Procedural Semantics for Fuzzy Disjunctive Programs on Residuated Lattices

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Abstract. In the paper, we present a procedural semantics for fuzzy disjunctive programs - sets of graded implications of the form:

$$(h_1 \vee \cdots \vee h_n \longleftarrow b_1 \,\&\, \cdots \,\&\, b_m, c) \qquad (n > 0, m \geq 0)$$

where h_i, b_j are atoms and c a truth value from a complete residuated lattice $\boldsymbol{L} = (L, \leq, \vee, \wedge, *, \Rightarrow, \mathbf{0}, \mathbf{1})$. A graded implication can be understood as a means of the representation of incomplete and uncertain information; the incompleteness is formalised by the disjunction in the consequent of this implication, while the uncertainty by its truth degree. In the end, the coincidence of the procedural and generalised fixpoint semantics proposed in [18, 19], will be reached.
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1 Introduction

The complexity of the real world causes that our knowledge about it are unfortunately ambiguous. The complete and certain description of even a 'small' part of the reality requires much more detailed information than humans or computer systems are capable to recognise simultaneously. From the philosophical point of view, such the exhaustive description of the world is an unreachable platonic ideal. Nevertheless, humans can understand complex real systems thanks to their ability of maintaining only a generic comprehension of them and of approximative reasoning about them. Generally speaking, the ambiguity in our knowledge is of various nature; we could roughly distinguish incompleteness and uncertainty of information. Consequently, in knowledge representation and commonsense reasoning, we should be able to handle such incomplete and uncertain information. Attempts to solve many real-world problems bring us to storing and retrieving incomplete and uncertain knowledge in deductive databases in order to represent an incomplete and uncertain model of the world, and to carry out a reasonable inference of new facts from this model. Expert systems, cognitive robots, mechatronic systems, sensor data, sound and image databases, temporal indeterminacy are only a few of the fields dealing with incomplete and uncertain information. In recent years, disjunctive and multivalued, annotated logic programming have been recognised as powerful tools for maintenance of such knowledge bases.

In disjunctive logic programming [35, 37, 38, 7], we allow the heads of clauses to be literal disjunctions in contrast to the case of Horn programs, where only single atoms (literals) may occur in the heads of clauses. At first, positive disjunctive programs were studied, among others in [32, 31, 34, 35]. A positive disjunctive program is a set of implications of the form

$$a_1 \vee \cdots \vee a_n \longleftarrow b_1 \wedge \cdots \wedge b_m$$

where a_i and b_j are atoms. If we infer a disjunction, e.g. $p \vee q$, from a given positive disjunctive program P, this disjunction can be viewed as some kind of incomplete information. We know that $p \vee q$ is derivable from the program P, but we do not know whether p or q themselves are derivable. This is the main difference to the Horn case, where we are interested only in complete information represented by single atoms, in our case by p and q. So, disjunctions can be used for a suitable representation of incomplete information. For positive disjunctive programs, Minker, Rajasekar, and Zanon introduced minimal model, fixpoint, and procedural semantics (based on *SLI*-resolution) [34, 35].

Positive disjunctive programs can be augmented by some kind of negation-by-default. Such programs are called normal disjunctive programs and they consist of clauses of the form

$$a_1 \vee \cdots \vee a_n \longleftarrow b_1 \wedge \cdots \wedge b_m \wedge not\, c_1 \wedge \cdots \wedge not\, c_k$$

where a_i, b_j, and c_o are atoms and *not* is a negation-by-default operator. Several semantics for normal disjunctive programs were introduced. Many of them are extensions of semantics for normal (Horn) programs: *the disjunctive well-founded semantics* (*DWFS*) [50], *the generalized disjunctive well-founded semantics* (*GWFS*) [3, 4], WF^3 [5], *the disjunctive perfect model semantics* (*DPMS*) [42, 13], *the stationary semantics* [43], *the disjunctive stable semantics* [15, 43], *the disjunctive partial stable semantics* [43], and *the static semantics* [43].

Negation-by-default has a limited use, because *not a is true* does not mean the presence of some knowledge asserting the falsehood of a, but the absence of knowledge asserting the truth of a. Therefore, it is worth extending normal disjunctive programs by classical negation. Explicit use of classical negation allows to express *when an atom is false* in addition to *when an atom is true*, which enhances the expressive power of normal disjunctive programs. Such programs are called extended disjunctive programs and they consist of clauses of the form

$$l_1 \vee \cdots \vee l_n \longleftarrow l_{n+1} \wedge \cdots \wedge l_m \wedge not\, l_{m+1} \wedge \cdots \wedge not\, l_k$$

where l_i are literals. Hence, extended clauses contain the two kinds of negation: classical and default. Semantics of logic programs dealing with classical (strong) negation can be found in [15, 1, 43], among others. In [36], a method how to obtain a semantics for extended disjunctive programs from a semantics for normal disjunctive programs, is described. This method is based on the *prime-*$\perp$ and *neg* transformations, which capture the intended meaning of original extended disjunctive programs after transformation to normal disjunctive programs. Another approach is described in [43]. Przymusinski introduced *the Autoepistemic Logic of Knowledge and Beliefs* ($AELB$), augmenting Moore's *Autoepistemic Logic* [39], as a unifying framework for various semantics of logic programs. Most semantics can be obtained by means of a suitable translation of logic programs into $AELB$.

