(S) \neq \emptyset$; by (101), for all $c, d \in \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd d$ or ($c = d$ or $c \triangleq d$) or $d \lhd c$, there exists $\alpha_0^* \leq \kappa < \gamma_2$ and $\delta_2(\alpha_0^*) \in M$ is a maximal element of M with respect to $\lhd$; for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\} \subseteq \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$, either $c \lhd \exists x\, b$ or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$, $\delta_2(\alpha_0^*) \lhd \exists x\, b$; we have that there does not exist a contradiction of $\mathbb{S}$; either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$, $c \lhd \exists x\, b$ or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$. Let $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\} = (\mathit{tcons}(S) - \{\mathit{0}, \mathit{1}\}) \cup \delta_2[\kappa+1]$, $\delta_2(\alpha_0^*) \lhd c$, $c \lhd \exists x\, b$. We get two cases for c.

Case 2.1.1: $c \in \mathit{tcons}(S) - \{\mathit{0}, \mathit{1}\}$. Then $c \lhd \delta_2(\beta^{**})$, $\delta_2(\alpha_0^*) \lhd \delta_2(\beta^{**})$; we have $\delta_2(\beta^{**}) \in M$; which is a contradiction that $\delta_2(\alpha_0^*)$ is a maximal element of M with respect to $\lhd$.

Case 2.1.2: $c \in \delta_2[\kappa+1]$. Then there exists $\alpha_c \leq \kappa$ and $\delta_2(\alpha_c) = c$, $\delta_2(\alpha_c) = c \lhd \exists x\, b$, $c = \delta_2(\alpha_c) \in M$; we have $\delta_2(\alpha_0^*) \lhd c$; which is a contradiction that $\delta_2(\alpha_0^*)$ is a maximal element of M with respect to $\lhd$.

Hence, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$; $\delta_2(\alpha_0^*) \lhd \exists x\, b$, by (105) for $\delta_2(\alpha_0^*)$, $\exists x\, b$, $\mathcal{V}(\exists x\, b) > \mathcal{V}(\delta_2(\alpha_0^*))$. We put $D = \mathcal{V}(\exists x\, b) - \mathcal{V}(\delta_2(\alpha_0^*))$, $0 < D \leq 1$. Then, by (111) for α_0^*, there exists $\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))) \in \mathcal{U}_{\mathfrak{A}}$ and

$$\begin{aligned}&b(x/\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \in \mathcal{B},\\ &\delta_2(\alpha_0^*) \lhd b(x/\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))),\\ &b(x/\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \lhd \exists x\, b;\end{aligned}$$

$b(x/\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \in \mathcal{B} - \mathit{tcons}(S)$, there exists $\kappa < \alpha_0^{**} < \gamma_2$ and $\delta_2(\alpha_0^{**}) = b(x/\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))))$, $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha_0^{**})$, $\delta_2(\alpha_0^{**}) \lhd \exists x\, b$.

Case 2.2: There does not exist $c^* \in \mathit{tcons}(S)$ and $c^* \lhd \exists x\, b$. We put $\kappa = \mathit{max}(\alpha^*, \beta^*) < \gamma_2 \leq \omega$ and $M = \{\delta_2(\alpha) \mid \alpha \leq \kappa < \gamma_2 \leq \omega, \delta_2(\alpha) \lhd \exists x\, b\} \subseteq_{\mathcal{F}} \mathcal{B} - \mathit{tcons}(S)$. Hence, $\beta^* \leq \kappa$ and $\delta_2(\beta^*) \in M \neq \emptyset$; we have, for all $c, d \in \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd d$ or ($c = d$ or $c \triangleq d$) or $d \lhd c$; there exists $\alpha_0^* \leq \kappa < \gamma_2$ and $\delta_2(\alpha_0^*) \in M$ is a maximal element of M with respect to $\lhd$; for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\} \subseteq \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$, either $c \lhd \exists x\, b$ or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$, $\delta_2(\alpha_0^*) \lhd \exists x\, b$; we have that there does not exist a contradiction of $\mathbb{S}$; either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd c$, $c \lhd \exists x\, b$ or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$. Let $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\} = (\mathit{tcons}(S) - \{\mathit{0}, \mathit{1}\}) \cup \delta_2[\kappa+1]$, $\delta_2(\alpha_0^*) \lhd c$, $c \lhd \exists x\, b$. We get two cases for c.

Case 2.2.1: $c \in \mathit{tcons}(S) - \{\mathit{0}, \mathit{1}\}$. Then $c \in \mathit{tcons}(S) - \{\mathit{0}, \mathit{1}\} \subseteq \mathit{tcons}(S)$; we have $c \lhd \exists x\, b$; which is a contradiction that there does not exist $c^* \in \mathit{tcons}(S)$ and $c^* \lhd \exists x\, b$.

Case 2.2.2: $c \in \delta_2[\kappa+1]$. Then there exists $\alpha_c \leq \kappa$ and $c = \delta_2(\alpha_c)$, $\delta_2(\alpha_c) = c \lhd \exists x\, b$, $c = \delta_2(\alpha_c) \in M$; we have $\delta_2(\alpha_0^*) \lhd c$; which is a contradiction that $\delta_2(\alpha_0^*)$ is a maximal element of M with respect to $\lhd$.

Hence, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$; $\delta_2(\alpha_0^*) \lhd \exists x\, b$, by (105) for $\delta_2(\alpha_0^*)$, $\exists x\, b$, $\mathcal{V}(\exists x\, b) > \mathcal{V}(\delta_2(\alpha_0^*))$. We put $D = \mathcal{V}(\exists x\, b) - \mathcal{V}(\delta_2(\alpha_0^*))$, $0 < D \leq 1$. Then, by (111) for α_0^*, there exists $\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))) \in \mathcal{U}_{\mathfrak{A}}$ and

$$\begin{aligned}&b(x/\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \in \mathcal{B},\\ &\delta_2(\alpha_0^*) \lhd b(x/\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))),\\ &b(x/\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \lhd \exists x\, b;\end{aligned}$$

$b(x/\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*)))) \in \mathcal{B} - \mathit{tcons}(S)$, there exists $\kappa < \alpha_0^{**} < \gamma_2$ and $\delta_2(\alpha_0^{**}) = b(x/\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))))$, $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha_0^{**})$, $\delta_2(\alpha_0^{**}) \lhd \exists x\, b$.

So, in both Cases 2.1 and 2.2, there exist $\alpha^*, \alpha_0^* \leq \kappa < \alpha_0^{**} < \gamma_2$, $\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))) \in \mathcal{U}_{\mathfrak{A}}$, $0 < D = \mathcal{V}(\exists x\, b) - \mathcal{V}(\delta_2(\alpha_0^*)) \leq 1$ such that $\delta_2(\alpha_0^*) \lhd \exists x\, b = \delta_2(\alpha^*)$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$, $\delta_2(\alpha_0^{**}) = b(x/\tilde{w}_0(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_0^*))))$, $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha_0^{**})$, $\delta_2(\alpha_0^{**}) \lhd \exists x\, b$.