An interesting approach to uncertainty (fuzziness, vagueness, and imprecision) is that of multivalued and annotated logic programming. When we derive some formulae from a given program, they do not represent certain information in general, since some truth value or degree, e.g. from the unit interval $[0,1]$, is assigned to each formula. So, derived formulae are only partly true in models of the given program. Roughly speaking, the main difference between multivalued and annotated logic programming is that implications in the multivalued approach have truth values (weights) associated with them as a whole in contrast to the annotated approach, where truth values are associated with each literal of an implication. This yields some advantages and disadvantages to both the approaches. Associating truth values with implications is intuitive and natural, yet the problem how truth values should be propagated across implications arises. On the other hand, annotated logics have a simpler formalism that is closer to classical logic. They use the classical definition of implication: $I \models F_1 \longleftarrow F_2$ iff $I \models F_1$ or $I \not\models F_2$. Also many problems in non-monotonic reasoning can be formulated more easily via non-monotonic annotated logics. This dichotomy may be partly removed by an appropriate generalisation of the concept of an annotation so that we are able to capture the propagation of truth values across implications. Annotated logics were introduced in [47], later developed in [21], and generalised in [22]. The multivalued approach can be roughly divided into the following subdomains:

- bilattice-based logics [14, 16, etc.]; and
- quantitative rule-based programs [40, etc.]:
  - heuristic, non-numerical expressing uncertainty [8, 12, etc.];
  - numerical expressing uncertainty and using e.g. Dempster-Shafer calculus [20, 45, 10, etc.], using fuzzy logic [27, 26, 6, 49, 46, etc.];
  - numerical expressing uncertainty and relying on probability theory [25, 24, 9, 23, 41, etc.].

Recently, multi-adjoint logic programs have been introduced in [28–30], where model, fixpoint, and procedural semantics for these programs have been proposed. Multi-adjoint logic programs allow simultaneous use of several implications in the rules and rather general connectives in the bodies.

In the paper, we shall aim to combine both the disjunctive and multivalued approaches and provide some formalisation of reasoning with incomplete and uncertain information represented by graded implications (disjunctions, $m = 0$) of the form:

$$(h_1 \vee \cdots \vee h_n \longleftarrow b_1 \mathbin{\&} \cdots \mathbin{\&} b_m, c) \qquad (n > 0, m \geq 0)$$

connective	truth function
$\vee$	$\vee$
$\&$	$*$
$\rightarrow$	$\Rightarrow$

Table 1. Truth functions.

where h_i, b_j are atoms and c a truth value from a complete residuated lattice $\boldsymbol{L} = (L, \leq, \vee, \wedge, *, \Rightarrow, \mathbf{0}, \mathbf{1})$ with a residuum $\Rightarrow$. More precisely, we shall suppose that

- $\boldsymbol{L} = (L, \leq, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a complete lattice;
- the supremum operator $\vee$ and the infimum operator $\wedge$ are infinitely distributive, i.e. for all $K \subseteq L$, $a \in L$,

$$a \vee \bigwedge K = \bigwedge_{k \in K} (a \vee k),$$
$$a \wedge \bigvee K = \bigvee_{k \in K} (a \wedge k);$$

- the binary operation $*$ over L is commutative, associative, and non-decreasing in both the arguments; its neutral element is $\mathbf{1}$, i.e. $\mathbf{1} * a = a$;
- the binary operation $\Rightarrow$ over L is non-increasing in the first argument and non-decreasing in the second one;
- the following condition of adjunction holds: for all $\alpha, \beta, \gamma \in L$,

$$\alpha * \beta \leq \gamma \text{ iff } \alpha \leq \beta \Rightarrow \gamma.$$

The truth functions of the connectives are defined in Tab. 1 in the usual way. Fuzzy disjunctive programs, from which we shall infer incomplete and uncertain information, will be viewed as sets of graded implications. Similar approaches can be found in [26, 27, 23, 24].

2 Declarative and fixpoint semantics

In this section, we generalise the declarative and fixpoint semantics for fuzzy disjunctive programs proposed in [18]. We proceed from the declarative one:

Definition 1 (Declarative semantics). *Let P be a fuzzy disjunctive program of $\mathcal{L}$. $\mathcal{DS}(P) = \{D \mid D$ is a graded disjunction of $\mathcal{L}$ and $P \models D\}$.*

Consider a complete lattice $\boldsymbol{L} = (L, \leq, \vee, \wedge, \mathbf{0}, \mathbf{1})$. Let $T : L \longrightarrow L$ be a mapping. We say T is monotonic iff for all $x, y \in L$, $x \leq y \Longrightarrow T(x) \leq T(y)$. T is said to be ω-continuous iff for all ω-chains $X \subseteq L$ ($X = \{x_i \mid i \in \omega\}$ is simply ordered by $\leq$), $\bigvee\{T(x) \mid x \in X\} = T(\bigvee X)$. We denote the α-th iteration power of T on $x \in L$ as $T^\alpha(x)$. $T^\alpha(\mathbf{0})$ will be abbreviated by T^α. We say $x \in L$ is a fixpoint of T iff $T(x) = x$.