We next prove the following statement:

For all $n \geq 1$, there exist $\kappa, \alpha_{n-1}^* < \alpha_n^* < \alpha_n^{**} < \gamma_2$, $\tilde{w}_n(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_n^*))) \in \mathcal{U}_{\mathfrak{A}}$ such that (113)

$$\begin{aligned}&\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha_n^*), \delta_2(\alpha_n^*) \lhd \exists x\, b,\\ &\text{for all } c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_n^*]) - \{\mathit{0}, \mathit{1}\},\\ &\quad \text{either } c \lhd \delta_2(\alpha_{n-1}^*) \text{ or } (c = \delta_2(\alpha_{n-1}^*) \text{ or } c \triangleq \delta_2(\alpha_{n-1}^*)) \text{ or } (c = \exists x\, b \text{ or } c \triangleq \exists x\, b) \text{ or } \exists x\, b \lhd c,\\ &\delta_2(\alpha_n^{**}) = b(x/\tilde{w}_n(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_n^*)))),\\ &\delta_2(\alpha_n^*) \lhd \delta_2(\alpha_n^{**}), \delta_2(\alpha_n^{**}) \lhd \exists x\, b,\\ &\mathcal{V}(\delta_2(\alpha_n^*)) = \mathcal{V}(\exists x\, b) - \frac{D}{2^n}, \mathcal{V}(\delta_2(\alpha_n^*)) < \mathcal{V}(\delta_2(\alpha_n^{**})) < \mathcal{V}(\exists x\, b).\end{aligned}$$

We proceed by induction on $1 \leq n$.

Case 2.3 (the base case): $n = 1$. We have $\alpha^*, \alpha_0^* \leq \kappa < \alpha_0^{**} < \gamma_2$, $\delta_2(\alpha_0^*) \lhd \exists x\, b = \delta_2(\alpha^*)$, $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha_0^{**})$, $\delta_2(\alpha_0^{**}) \lhd \exists x\, b$, $0 < D = \mathcal{V}(\exists x\, b) - \mathcal{V}(\delta_2(\alpha_0^*)) \leq 1$, for all $c, d \in \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd d$ or ($c = d$ or $c \triangleq d$) or $d \lhd c$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$. We put $M_1 = \{\alpha \,|\, \kappa < \alpha \leq \alpha_0^{**} < \gamma_2 \leq \omega, \delta_2(\alpha_0^*) \lhd \delta_2(\alpha), \delta_2(\alpha) \lhd \exists x\, b\} \subseteq_{\mathcal{F}} \gamma_2$. Then $\alpha_0^{**} \in M_1 \neq \emptyset$, there exists $\kappa < \alpha_1^* \leq \alpha_0^{**} < \gamma_2$ and $\alpha_1^* = \mathit{min}\, M_1$, $\alpha_1^* \in M_1$, $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha_1^*)$, $\delta_2(\alpha_1^*) \lhd \exists x\, b$, for all $\kappa < \alpha < \alpha_1^* < \gamma_2$, either $\delta_2(\alpha) \lhd \delta_2(\alpha_0^*)$ or ($\delta_2(\alpha) = \delta_2(\alpha_0^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha)$, either $\delta_2(\alpha) \lhd \exists x\, b$ or ($\delta_2(\alpha) = \exists x\, b$ or $\delta_2(\alpha) \triangleq \exists x\, b$) or $\exists x\, b \lhd \delta_2(\alpha)$; we have that $\delta_2(\alpha_0^*) \lhd \exists x\, b$, there does not exist a contradiction of $\mathbb{S}$; either $\delta_2(\alpha) \lhd \delta_2(\alpha_0^*)$ or ($\delta_2(\alpha) = \delta_2(\alpha_0^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_0^*)$) or $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha)$, $\delta_2(\alpha) \lhd \exists x\, b$ or ($\delta_2(\alpha) = \exists x\, b$ or $\delta_2(\alpha) \triangleq \exists x\, b$) or $\exists x\, b \lhd \delta_2(\alpha)$. Let $\kappa < \alpha < \alpha_1^* \leq \alpha_0^{**} < \gamma_2$, $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha)$, $\delta_2(\alpha) \lhd \exists x\, b$. Then $\alpha \in M_1$, $\alpha < \alpha_1^* = \mathit{min}\, M_1$, which is a contradiction. Hence, for all $\kappa < \alpha < \alpha_1^*$, either $\delta_2(\alpha) \lhd \delta_2(\alpha_0^*)$ or ($\delta_2(\alpha) = \delta_2(\alpha_0^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_0^*)$) or ($\delta_2(\alpha) = \exists x\, b$ or $\delta_2(\alpha) \triangleq \exists x\, b$) or $\exists x\, b \lhd \delta_2(\alpha)$, for all $c \in ((\mathit{tcons}(S) \cup \delta_2[\kappa+1]) - \{\mathit{0}, \mathit{1}\}) \cup \{\delta_2(\alpha) \,|\, \kappa < \alpha < \alpha_1^*\} = (\mathit{tcons}(S) \cup \delta_2[\alpha_1^*]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$; we have $\alpha_1^* < \gamma_2$; $\alpha_1^* + 1 \leq \gamma_2$ is a successor ordinal; we have that $\delta_2(\alpha_0^*) \lhd \delta_2(\alpha_1^*)$, $\delta_2(\alpha_1^*) \lhd \exists x\, b$, there does not exist a contradiction of $\mathbb{S}$; $\mathit{0} \not\triangleq \delta_2(\alpha_1^*)$, $\mathit{1} \not\triangleq \delta_2(\alpha_1^*)$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_1^*]) - \{\mathit{0}, \mathit{1}\}$, $c \not\triangleq \delta_2(\alpha_1^*)$, for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_1^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_1^*})$, $c \not\triangleq \delta_2(\alpha_1^*)$, $\mathbb{E}_{\alpha_1^*} = \emptyset$; $\alpha^*, \alpha_0^* \leq \kappa < \alpha_1^*$, $\delta_2(\alpha^*), \delta_2(\alpha_0^*) \in \mathit{tcons}(S) \cup \delta_2[\alpha_1^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_1^*})$, $\mathcal{V}_{\alpha_1^*}(\mathit{0}) = 0 \leq \mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*))$, $\mathit{1} \ntriangleleft \delta_2(\alpha_1^*)$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_1^*]) - \{\mathit{0}, \mathit{1}\}$ and $c \lhd \delta_2(\alpha_1^*)$, either $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$), for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_1^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_1^*})$ and $c \lhd \delta_2(\alpha_1^*)$, $c \lhd \delta_2(\alpha_0^*)$ or ($c = \delta_2(\alpha_0^*)$ or $c \triangleq \delta_2(\alpha_0^*)$) or $c = \mathit{0}$, by (102) for α_1^*, c, $\delta_2(\alpha_0^*)$, either $\mathcal{V}_{\alpha_1^*}(c) < \mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*))$ or $\mathcal{V}_{\alpha_1^*}(c) = \mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*))$; $\mathcal{V}_{\alpha_1^*}(c) \leq \mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*))$; $\mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*)) \in \mathbb{D}_{\alpha_1^*}$, $\bigvee \mathbb{D}_{\alpha_1^*} = \mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*))$; $\exists x\, b = \delta_2(\alpha^*) \in \mathit{dom}(\mathcal{V}_{\alpha_1^*})$, $\mathcal{V}_{\alpha_1^*}(\exists x\, b) \leq 1 = \mathcal{V}_{\alpha_1^*}(\mathit{1})$, $\delta_2(\alpha_1^*) \ntriangleleft \mathit{0}$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_1^*]) - \{\mathit{0}, \mathit{1}\}$ and $\delta_2(\alpha_1^*) \lhd c$, either ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$, for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_1^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_1^*})$ and $\delta_2(\alpha_1^*) \lhd c$, ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$ or $c = \mathit{1}$, by (102) for α_1^*, $\exists x\, b$, c, either $\mathcal{V}_{\alpha_1^*}(\exists x\, b) = \mathcal{V}_{\alpha_1^*}(c)$ or $\mathcal{V}_{\alpha_1^*}(\exists x\, b) < \mathcal{V}_{\alpha_1^*}(c)$; $\mathcal{V}_{\alpha_1^*}(\exists x\, b) \leq \mathcal{V}_{\alpha_1^*}(c)$; $\mathcal{V}_{\alpha_1^*}(\exists x\, b) \in \mathbb{U}_{\alpha_1^*}$, $\bigwedge \mathbb{U}_{\alpha_1^*} = \mathcal{V}_{\alpha_1^*}(\exists x\, b)$; we have $\mathbb{E}_{\alpha_1^*} = \emptyset$; $\mathcal{V}(\delta_2(\alpha_1^*)) \stackrel{(94)}{=} \mathcal{V}_{\alpha_1^*+1}(\delta_2(\alpha_1^*)) = \dfrac{\bigvee \mathbb{D}_{\alpha_1^*} + \bigwedge \mathbb{U}_{\alpha_1^*}}{2} = \dfrac{\mathcal{V}_{\alpha_1^*}(\delta_2(\alpha_0^*)) + \mathcal{V}_{\alpha_1^*}(\exists x\, b)}{2} \stackrel{(94)}{=} \dfrac{\mathcal{V}(\delta_2(\alpha_0^*)) + \mathcal{V}(\exists x\, b)}{2} = \mathcal{V}(\exists x\, b) - \dfrac{\mathcal{V}(\exists x\, b) - \mathcal{V}(\delta_2(\alpha_0^*))}{2} = \mathcal{V}(\exists x\, b) - \dfrac{D}{2}$; we have $\delta_2(\alpha_1^*) \lhd \exists x\, b$; by (111) for α_1^*, there exists $\tilde{w}_1(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*))) \in \mathcal{U}_{\mathfrak{A}}$ and

$$b(x/\tilde{w}_1(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*)))) \in \mathcal{B},$$
$$\delta_2(\alpha_1^*) \lhd b(x/\tilde{w}_1(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*)))),$$
$$b(x/\tilde{w}_1(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*)))) \lhd \exists x\, b;$$

$b(x/\tilde{w}_1(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*)))) \in \mathcal{B} - \mathit{tcons}(S)$, $\delta_2(\alpha_1^*) \ntriangleleft \delta_2(\alpha_1^*)$, there exists $\kappa, \alpha_0^* < \alpha_1^* < \alpha_1^{**} < \gamma_2$ and $\delta_2(\alpha_1^{**}) = b(x/\tilde{w}_1(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_1^*))))$, $\delta_2(\alpha_1^*) \lhd \delta_2(\alpha_1^{**})$, $\delta_2(\alpha_1^{**}) \lhd \exists x\, b$, by (105) for $\delta_2(\alpha_1^*)$, $\delta_2(\alpha_1^{**})$, $\exists x\, b$, $\mathcal{V}(\delta_2(\alpha_1^*)) < \mathcal{V}(\delta_2(\alpha_1^{**}))$, $\mathcal{V}(\delta_2(\alpha_1^{**})) < \mathcal{V}(\exists x\, b)$; (113) holds.