Proposition 1 (Fixpoint Theorem (Knaster, Tarski)**).** *Let $T : L \longrightarrow L$ be a mapping over a complete lattice $\boldsymbol{L} = (L, \leq, \vee, \wedge, \mathbf{0}, \mathbf{1})$.*

i) *If T is monotonic, then there exists the least fixpoint of T, denoted as $lfp(T)$.*
ii) *If T is ω-continuous, then $lfp(T) = T^\omega$.*

Proof. See [48]. □

An open formula ϕ' is a variant of an open formula ϕ iff there exists a variable renaming ρ such that $\phi' = \phi\rho$. By $Atoms(\phi)$ we denote the set of all the atoms occurring in the formula ϕ. A disjunction, D, of atoms is called a subdisjunction of another disjunction, D', of atoms, written as $D \sqsubseteq D'$, iff $Atoms(D) \subseteq Atoms(D')$. Moreover, we say that D is a disjunctive factor of D' iff $Atoms(D) = Atoms(D')$ and every atom from $Atoms(D)$ has exactly one occurrence in D. A graded disjunction (D, c) is a disjunctive factor of a graded disjunction (D', c) iff D is a disjunctive factor of D'. A graded implication $(H \longleftarrow B, c)$ is an implication factor of a graded implication $(H' \longleftarrow B, c)$ iff H is a disjunctive factor of H'. Denote the set of all graded disjunctions (of atoms) of $\mathcal{L}$ as $Dis_{\mathcal{L}}$. Consider the following hyperresolution operator $\mathcal{C}_P$:

Definition 2. *Let P be a fuzzy disjunctive program of $\mathcal{L}$ and $I \subseteq Dis_{\mathcal{L}}$. A graded disjunction (D, c) of $\mathcal{L}$ is said to be an immediate consequence of I and P iff there exist*

- *$(H \longleftarrow b_1 \,\&\cdots\&\, b_m, cv_0)$, an implication factor of a variant of a graded implication in P;*
- *$(C_i \vee D_i, cv_i)$, $i = 1, \dots, m$, disjunctive factors of some variants of graded disjunctions in I;*
  the implication and disjunctions do not share variables in common;
- *$\theta = mgu((\underbrace{b_1 \vee \cdots \vee b_1}_{|C_1|}, \dots, \underbrace{b_m \vee \cdots \vee b_m}_{|C_m|}), (C_1, \dots, C_m))$,*
  $dom(\theta) = vars(b_1, \dots, b_m, C_1, \dots, C_m)$;
- *θ', its regular extension to $vars(H \longleftarrow b_1 \,\&\cdots\&\, b_m) \cup \bigcup_{i=1}^m vars(C_i \vee D_i)$;*

such that

- *D is a disjunctive factor of $(H \vee \bigvee_{i=1}^m D_i)\theta'$; and*
- *$c = *_{i=0}^m cv_i$.*

The hyperresolution operator $\mathcal{C}_P$ is defined as follows: $\mathcal{C}_P : 2^{Dis_{\mathcal{L}}} \longrightarrow 2^{Dis_{\mathcal{L}}}$, $\mathcal{C}_P(I) = \{D \mid D$ is an immediate consequence of I and $P\}$.

From Definition 2, we immediately get that $\mathcal{C}_P$ is monotonic and ω-continuous. So, the fixpoint semantics is based on the least fixpoint $\mathcal{C}_P^\omega$ of the hyperresolution operator:

Definition 3 (Fixpoint semantics). *Let P be a fuzzy disjunctive program of $\mathcal{L}$. $\mathcal{FS}(P) = \{(D, c_d) \mid (D, c_d)$ is a graded disjunction of $\mathcal{L}$, $c_d \leq \bigvee\{c'_d \mid (D', c'_d) \in \mathcal{C}_P^\omega, \vartheta \in Subst_{\mathcal{L}}, D'\vartheta \sqsubseteq D\}\}$.*

We close the section with the equivalence theorem:

Theorem 1 (Equivalence Theorem). *Let P be a fuzzy disjunctive program of $\mathcal{L}$. For any graded disjunction of $\mathcal{L}$, (D, c_d), $P \models (D, c_d)$ if and only if* $c_d \leq \bigvee\{c'_d \,|\, (D', c'_d) \in \mathcal{C}_P^\omega, \vartheta \in \mathit{Subst}_{\mathcal{L}}, D'\vartheta \sqsubseteq D\}$.