Case 2.4 (the induction case): $n > 1$. We have $\alpha^* \leq \kappa < \gamma_2$, $\exists x\, b = \delta_2(\alpha^*)$, for all $c, d \in \mathcal{B} - \{\mathit{0}, \mathit{1}\}$, either $c \lhd d$ or ($c = d$ or $c \triangleq d$) or $d \lhd c$. We get by induction hypothesis for $n-1$ that there exist $\kappa, \alpha_{n-2}^* < \alpha_{n-1}^* < \alpha_{n-1}^{**} < \gamma_2$, $\tilde{w}_{n-1}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_{n-1}^*))) \in \mathcal{U}_{\mathfrak{A}}$ satisfying $\delta_2(\alpha_{n-2}^*) \lhd \delta_2(\alpha_{n-1}^*)$, $\delta_2(\alpha_{n-1}^*) \lhd \exists x\, b$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_{n-1}^*]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \delta_2(\alpha_{n-2}^*)$ or ($c = \delta_2(\alpha_{n-2}^*)$ or $c \triangleq \delta_2(\alpha_{n-2}^*)$) or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$, $\delta_2(\alpha_{n-1}^{**}) = b(x/\tilde{w}_{n-1}(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha_{n-1}^*))))$, $\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha_{n-1}^{**})$, $\delta_2(\alpha_{n-1}^{**}) \lhd \exists x\, b$, $\mathcal{V}(\delta_2(\alpha_{n-1}^*)) = \mathcal{V}(\exists x\, b) - \dfrac{D}{2^{n-1}}$. We put $M_n = \{\alpha \,|\, \alpha_{n-1}^* < \alpha \leq \alpha_{n-1}^{**} < \gamma_2 \leq \omega, \delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha), \delta_2(\alpha) \lhd \exists x\, b\} \subseteq_{\mathcal{F}} \gamma_2$. Then $\alpha_{n-1}^{**} \in M_n \neq \emptyset$, there exists $\alpha_{n-1}^* < \alpha_n^* \leq \alpha_{n-1}^{**} < \gamma_2$ and $\alpha_n^* = \mathit{min}\, M_n$, $\alpha_n^* \in M_n$, $\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha_n^*)$, $\delta_2(\alpha_n^*) \lhd \exists x\, b$, for all $\alpha_{n-1}^* \leq \alpha < \alpha_n^* < \gamma_2$, either $\delta_2(\alpha) \lhd \delta_2(\alpha_{n-1}^*)$ or ($\delta_2(\alpha) = \delta_2(\alpha_{n-1}^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_{n-1}^*)$) or $\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha)$, either $\delta_2(\alpha) \lhd \exists x\, b$ or ($\delta_2(\alpha) = \exists x\, b$ or $\delta_2(\alpha) \triangleq \exists x\, b$) or $\exists x\, b \lhd \delta_2(\alpha)$; we have that $\delta_2(\alpha_{n-1}^*) \lhd \exists x\, b$, there does not exist a contradiction of $\mathbb{S}$; either $\delta_2(\alpha) \lhd \delta_2(\alpha_{n-1}^*)$ or ($\delta_2(\alpha) = \delta_2(\alpha_{n-1}^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_{n-1}^*)$) or $\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha)$, $\delta_2(\alpha) \lhd \exists x\, b$ or ($\delta_2(\alpha) = \exists x\, b$ or $\delta_2(\alpha) \triangleq \exists x\, b$) or $\exists x\, b \lhd \delta_2(\alpha)$. Let $\alpha_{n-1}^* < \alpha < \alpha_n^* \leq \alpha_{n-1}^{**} < \gamma_2$, $\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha)$, $\delta_2(\alpha) \lhd \exists x\, b$. Then $\alpha \in M_n$, $\alpha < \alpha_n^* = \mathit{min}\, M_n$, which is a contradiction. Hence, for all $\alpha_{n-1}^* \leq \alpha < \alpha_n^*$, either $\delta_2(\alpha) \lhd \delta_2(\alpha_{n-1}^*)$ or ($\delta_2(\alpha) = \delta_2(\alpha_{n-1}^*)$ or $\delta_2(\alpha) \triangleq \delta_2(\alpha_{n-1}^*)$) or ($\delta_2(\alpha) = \exists x\, b$ or $\delta_2(\alpha) \triangleq \exists x\, b$) or $\exists x\, b \lhd \delta_2(\alpha)$; we have $\delta_2(\alpha_{n-2}^*) \lhd \delta_2(\alpha_{n-1}^*)$; for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_{n-1}^*]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \delta_2(\alpha_{n-1}^*)$ or ($c = \delta_2(\alpha_{n-1}^*)$ or $c \triangleq \delta_2(\alpha_{n-1}^*)$) or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$, for all $c \in ((\mathit{tcons}(S) \cup \delta_2[\alpha_{n-1}^*]) - \{\mathit{0}, \mathit{1}\}) \cup \{\delta_2(\alpha) \,|\, \alpha_{n-1}^* \leq \alpha < \alpha_n^*\} = (\mathit{tcons}(S) \cup \delta_2[\alpha_n^*]) - \{\mathit{0}, \mathit{1}\}$, either $c \lhd \delta_2(\alpha_{n-1}^*)$ or ($c = \delta_2(\alpha_{n-1}^*)$ or $c \triangleq \delta_2(\alpha_{n-1}^*)$) or ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$; we have $\alpha_n^* < \gamma_2$; $\alpha_n^* + 1 \leq \gamma_2$ is a successor ordinal; we have that $\delta_2(\alpha_{n-1}^*) \lhd \delta_2(\alpha_n^*)$, $\delta_2(\alpha_n^*) \lhd \exists x\, b$, there does not exist a contradiction of $\mathbb{S}$; $\mathit{0} \not\triangleq \delta_2(\alpha_n^*)$, $\mathit{1} \not\triangleq \delta_2(\alpha_n^*)$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_n^*]) - \{\mathit{0}, \mathit{1}\}$, $c \not\triangleq \delta_2(\alpha_n^*)$, for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_n^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_n^*})$, $c \not\triangleq \delta_2(\alpha_n^*)$, $\mathbb{E}_{\alpha_n^*} = \emptyset$; $\alpha^* \leq \kappa < \alpha_{n-1}^* < \alpha_n^*$, $\delta_2(\alpha^*), \delta_2(\alpha_{n-1}^*) \in \mathit{tcons}(S) \cup \delta_2[\alpha_n^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_n^*})$, $\mathcal{V}_{\alpha_n^*}(\mathit{0}) = 0 \leq \mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*))$, $\mathit{1} \ntriangleleft \delta_2(\alpha_n^*)$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_n^*]) - \{\mathit{0}, \mathit{1}\}$ and $c \lhd \delta_2(\alpha_n^*)$, either $c \lhd \delta_2(\alpha_{n-1}^*)$ or ($c = \delta_2(\alpha_{n-1}^*)$ or $c \triangleq \delta_2(\alpha_{n-1}^*)$), for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_n^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_n^*})$ and $c \lhd \delta_2(\alpha_n^*)$, $c \lhd \delta_2(\alpha_{n-1}^*)$ or ($c = \delta_2(\alpha_{n-1}^*)$ or $c \triangleq \delta_2(\alpha_{n-1}^*)$) or $c = \mathit{0}$, by (102) for α_n^*, c, $\delta_2(\alpha_{n-1}^*)$, either $\mathcal{V}_{\alpha_n^*}(c) < \mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*))$ or $\mathcal{V}_{\alpha_n^*}(c) = \mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*))$; $\mathcal{V}_{\alpha_n^*}(c) \leq \mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*))$; $\mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*)) \in \mathbb{D}_{\alpha_n^*}$, $\bigvee \mathbb{D}_{\alpha_n^*} = \mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*))$; $\exists x\, b = \delta_2(\alpha^*) \in \mathit{dom}(\mathcal{V}_{\alpha_n^*})$, $\mathcal{V}_{\alpha_n^*}(\exists x\, b) \leq 1 = \mathcal{V}_{\alpha_n^*}(\mathit{1})$, $\delta_2(\alpha_n^*) \ntriangleleft \mathit{0}$, for all $c \in (\mathit{tcons}(S) \cup \delta_2[\alpha_n^*]) - \{\mathit{0}, \mathit{1}\}$ and $\delta_2(\alpha_n^*) \lhd c$, either ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$, for all $c \in \mathit{tcons}(S) \cup \delta_2[\alpha_n^*] \stackrel{(94)}{=} \mathit{dom}(\mathcal{V}_{\alpha_n^*})$ and $\delta_2(\alpha_n^*) \lhd c$, ($c = \exists x\, b$ or $c \triangleq \exists x\, b$) or $\exists x\, b \lhd c$ or $c = \mathit{1}$, by (102) for α_n^*, $\exists x\, b$, c, either $\mathcal{V}_{\alpha_n^*}(\exists x\, b) = \mathcal{V}_{\alpha_n^*}(c)$ or $\mathcal{V}_{\alpha_n^*}(\exists x\, b) < \mathcal{V}_{\alpha_n^*}(c)$; $\mathcal{V}_{\alpha_n^*}(\exists x\, b) \leq \mathcal{V}_{\alpha_n^*}(c)$; $\mathcal{V}_{\alpha_n^*}(\exists x\, b) \in \mathbb{U}_{\alpha_n^*}$, $\bigwedge \mathbb{U}_{\alpha_n^*} = \mathcal{V}_{\alpha_n^*}(\exists x\, b)$; we have $\mathbb{E}_{\alpha_n^*} = \emptyset$; $\mathcal{V}(\delta_2(\alpha_n^*)) \stackrel{(94)}{=} \mathcal{V}_{\alpha_n^*+1}(\delta_2(\alpha_n^*)) = \dfrac{\bigvee \mathbb{D}_{\alpha_n^*} + \bigwedge \mathbb{U}_{\alpha_n^*}}{2} = \dfrac{\mathcal{V}_{\alpha_n^*}(\delta_2(\alpha_{n-1}^*)) + \mathcal{V}_{\alpha_n^*}(\exists x\, b)}{2} \stackrel{(94)}{=} \dfrac{\mathcal{V}(\delta_2(\alpha_{n-1}^*)) + \mathcal{V}(\exists x\, b)}{2} = \mathcal{V}(\exists x\, b) - \dfrac{\mathcal{V}(\exists x\, b) - \mathcal{V}(\delta_2(\alpha_{n-1}^*))}{2} = \mathcal{V}(\exists x\, b) - \dfrac{\mathcal{V}(\exists x\, b) - (\mathcal{V}(\exists x\, b) - \frac{D}{2^{n-1}})}{2} = \mathcal{V}(\exists x\, b) - \dfrac{D}{2^n}$; we have $\delta_2(\alpha_n^*) \lhd \exists x\, b$; by

(111) for α^*_n, there exists $\tilde{w}_n(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha^*_n))) \in \mathcal{U}_{\mathfrak{A}}$ and

$$\begin{gathered}b(x/\tilde{w}_n(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha^*_n)))) \in \mathcal{B}, \\ \delta_2(\alpha^*_n) \lhd b(x/\tilde{w}_n(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha^*_n)))), \\ b(x/\tilde{w}_n(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha^*_n)))) \lhd \exists x\, b;\end{gathered}$$

$b(x/\tilde{w}_n(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha^*_n)))) \in \mathcal{B} - \mathit{tcons}(S)$, $\delta_2(\alpha^*_n) \not\lhd \delta_2(\alpha^*_n)$, there exists $\kappa, \alpha^*_{n-1} < \alpha^*_n < \alpha^{**}_n < \gamma_2$ and $\delta_2(\alpha^{**}_n) = b(x/\tilde{w}_n(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha^*_n))))$, $\delta_2(\alpha^*_n) \lhd \delta_2(\alpha^{**}_n)$, $\delta_2(\alpha^{**}_n) \lhd \exists x\, b$, by (105) for $\delta_2(\alpha^*_n)$, $\delta_2(\alpha^{**}_n)$, $\exists x\, b$, $\mathcal{V}(\delta_2(\alpha^*_n)) < \mathcal{V}(\delta_2(\alpha^{**}_n))$, $\mathcal{V}(\delta_2(\alpha^{**}_n)) < \mathcal{V}(\exists x\, b)$; (113) holds.

So, in both Cases 2.3 and 2.4, (113) holds. The induction is completed. Thus, (113) holds.

Then, by (113),

$$\begin{gathered}\bigvee_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u)) \geq \bigvee_{n \geq 1} \mathcal{V}(b(x/\tilde{w}_n(\mathit{freetermseq}(\exists x\, b), \mathit{freetermseq}(\delta_2(\alpha^*_n))))) = \\ \bigvee_{n \geq 1} \mathcal{V}(\delta_2(\alpha^{**}_n)) \geq \bigvee_{n \geq 1} \mathcal{V}(\delta_2(\alpha^*_n)) = \bigvee_{n \geq 1} \left(\mathcal{V}(\exists x\, b) - \frac{D}{2^n}\right) = \mathcal{V}(\exists x\, b) - \bigwedge_{n \geq 1} \frac{D}{2^n} = \mathcal{V}(\exists x\, b),\end{gathered}$$

$\mathcal{V}(\exists x\, b) = \bigvee_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u))$.

So, in both Cases 1 and 2, $\mathcal{V}(\exists x\, b) = \bigvee_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u))$; $\mathcal{V}(\exists x\, b) = \bigvee_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u))$. Thus, (107) holds.

We put

$$f^{\mathfrak{A}}(u_1, \dots, u_\tau) = \begin{cases} f(u_1, \dots, u_\tau) & \mathit{if}\ f \in \mathit{Func}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}, \\ cn^* & \mathit{else}, \end{cases} \qquad f \in \mathit{Func}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}, u_i \in \mathcal{U}_{\mathfrak{A}};$$

$$p^{\mathfrak{A}}(u_1, \dots, u_\tau) = \begin{cases} \mathcal{V}(p(u_1, \dots, u_\tau)) & \mathit{if}\ p(u_1, \dots, u_\tau) \in \mathcal{B}, \\ 0 & \mathit{else}, \end{cases} \qquad p \in \mathit{Pred}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}, u_i \in \mathcal{U}_{\mathfrak{A}};$$

$$\mathfrak{A} = \left(\mathcal{U}_{\mathfrak{A}}, \{f^{\mathfrak{A}} \,|\, f \in \mathit{Func}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}\}, \{p^{\mathfrak{A}} \,|\, p \in \mathit{Pred}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}\}\right), \text{ an interpretation for } \mathcal{L} \cup \tilde{\mathbb{W}} \cup P.$$

For all $C \in S$ and $e \in \mathcal{S}_{\mathfrak{A}}$, $C(e|_{\mathit{freevars}(C)}) \in \mathit{clo}^{\mathcal{BH}}(S)$. (114)