Proof. Consider the closure operator $\mathcal{C}_P$. Since $\mathcal{C}_P$ is continuous, at worst, we reach the least fixpoint in the ω-th step of the iteration of the operator from $\emptyset$, denoted as $\mathcal{C}_P^\omega$.

Next, we prove that

$$P \models \mathcal{C}_P^\omega. \tag{1}$$

We need to show that for any $\boldsymbol{L}$-model $\mathfrak{N}$ of P and $\alpha \in \omega$

$$\mathfrak{N} \models \mathcal{C}_P^\alpha. \tag{2}$$

We proceed by induction on α.

We construct a characteristic model [17, 19] $\mathfrak{M} = (\mathcal{U}_\mathfrak{M}, \mathcal{B}_\mathfrak{M})$ of P for $\mathcal{L}$ satisfying the condition:

For any graded disjunction (D, c_d) of $\mathcal{L}$,

$$\text{if } \mathfrak{M} \models^\vee (D, c_d), \text{ then } P \models (D, c_d). \tag{3}$$

Let $\mathcal{A}_\mathfrak{M} = \{a_n \,|\, n \in \omega\}$ where a_n are new pairwise different constant symbols not occurring in $\mathcal{L}$. We let $\mathcal{U}_\mathfrak{M} = \mathit{GTerm}_{\mathit{Func}_\mathcal{L} \cup \mathcal{A}_\mathfrak{M}}$. Since $\mathcal{A}_\mathfrak{M}$ contains the constants a_n, $\mathcal{U}_\mathfrak{M} \neq \emptyset$. A function symbol $f \in \mathit{Func}_\mathcal{L}$ is interpreted as the function $f^\mathfrak{M} : \mathcal{U}_\mathfrak{M}^{ar(f)} \longrightarrow \mathcal{U}_\mathfrak{M}$, $f^\mathfrak{M}(u_1, \ldots, u_{ar(f)}) = f(u_1, \ldots, u_{ar(f)})$ where $u_1, \ldots, u_{ar(f)} \in \mathcal{U}_\mathfrak{M}$. Let D be a ground atom disjunction of $\mathcal{L} \cup \mathcal{A}_\mathfrak{M}$. Denote $\overline{D} = \overline{D[e]}$ for some variable assignment e in $\mathfrak{M}$. Notice that since D is ground, $\overline{D}$ does not depend on e. We put

$\mathcal{B}_\mathfrak{M} = \{(\overline{B}, c_b) \,|$
- *$\exists\,(D, c_d) \in \mathcal{C}_P^\omega$, $\exists\,\vartheta \in \mathit{Subst}_{\mathcal{L}\cup\mathcal{A}_\mathfrak{M}}$ such that $B = D\vartheta$ is a ground instance of D in the language $\mathcal{L} \cup \mathcal{A}_\mathfrak{M}$;*
- *$c_b = \bigcup\{c'_d \,|\, \exists\,(D', c'_d) \in \mathcal{C}_P^\omega$, $\exists\,\vartheta' \in \mathit{Subst}_{\mathcal{L}\cup\mathcal{A}_\mathfrak{M}}$, $D'\vartheta'$ is ground, $\overline{D'\vartheta'} \subseteq \overline{B}\} > \mathbf{0}$;*
- *$\forall\,(D^*, c_d^*) \in \mathcal{C}_P^\omega$ such that $\exists\,\vartheta^* \in \mathit{Subst}_{\mathcal{L}\cup\mathcal{A}_\mathfrak{M}}$ where $B^* = D^*\vartheta^*$ is a ground instance of D^* in the language $\mathcal{L} \cup \mathcal{A}_\mathfrak{M}$ and $\overline{B^*} \subset \overline{B}$, $c_b > \bigcup\{c'_d \,|\, \exists\,(D', c'_d) \in \mathcal{C}_P^\omega$, $\exists\,\vartheta' \in \mathit{Subst}_{\mathcal{L}\cup\mathcal{A}_\mathfrak{M}}$, $D'\vartheta'$ is ground, $\overline{D'\vartheta'} \subseteq \overline{B^*}\}\}$.*

It is a simple matter to see that $\mathcal{B}_\mathfrak{M}$ satisfies the conditions i) - iv) from the definition of fuzzy disjunctive $\boldsymbol{L}$-model.

The following auxiliary fact will be usefull later: Let $\overline{B}$ be a finite set of triples in $\mathfrak{M}$. Then

$$\bigcup\{c'_b \,|\, (\overline{B'}, c'_b) \in \mathcal{B}_\mathfrak{M}, \overline{B'} \subseteq \overline{B}\} = \\ \bigcup\{c'_d \,|\, \exists\,(D', c'_d) \in \mathcal{C}_P^\omega, \exists\,\vartheta' \in \mathit{Subst}_{\mathcal{L}\cup\mathcal{A}_\mathfrak{M}}, D'\vartheta' \text{ is ground}, \overline{D'\vartheta'} \subseteq \overline{B}\}. \tag{4}$$

By 2, we get that $\mathfrak{M} \models^{\vee} P$.