Let $C \in S$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $e : \mathit{Var}_{\mathcal{L}} \longrightarrow \mathcal{U}_{\mathfrak{A}}$, $\mathit{freevars}(C) \subseteq_{\mathcal{F}} \mathit{Var}_{\mathcal{L}}$, $e|_{\mathit{freevars}(C)} \in \mathit{Subst}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $\mathit{dom}(e|_{\mathit{freevars}(C)}) = \mathit{freevars}(C)$, $\mathit{range}(e|_{\mathit{freevars}(C)}) = \emptyset$; $e|_{\mathit{freevars}(C)}$ is applicable to C, $C(e|_{\mathit{freevars}(C)}) \in \mathit{GInst}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}(S)$, $\{e(x) \,|\, x \in \mathit{freevars}(C)\} \subseteq_{\mathcal{F}} \mathit{GTerm}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$; there exists $\{\tilde{w}_j \,|\, 1 \leq j \leq m\} = \mathit{funcs}(\{e(x) \,|\, x \in \mathit{freevars}(C)\}) \cap \tilde{\mathbb{W}}^* \subseteq_{\mathcal{F}} \tilde{\mathbb{W}}^* = \mathit{funcs}(\mathbb{S}) \cap \tilde{\mathbb{W}}$; for all $1 \leq j \leq m$, there exists $a_j \in \mathit{atoms}(\mathbb{S}) \cup \mathit{qatoms}(\mathbb{S}) \subseteq \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S)) \cup \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$ and $\tilde{w}_j \in \mathit{funcs}(a_j)$; $\mathit{0} \in \mathcal{B} \subseteq \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S)) \cup \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$; $\emptyset \neq \{\mathit{0}\} \cup \{a_j \,|\, 1 \leq j \leq m\} \subseteq_{\mathcal{F}} \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S)) \cup \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$, by (99) for $\{\mathit{0}\} \cup \{a_j \,|\, 1 \leq j \leq m\}$, there exist a deduction $\mathcal{D} = C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq \mathit{GOrdCl}_{\mathcal{L}_n}$, and $\{\mathit{0}\} \cup \{a_j \,|\, 1 \leq j \leq m\} \subseteq \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n) \subseteq \mathit{GAtom}_{\mathcal{L}_n} \cup \mathit{QAtom}_{\mathcal{L}_n}$, $\mathit{funcs}(\{e(x) \,|\, x \in \mathit{freevars}(C)\}) \subseteq \mathit{Func}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $\mathit{Func}_{\mathcal{L}_n} \supseteq \bigcup_{j=1}^m \mathit{funcs}(a_j) \supseteq \{\tilde{w}_j \,|\, 1 \leq j \leq m\} = \mathit{funcs}(\{e(x) \,|\, x \in \mathit{freevars}(C)\}) \cap \tilde{\mathbb{W}}^*$, $\mathit{Func}_{\mathcal{L}_n} \supseteq \mathit{Func}_{\mathcal{L} \cup P} \supseteq \mathit{funcs}(\{e(x) \,|\, x \in \mathit{freevars}(C)\}) \cap \mathit{Func}_{\mathcal{L} \cup P}$, $\mathit{Func}_{\mathcal{L}_n} \supseteq (\mathit{funcs}(\{e(x) \,|\, x \in \mathit{freevars}(C)\}) \cap \tilde{\mathbb{W}}^*) \cup (\mathit{funcs}(\{e(x) \,|\, x \in \mathit{freevars}(C)\}) \cap \mathit{Func}_{\mathcal{L} \cup P}) = \mathit{funcs}(\{e(x) \,|\, x \in \mathit{freevars}(C)\}) \cap (\tilde{\mathbb{W}}^* \cup \mathit{Func}_{\mathcal{L} \cup P}) = \mathit{funcs}(\{e(x) \,|\, x \in \mathit{freevars}(C)\}) \cap \mathit{Func}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P} = \mathit{funcs}(\{e(x) \,|\, x \in \mathit{freevars}(C)\})$, $\{e(x) \,|\, x \in \mathit{freevars}(C)\} \subseteq \mathit{GTerm}_{\mathcal{L}_n}$, $e|_{\mathit{freevars}(C)} \in \mathit{Subst}_{\mathcal{L}_n}$, $C(e|_{\mathit{freevars}(C)}) \in \mathit{GInst}_{\mathcal{L}_n}(S)$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n$, $C_{n+1} = C(e|_{\mathit{freevars}(C)}) \in \mathit{GInst}_{\mathcal{L}_n}(S) \subseteq \mathit{ordtcons}(S) \cup \mathit{GInst}_{\mathcal{L}_n}(S) \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} \subseteq \mathit{GOrdCl}_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Hence, $C(e|_{\mathit{freevars}(C)}) = C_{n+1} \in \mathit{clo}^{\mathcal{BH}}(S)$; (114) holds.

Let $a = p(t_1, \dots, t_\tau) \in \mathit{atoms}(\mathbb{S}) \subseteq \mathcal{B}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $a = p(t_1, \dots, t_\tau) \in \mathit{GAtom}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $t_i \in \mathit{GTerm}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P} = \mathcal{U}_{\mathfrak{A}}$; for all $1 \leq i \leq \tau$, $\|t_i\|^{\mathfrak{A}}_e = t_i$; the proof is by induction on t_i; $\|a\|^{\mathfrak{A}}_e = \|p(t_1, \dots, t_\tau)\|^{\mathfrak{A}}_e = p^{\mathfrak{A}}(\|t_1\|^{\mathfrak{A}}_e, \dots, \|t_\tau\|^{\mathfrak{A}}_e) = p^{\mathfrak{A}}(t_1, \dots, t_\tau) = \mathcal{V}(p(t_1, \dots, t_\tau)) = \mathcal{V}(a)$. Let $a = \forall x\, p(t_0, \dots, t_\tau) \in \mathit{qatoms}(\mathbb{S}) \subseteq \mathcal{B}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $a = \forall x\, p(t_0, \dots, t_\tau) \in \mathit{QAtom}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $\mathit{freevars}(a) = \emptyset$, $\mathit{vars}(p(t_0, \dots, t_\tau)) = \{x\}$, either $t_i = x$ or $t_i \in \mathit{GTerm}_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P} = \mathcal{U}_{\mathfrak{A}}$; for all $i \leq \tau$ and $u \in \mathcal{U}_{\mathfrak{A}}$, either $t_i(x/u) = x(x/u) = u \in \mathcal{U}_{\mathfrak{A}}$ or $t_i(x/u) = t_i \in \mathcal{U}_{\mathfrak{A}}$; $t_i(x/u) \in \mathcal{U}_{\mathfrak{A}}$, $\|t_i\|^{\mathfrak{A}}_{e[x/u]} = t_i(x/u)$; the proof is by case analysis and induction on t_i; $a = \forall x\, p(t_0, \dots, t_\tau) \in \mathit{qatoms}(\mathbb{S}) \subseteq \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$. Let $u \in \mathcal{U}_{\mathfrak{A}}$. Then, by (106) for $\forall x\, p(t_0, \dots, t_\tau)$, u, $p(t_0, \dots, t_\tau)(x/u) \in \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S))$, by (97) for $\forall x\, p(t_0, \dots, t_\tau)$, $p(t_0, \dots, t_\tau)(x/u)$, there exist a deduction $\mathcal{D} = C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq \mathit{GOrdCl}_{\mathcal{L}_n}$, and $\forall x\, p(t_0, \dots, t_\tau), p(t_0, \dots, t_\tau)(x/u) \in \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n)$, $\forall x\, p(t_0, \dots, t_\tau) \in \mathit{qatoms}^{\forall}(S_n) \subseteq \mathit{QAtom}_{\mathcal{L}_n}$, $p(t_0, \dots, t_\tau)(x/u) \in \mathit{atoms}(S_n) \subseteq \mathit{GAtom}_{\mathcal{L}_n}$, $u \in \mathit{GTerm}_{\mathcal{L}_n}$. We put $\gamma = x/u \in \mathit{Subst}_{\mathcal{L}_n}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(p(t_0, \dots, t_\tau))$. Hence, $p(t_0, \dots, t_\tau)\gamma = p(t_0, \dots, t_\tau)(x/u)$, using Rule (45) with respect to $\mathcal{L}_n$, S_n, we derive $\forall x\, p(t_0,\dots,t_\tau) \prec p(t_0,\dots,t_\tau)\gamma \vee \forall x\, p(t_0,\dots,t_\tau) \eqcirc p(t_0,\dots,t_\tau)\gamma = \forall x\, p(t_0,\dots,t_\tau) \prec p(t_0,\dots,t_\tau)(x/u) \vee \forall x\, p(t_0,\dots,t_\tau) \eqcirc p(t_0,\dots,t_\tau)(x/u) \in \mathit{GOrdCl}_{\mathcal{L}_n}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n$, $C_{n+1} = \forall x\, p(t_0,\dots,t_\tau) \prec p(t_0,\dots,t_\tau)(x/u) \vee \forall x\, p(t_0,\dots,t_\tau) \eqcirc p(t_0,\dots,t_\tau)(x/u) \in \mathit{GOrdCl}_{\mathcal{L}_n} \subseteq \mathit{GOrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $C_{n+1} \in \mathit{clo}^{\mathcal{BH}}(S)$, $\forall x\, p(t_0,\dots,t_\tau) \prec p(t_0,\dots,t_\tau)(x/u) \in \mathbb{S}$ or $\forall x\, p(t_0,\dots,t_\tau) \eqcirc p(t_0,\dots,t_\tau)(x/u) \in \mathbb{S}$, for both the cases $\forall x\, p(t_0,\dots,t_\tau) \prec p(t_0,\dots,t_\tau)(x/u) \in \mathbb{S}$ and $\forall x\, p(t_0,\dots,t_\tau) \eqcirc p(t_0,\dots,t_\tau)(x/u) \in \mathbb{S}$, $p(t_0,\dots,t_\tau)(x/u) \in \mathit{atoms}(\mathbb{S})$; $p(t_0(x/u),\dots,t_\tau(x/u)) = p(t_0,\dots,t_\tau)(x/u) \in \mathit{atoms}(\mathbb{S}) \subseteq \mathcal{B}$. Hence, $\|a\|_e^{\mathfrak{A}} = \|\forall x\, p(t_0,\dots,t_\tau)\|_e^{\mathfrak{A}} = \bigwedge_{u\in\mathcal{U}_{\mathfrak{A}}} \|p(t_0,\dots,t_\tau)\|_{e[x/u]}^{\mathfrak{A}} = \bigwedge_{u\in\mathcal{U}_{\mathfrak{A}}} p^{\mathfrak{A}}(\|t_0\|_{e[x/u]}^{\mathfrak{A}},\dots,\|t_\tau\|_{e[x/u]}^{\mathfrak{A}}) = \bigwedge_{u\in\mathcal{U}_{\mathfrak{A}}} p^{\mathfrak{A}}(t_0(x/u),\dots,t_\tau(x/u)) = \bigwedge_{u\in\mathcal{U}_{\mathfrak{A}}} \mathcal{V}(p(t_0(x/u),\dots,t_\tau(x/u))) = \bigwedge_{u\in\mathcal{U}_{\mathfrak{A}}} \mathcal{V}(p(t_0,\dots,t_\tau)(x/u)) \overset{(107)}{=\!=} \mathcal{V}(\forall x\, p(t_0,\dots,t_\tau)) = \mathcal{V}(a)$. Let $a = \exists x\, p(t_0,\dots,t_\tau) \in \mathit{qatoms}(\mathbb{S}) \subseteq \mathcal{B}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $a = \exists x\, p(t_0,\dots,t_\tau) \in \mathit{QAtom}_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}$, $\mathit{freevars}(a) = \emptyset$, $\mathit{vars}(p(t_0,\dots,t_\tau)) = \{x\}$, either $t_i = x$ or $t_i \in \mathit{GTerm}_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P} = \mathcal{U}_{\mathfrak{A}}$; we have, for all $i \le \tau$ and $u \in \mathcal{U}_{\mathfrak{A}}$, $\|t_i\|_{e[x/u]}^{\mathfrak{A}} = t_i(x/u)$; $a = \exists x\, p(t_0,\dots,t_\tau) \in \mathit{qatoms}(\mathbb{S}) \subseteq \mathit{qatoms}(\mathit{clo}^{\mathcal{BH}}(S))$. Let $u \in \mathcal{U}_{\mathfrak{A}}$. Then, by (106) for $\exists x\, p(t_0,\dots,t_\tau)$, u, $p(t_0,\dots,t_\tau)(x/u) \in \mathit{atoms}(\mathit{clo}^{\mathcal{BH}}(S))$, by (97) for $\exists x\, p(t_0,\dots,t_\tau)$, $p(t_0,\dots,t_\tau)(x/u)$, there exist a deduction $\mathcal{D} = C_1,\dots,C_n$, $n \ge 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq \mathit{GOrdCl}_{\mathcal{L}_n}$, and $\exists x\, p(t_0,\dots,t_\tau), p(t_0,\dots,t_\tau)(x/u) \in \mathit{atoms}(S_n) \cup \mathit{qatoms}(S_n)$, $\exists x\, p(t_0,\dots,t_\tau) \in \mathit{qatoms}^{\exists}(S_n) \subseteq \mathit{QAtom}_{\mathcal{L}_n}$, $p(t_0,\dots,t_\tau)(x/u) \in \mathit{atoms}(S_n) \subseteq \mathit{GAtom}_{\mathcal{L}_n}$, $u \in \mathit{GTerm}_{\mathcal{L}_n}$. We put $\gamma = x/u \in \mathit{Subst}_{\mathcal{L}_n}$, $\mathit{dom}(\gamma) = \{x\} = \mathit{vars}(p(t_0,\dots,t_\tau))$. Hence, $p(t_0,\dots,t_\tau)\gamma = p(t_0,\dots,t_\tau)(x/u)$, using Rule (46) with respect to $\mathcal{L}_n$, S_n, we derive $p(t_0,\dots,t_\tau)\gamma \prec \exists x\, p(t_0,\dots,t_\tau) \vee p(t_0,\dots,t_\tau)\gamma \eqcirc \exists x\, p(t_0,\dots,t_\tau) = p(t_0,\dots,t_\tau)(x/u) \prec \exists x\, p(t_0,\dots,t_\tau) \vee p(t_0,\dots,t_\tau)(x/u) \eqcirc \exists x\, p(t_0,\dots,t_\tau) \in \mathit{GOrdCl}_{\mathcal{L}_n}$. We put $\mathcal{L}_{n+1} = \mathcal{L}_n$, $C_{n+1} = p(t_0,\dots,t_\tau)(x/u) \prec \exists x\, p(t_0,\dots,t_\tau) \vee p(t_0,\dots,t_\tau)(x/u) \eqcirc \exists x\, p(t_0,\dots,t_\tau) \in \mathit{GOrdCl}_{\mathcal{L}_n} \subseteq \mathit{GOrdCl}_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $\mathcal{D}' = \mathcal{D}, C_{n+1}$, $S_{n+1} = S_n \cup \{C_{n+1}\} \subseteq \mathit{GOrdCl}_{\mathcal{L}_n} = \mathit{GOrdCl}_{\mathcal{L}_{n+1}}$; $\mathcal{D}'$ is a deduction of C_{n+1} from S by basic order hyperresolution. Then $C_{n+1} \in \mathit{clo}^{\mathcal{BH}}(S)$, $p(t_0,\dots,t_\tau)(x/u) \prec \exists x\, p(t_0,\dots,t_\tau) \in \mathbb{S}$ or $p(t_0,\dots,t_\tau)(x/u) \eqcirc \exists x\, p(t_0,\dots,t_\tau) \in \mathbb{S}$, for both the cases $p(t_0,\dots,t_\tau)(x/u) \prec \exists x\, p(t_0,\dots,t_\tau) \in \mathbb{S}$ and $p(t_0,\dots,t_\tau)(x/u) \eqcirc \exists x\, p(t_0,\dots,t_\tau) \in \mathbb{S}$, $p(t_0,\dots,t_\tau)(x/u) \in \mathit{atoms}(\mathbb{S})$; $p(t_0(x/u),\dots,t_\tau(x/u)) = p(t_0,\dots,t_\tau)(x/u) \in \mathit{atoms}(\mathbb{S}) \subseteq \mathcal{B}$. Hence, $\|a\|_e^{\mathfrak{A}} = \|\exists x\, p(t_0,\dots,t_\tau)\|_e^{\mathfrak{A}} = \bigvee_{u\in\mathcal{U}_{\mathfrak{A}}} \|p(t_0,\dots,t_\tau)\|_{e[x/u]}^{\mathfrak{A}} = \bigvee_{u\in\mathcal{U}_{\mathfrak{A}}} p^{\mathfrak{A}}(\|t_0\|_{e[x/u]}^{\mathfrak{A}},\dots,\|t_\tau\|_{e[x/u]}^{\mathfrak{A}}) = \bigvee_{u\in\mathcal{U}_{\mathfrak{A}}} p^{\mathfrak{A}}(t_0(x/u),\dots,t_\tau(x/u)) = \bigvee_{u\in\mathcal{U}_{\mathfrak{A}}} \mathcal{V}(p(t_0(x/u),\dots,t_\tau(x/u))) = \bigvee_{u\in\mathcal{U}_{\mathfrak{A}}} \mathcal{V}(p(t_0,\dots,t_\tau)(x/u)) \overset{(107)}{=\!=} \mathcal{V}(\exists x\, p(t_0,\dots,t_\tau)) = \mathcal{V}(a)$. So, for all $a \in \mathcal{B}$ and $e \in \mathcal{S}_{\mathfrak{A}}$, for all the three cases $a \in \mathit{atoms}(\mathbb{S}) \subseteq \mathcal{B}$, $a \in \mathit{qatoms}^{\forall}(\mathbb{S}) \subseteq \mathcal{B}$, $a \in \mathit{qatoms}^{\exists}(\mathbb{S}) \subseteq \mathcal{B}$, $\|a\|_e^{\mathfrak{A}} = \mathcal{V}(a)$; $\|a\|_e^{\mathfrak{A}} = \mathcal{V}(a)$. Let $l = \varepsilon_1 \eqcirc \varepsilon_2 \in \mathbb{S}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $\varepsilon_1, \varepsilon_2 \in \mathcal{B}$, $\varepsilon_1 \triangleq \varepsilon_2$, by (105) for ε_1, ε_2, $\mathcal{V}(\varepsilon_1) = \mathcal{V}(\varepsilon_2)$, $\|l\|_e^{\mathfrak{A}} = \|\varepsilon_1 \eqcirc \varepsilon_2\|_e^{\mathfrak{A}} = \|\varepsilon_1\|_e^{\mathfrak{A}} \boldsymbol{\eqcirc} \|\varepsilon_2\|_e^{\mathfrak{A}} = \mathcal{V}(\varepsilon_1) \boldsymbol{\eqcirc} \mathcal{V}(\varepsilon_2) = 1$. Let $l = \varepsilon_1 \prec \varepsilon_2 \in \mathbb{S}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $\varepsilon_1, \varepsilon_2 \in \mathcal{B}$, $\varepsilon_1 \lhd \varepsilon_2$, by (105) for ε_1, ε_2, $\mathcal{V}(\varepsilon_1) < \mathcal{V}(\varepsilon_2)$, $\|l\|_e^{\mathfrak{A}} = \|\varepsilon_1 \prec \varepsilon_2\|_e^{\mathfrak{A}} = \|\varepsilon_1\|_e^{\mathfrak{A}} \boldsymbol{\prec} \|\varepsilon_2\|_e^{\mathfrak{A}} = \mathcal{V}(\varepsilon_1) \boldsymbol{\prec} \mathcal{V}(\varepsilon_2) = 1$. So, for all $l \in \mathbb{S}$ and $e \in \mathcal{S}_{\mathfrak{A}}$, for both the cases $l = \varepsilon_1 \eqcirc \varepsilon_2 \in \mathbb{S}$ and $l = \varepsilon_1 \prec \varepsilon_2 \in \mathbb{S}$, $\|l\|_e^{\mathfrak{A}} = 1$; $\|l\|_e^{\mathfrak{A}} = 1$. Let $C \in S \subseteq \mathit{OrdCl}_{\mathcal{L}\cup P}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $e : \mathit{Var}_{\mathcal{L}} \longrightarrow \mathcal{U}_{\mathfrak{A}}$, $\mathit{freevars}(C) \subseteq_{\mathcal{F}} \mathit{Var}_{\mathcal{L}}$, $e|_{\mathit{freevars}(C)} \in \mathit{Subst}_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}$, $\mathit{dom}(e|_{\mathit{freevars}(C)}) = \mathit{freevars}(C)$, $\mathit{range}(e|_{\mathit{freevars}(C)}) = \emptyset$; $e|_{\mathit{freevars}(C)}$ is applicable to C; by (114) for C, e, $C(e|_{\mathit{freevars}(C)}) \in \mathit{clo}^{\mathcal{BH}}(S)$, there exists $l^* \in C(e|_{\mathit{freevars}(C)})$ and $l^* \in \mathbb{S}$, $\|l^*\|_e^{\mathfrak{A}} = 1$; there exists $l^{**} \in C \in \mathit{OrdCl}_{\mathcal{L}\cup P}$ and $l^{**} \in \mathit{OrdLit}_{\mathcal{L}\cup P} \subseteq \mathit{OrdLit}_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}$, $\mathit{freevars}(l^{**}) \subseteq \mathit{freevars}(C)$; $e|_{\mathit{freevars}(l^{**})}$ is applicable to l^{**}, $l^{**}(e|_{\mathit{freevars}(l^{**})}) = l^*$; for all $t \in \mathit{Term}_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}$, $a \in \mathit{Atom}_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P} \cup \mathit{QAtom}_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}$, $l \in \mathit{OrdLit}_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}$, $\|t\|_e^{\mathfrak{A}} = t(e|_{\mathit{vars}(t)}) = \|t(e|_{\mathit{vars}(t)})\|_e^{\mathfrak{A}}$, $\|a\|_e^{\mathfrak{A}} = \|a(e|_{\mathit{freevars}(a)})\|_e^{\mathfrak{A}}$, $\|l\|_e^{\mathfrak{A}} = \|l(e|_{\mathit{freevars}(l)})\|_e^{\mathfrak{A}}$; the proof is by induction on t and by definition; $\|l^{**}\|_e^{\mathfrak{A}} = \|l^{**}(e|_{\mathit{freevars}(l^{**})})\|_e^{\mathfrak{A}} = \|l^*\|_e^{\mathfrak{A}} = 1$; $\mathfrak{A} \models_e C$; $\mathfrak{A} \models S$, $\mathfrak{A}|_{\mathcal{L}\cup P} \models S$; S is satisfiable. The theorem is proved. □