It remains to prove the main property 3 of $\mathfrak{M}$ and the point ii) of the theorem. Let (D, c_d) be a graded disjunction of $\mathcal{L}$. Suppose $\mathfrak{M} \models^{\vee} (D, c_d)$. Consider a variable assignment e^* such that for all $v \neq v' \in vars(D)$, $e^*(v) \neq e^*(v') \in \mathcal{A}_{\mathfrak{M}}$. So, we assign new pairwise different constants not occurring in $\mathcal{L}$ to the variables in D. We can see that $\overline{D[e^*]} = \overline{De^*|_{vars(D)}}$. Since $\mathfrak{M} \models^{\vee} D[e^*]$,

$$c_d \leq \bigcup \{c'_b \mid (\overline{B'}, c'_b) \in \mathcal{B}_{\mathfrak{M}}, \overline{B'} \subseteq \overline{De^*|_{vars(D)}}\}.$$

By the fact 4, we get

$$c_d \leq \bigcup \{c'_d \mid \exists\, (D', c'_d) \in \mathcal{C}^{\omega}_P, \exists\, \vartheta' \in Subst_{\mathcal{L} \cup \mathcal{A}_{\mathfrak{M}}},\ dom(\vartheta') = vars(D'), D'\vartheta' \text{ is ground}, \overline{D'\vartheta'} \subseteq \overline{De^*|_{vars(D)}}\}.$$

It is evident that $Atoms(D'\vartheta') \subseteq Atoms(De^*|_{vars(D)})$. This implies that there exists $\vartheta \in Subst_{\mathcal{L}}$, $dom(\vartheta) = vars(D')$, such that $Atoms(D'\vartheta) \subseteq Atoms(D)$.

This yields

$$c_d \leq \bigcup \{c'_d \mid \exists\, (D', c'_d) \in \mathcal{C}^{\omega}_P, \exists\, \vartheta \in Subst_{\mathcal{L}}, D'\vartheta \sqsubseteq D\}. \tag{5}$$

if (D, c_d) is a graded strong literal disjunction of $\mathcal{L}$ and $P \models (D, c_d)$, then $\mathfrak{M} \models^{\vee} (D, c_d)$ ($\mathfrak{M}$ is a fuzzy disjunctive $\boldsymbol{L}$-model of P), and consequently the property 5 holds. So, the left-right direction of the point ii) of the theorem holds.

The property 3 and the right-left direction of the point ii) are proved as follows: Let $\mathfrak{M} \models^{\vee} (D, c_d)$. We show that $P \models (D, c_d)$. Denote

$$C'_d = \{c'_d \mid \exists\, (D', c'_d) \in \mathcal{C}^{\omega}_P, \exists\, \vartheta \in Subst_{\mathcal{L}}, D'\vartheta \sqsubseteq D\},$$
$$c^*_d = \bigcup C'_d.$$

Since $\mathfrak{M} \models^{\vee} (D, c_d)$, by the above considerations, $c_d \leq c^*_d$. Note that this is also the assumption of the right-left direction of the point ii). We have proved that for every fuzzy disjunctive $\boldsymbol{L}$-model $\mathfrak{N}$, if $\mathfrak{N} \models^{\vee} P$, then $\mathfrak{N} \models^{\vee} (D', c'_d)$, $(D', c'_d) \in \mathcal{C}^{\omega}_P$, and also $\mathfrak{N} \models^{\vee} (D'\vartheta, c'_d)$ where $\vartheta \in Subst_{\mathcal{L}}$ and $D'\vartheta \sqsubseteq D$. This yields $\mathfrak{N} \models^{\vee} (D, c'_d)$ for every $c'_d \in C'_d$ because of $\overline{D'\vartheta[e]} \subseteq \overline{D[e]}$ for any variable assignment e in $\mathfrak{N}$. We next show that $\mathfrak{N} \models^{\vee} (D, c^*_d)$. Let e be a variable assignment in $\mathfrak{N}$. Denote

$$c^{\#}_d = \bigcup \{c''_d \mid (\overline{D''}, c''_d) \in \mathcal{B}_{\mathfrak{N}}, \overline{D''} \subseteq \overline{D[e]}\}.$$

Since $\mathfrak{N} \models^{\vee} (D, c'_d)[e]$, $c'_d \in C'_d$, $c'_d \leq c^{\#}_d$. Hence, $c_d \leq c^*_d = \bigcup C'_d \leq c^{\#}_d$. So, $\mathfrak{N} \models^{\vee} (D, c^*_d)[e]$. We obtain that $\mathfrak{N} \models^{\vee} (D, c^*_d)$ and $\mathfrak{N} \models^{\vee} (D, c_d)$. Thus $P \models^{\vee} (D, c_d)$. The property 3 and the right-left direction of the point ii) of the theorem hold.

Let (Im, c_i) be a graded implication of $\mathcal{L}$. Then the following are equivalent:

$$\mathfrak{M} \models^{\vee} (Im, c_i) \iff P \models (Im, c_i).$$

Hence, $\mathfrak{M}$ is a characteristic model of P for $\mathcal{L}$ and the point i) is proved.

Thus the proof is completed. $\square$

As a consequence we obtain the coincidence of the semantics:

Corollary 1. *Let P be a fuzzy disjunctive program of $\mathcal{L}$.*

$$\mathcal{D}S(P) = \mathcal{F}S(P).$$

3 Procedural semantics

A most important aim of a resolution procedure is to provide an answer satisfying some suitable logical conditions for a query and program. For instance, in the case of Horn programs and *SLD*-resolution, an answer for a query $\longleftarrow G$ (G is a conjunction of atoms) and for a program P, is a substitution ϑ such that $G\vartheta$ is a common logical consequence of P and G. Concerning disjunctive programs, the question is more complex. Our intention is to find out all disjunctions being common logical consequences of a program and query disjunction. For this reason, we must consider not only instances of the query disjunction, but also superdisjunctions of these instances. For example, let $P = \{p(f(x)) \vee q(f(x)) \vee s(x)\}$ and a query be of the form $\longleftarrow p(y) \vee r(y)$. An answer is both the substitution $\{y/f(x), x/x\} = mgu(p(y), p(f(x)))$ and the remainder disjunction $q(f(x)) \vee s(x)$. Indeed, the resulting disjunction $q(f(x)) \vee s(x) \vee (p(y) \vee r(y))\{y/f(x)\}$, containing the instantiated query disjunction and the remainder, is a common logical consequence of P and the query disjunction.

In order to compute such compound answers, *ULSLD*-resolution has been developed for disjunctive programs on Boolean lattices in [18]. The abbreviation *ULSLD* stands for **U**n**L**imited **S**election rule driven **L**inear resolution for **D**isjunctions. To outline how the resolution works, let us consider Fig. 1, where an unlimited (fuzzy) derivation step is drawn. Let D_q be the selected disjunction by some selection rule R from the old goal. We choose some sequence of atoms $d_1, \dots, d_{|H|}$ from D_q and unify the disjunction $d_1 \vee \cdots \vee d_{|H|}$ with the subdisjunction H of the head of the implication. Let θ be a most general unifier. Then the resulting new goal is formed of the new subgoals of the form $D_q \vee b_i$ and the remaining ones. All the subgoals are instantiated by a regular extension θ' (of θ) to the variables appearing in the old goal and the implication. [1] A new part of an unlimited derivation step is a step remainder disjunction. In our example, the step remainder disjunction $Z\theta'$ consists of the remaining atoms from the head of the implication instantiated by θ'.

We next provide some generalisation of *ULSLD*-resolution to the case of residuated lattices. Let P be a fuzzy disjunctive program and $\longleftarrow D_1, \dots, D_k$ a goal (D_i are disjunctions of atoms). A refutation answer will consists of a refutation answer substitution θ, a refutation remainder $(rd_1, \dots, rd_k)$ (rd_i are disjunctions of atoms), and moreover of a refutation truth vector $(c_1, \dots, c_k)$ where $c_i \in \boldsymbol{L}$. Each remainder disjunction rd_i is associated with the corresponding subgoal D_i. By the remainder disjunctions, we supplement the subgoals

[1] They do not share common variables.

Old goal

$\longleftarrow D_1, \ldots, D_{q-1}, \qquad D_q, \qquad D_{q+1}, \ldots, D_k$

$$\boxed{d_1 \vee \cdots \vee d_{|H|}}$$

$$\theta$$

Implication

$Z \qquad \vee \qquad \boxed{H} \longleftarrow b_1 \,\&\, \cdots \,\&\, b_m$

New goal

$\longleftarrow (D_1, \ldots, D_{q-1},$

New subgoals $\begin{cases} D_q \vee b_1, \\ \vdots \quad \vdots \\ D_q \vee b_m, \end{cases}$

$D_{q+1}, \ldots, D_k)\theta'$

Step remainder disjunction

$Z\theta'$

Fig. 1. Unlimited fuzzy derivation step.

instantiated with the substitution θ. Also, each truth value c_i is assigned to the corresponding subgoal D_i so that

$$(rd_i \vee D_i\theta, c_i)$$

is a common fuzzy logical consequence of the program P and the graded disjunction (D_i, c_i).

We now describe *ULSLD*-resolution more precisely:

A finite sequence $D_1, \ldots, D_k$ $(k \geq 0)$ of disjunctions of atoms of $\mathcal{L}$ is called a goal of $\mathcal{L}$. We denote the goal by $\longleftarrow D_1, \ldots, D_k$. The empty goal is denoted as $\Box$.

By a selection rule R we mean a function which returns an index q, $1 \leq q \leq k$ for a non-empty goal $\longleftarrow D_1, \ldots, D_k$ $(k \geq 1)$. For the empty goal $\Box$, we put $R(\Box) = 0$. By $|D|$ we denote the number of the atoms (not necessarily different) occurring in the disjunction D. E.g. $|a \vee b \vee a| = 3$.

Definition 4. *For a given graded implication (Im, c) of $\mathcal{L}$, a selection rule R, a goal $\longleftarrow D_1, \ldots, D_k$ $(k \geq 1)$ of $\mathcal{L}$ where Im and $\longleftarrow D_1, \ldots, D_k$ do not share variables in common; an unlimited fuzzy derivation step is defined as follows:*

Let

- *Im be of the form $Z \vee H \longleftarrow b_1 \,\&\, \cdots \,\&\, b_m$ $(m \geq 0)$;*
- *$R(\longleftarrow D_1, \ldots, D_k) = q$, $1 \leq q \leq k$;*
- *there exist an mgu θ for the disjunctions H and $d_1 \vee \cdots \vee d_{|H|}$ where for $i = 1, \ldots, |H|$ $d_i \in Atoms(D_q)$, and $dom(\theta) = vars(H, d_1 \vee \cdots \vee d_{|H|})$;*

- θ' *be a regular extension of* θ *to* $vars(Im, \longleftarrow D_1, \dots, D_k)$.

The resulting new goal is of the form

$$\longleftarrow (D_1, \dots, D_{q-1}, D_q \vee b_1, \dots, D_q \vee b_m, D_{q+1}, \dots, D_k)\theta';$$

the step remainder disjunction is defined as $Z\theta'$.

The entire derivation step is denoted as:

$$\longleftarrow D_1, \dots, D_k \vdash\!\xrightarrow{Z\theta' \,|\, (Im, c),\, \theta',\, R} \longleftarrow (D_1, \dots, D_{q-1}, D_q \vee b_1, \dots, D_q \vee b_m, D_{q+1}, \dots, D_k)\theta'.$$

See Fig. 1.

Definition 5. *Let* P *be a fuzzy disjunctive program of* $\mathcal{L}$, $G_0 = \longleftarrow D_1, \dots, D_k$ $(k \geq 0)$ *be a goal of* $\mathcal{L}$, *and* R *a selection rule.*

A ULSLD-derivation for G_0 *of length* n $(n \geq 0)$ *is a finite sequence of goals* $G_0, \dots, G_n$ *satisfying*

$$\text{for } 0 \leq i < n : \ G_i \vdash\!\xrightarrow{rm_i \,|\, (Im_i, c_i),\, \theta'_i,\, R} G_{i+1}$$

where (Im_i, c_i) *is an implication factor of a variant of a graded implication in* P.
If $G_n = \Box$, *such a derivation is called a refutation. Given a refutation for* G_0, *we define the refutation answer substitution* ϑ, $dom(\vartheta) = vars(G_0)$, *the refutation remainder, a tuple of disjunctions* $(rd_1, \dots, rd_k)$, *and the refutation truth vector* $(cv_1, \dots, cv_k)$, $cv_i \in \boldsymbol{L}$, *by recursion on the length* n *of the refutation:*

- *If* $n = 0$, *then* $G_0 = \Box$ *and* $k = 0$. *We let the refutation answer substitution be* $\emptyset$, *the refutation remainder and truth vector be the empty tuples.*
- *If* $n \geq 1$, *then* $G_0 \neq \Box$ *and* $k \geq 1$. [2] *Let* $R(\longleftarrow D_1, \dots, D_k) = q$ $(1 \leq q \leq k)$ *and* Im_0 *be of the form* $H \longleftarrow b_1 \,\&\, \cdots \,\&\, b_m$. *Then the first derivation step is of the form:*

$$G_0 \vdash\!\xrightarrow{rm_0 \,|\, (Im_0, c_0),\, \theta'_0,\, R} G_1,$$

$$G_1 = \longleftarrow (D_1, \dots, D_{q-1}, D_q \vee b_1, \dots, D_q \vee b_m, D_{q+1}, \dots, D_k)\theta'_0.$$

  Denote the rest of the refutation of the length $n-1$ *as follows:*

$$G_1 \models\!\xRightarrow{\left.\begin{matrix}(rd'_1, \dots, rd'_{q-1}, rd'_{q_1}, \dots, rd'_{q_m}, rd'_{q+1}, \dots, rd'_k) \\ (cv'_1, \dots, cv'_{q-1}, cv'_{q_1}, \dots, cv'_{q_m}, cv'_{q+1}, \dots, cv'_k)\end{matrix}\right|\, P, \vartheta', R} \Box$$

  where, by recursion,
  – ϑ', $dom(\vartheta') = vars(G_1)$, *is the refutation answer substitution,*

[2] We may execute derivation steps only for non-empty goals.

– $(rd'_1, \ldots, rd'_{q-1}, rd'_{q_1}, \ldots, rd'_{q_m}, rd'_{q+1}, \ldots, rd'_k)$ *is the refutation remainder, and*
– $(cv'_1, \ldots, cv'_{q-1}, cv'_{q_1}, \ldots, cv'_{q_m}, cv'_{q+1}, \ldots, cv'_k)$ *the refutation truth vector for* G_1.

Denote by V_r $vars(rd'_1, \ldots, rd'_{q-1}, rd'_{q_1}, \ldots, rd'_{q_m}, rd'_{q+1}, \ldots, rd'_k)$. *Let* ϑ^* *be a regular extension of* ϑ' *to* $range(\theta'_0) \supseteq vars(G_1)$ *such that*

$$range(\vartheta^*|_{range(\theta'_0)-vars(G_1)}) \cap V_r = \emptyset.$$

Then

– $\vartheta = \theta'_0|_{vars(G_0)} \circ \vartheta^*$,
– $(rd_1, \ldots, rd_k) = (rd'_1, \ldots, rd'_{q-1}, rd'_q, rd'_{q+1}, \ldots, rd'_k)$ *where* rd'_q *is a disjunctive factor of* $rm_0\vartheta^* \vee rd'_{q_1} \vee \cdots \vee rd'_{q_m}$, *and*
– $(cv_1, \ldots, cv_k) = (cv'_1, \ldots, cv'_{q-1}, c_0 * *_{i=1}^{m} cv'_{q_i}, cv'_{q+1}, \ldots, cv'_k)$.

The entire refutation is denoted as

$$G_0 \left|\!\xrightarrow[]{\begin{smallmatrix}(rd_1,\ldots,rd_k)\\(cv_1,\ldots,cv_k)\end{smallmatrix}\Big| P, \vartheta, R}\right. \square.$$

Theorem 2 (Soundness and Completeness of *ULSLD*-resolution). *Let* P *be a fuzzy disjunctive program of* $\mathcal{L}$, R *a selection rule, and* $G = \longleftarrow D_1, \ldots, D_k$ $(k \geq 0)$ *be a goal of* $\mathcal{L}$.

There exist

- *graded disjunctions* $(C_1, cv_1), \ldots, (C_k, cv_k)$ *where* (C_i, cv_i) *is a disjunctive factor of a variant of a graded disjunction in* $\mathcal{C}_P^\omega$, $C_i = C^i \vee c_1^i \vee \cdots \vee c_{u^i}^i$ $(u^i \geq 0)$, *and for* $i \neq j$, $vars(C_i) \cap vars(C_j) = \emptyset$, $vars(C_i) \cap vars(G) = \emptyset$;
- *an mgu* θ *for the tuples*

$$\begin{matrix}(c_1^1,\ldots,c_{u^1}^1,\ldots,c_1^k,\ldots,c_{u^k}^k)\\(d_1^1,\ldots,d_{u^1}^1,\ldots,d_1^k,\ldots,d_{u^k}^k)\end{matrix} \tag{6}$$

where $d_j^i \in Atoms(D_i)$ *and* $dom(\theta) = vars(6)$;
- *and its regular extension* θ' *to* $vars(G, C_1, \ldots, C_k)$;

if and only if there exists a ULSLD-refutation

$$G \left|\!\xrightarrow[]{\begin{smallmatrix}(rd_1,\ldots,rd_k)\\(dv_1,\ldots,dv_k)\end{smallmatrix}\Big| P, \vartheta, R}\right. \square;$$

so that $\vartheta = \theta'|_{vars(G)}$ *and* $rd_i = C^i\theta'$, $dv_i = cv_i$ *for* $i = 1 \ldots k$.

Proof. If $G = \square$, $k = 0$, the theorem holds trivially:

There exist

- graded disjunctions $(C_1, cv_1), \dots, (C_0, cv_0)$ where (C_i, cv_i) is a disjunctive factor of a variant of a disjunction in $\mathcal{C}_P^\omega$, $C_i = C^i \vee c_1^i \vee \dots \vee c_{u^i}^i$ $(u^i \geq 0)$, and for $i \neq j$, $vars(C_i) \cap vars(C_j) = \emptyset$, $vars(C_i) \cap vars(G) = \emptyset$;
- an mgu $\theta = \emptyset$ for the tuples

$$\begin{array}{l} (c_1^1, \dots, c_{u^1}^1, \dots, c_1^0, \dots, c_{u^0}^0) \\ (d_1^1, \dots, d_{u^1}^1, \dots, d_1^0, \dots, d_{u^0}^0) \end{array} \tag{7}$$

  where $d_j^i \in Atoms(D_i)$ and $dom(\theta) = vars(7) = \emptyset$;
- and its regular extension $\theta' = \emptyset$ to $vars(\Box) = \emptyset$;

if and only if there exists a $ULSLD$-refutation

$$\Box \models\!\!\!\!\xrightarrow{\begin{smallmatrix}() \\ ()\end{smallmatrix}\Big| P, \emptyset, R} \Box;$$

so that $\emptyset = \emptyset|_\emptyset$ and $rd_i = C^i\theta'$, $dv_i = cv_i$ for $i = 1 \dots 0$.

We next prove the following statement: By this technical statement we can rewrite a $\mathcal{C}_P$ step to a $ULSLD$-refutation step and vice-versa, which we shall use in the proofs of $(\Longrightarrow)$ and $(\Longleftarrow)$.

Let $C_1, \dots, C_{q-1}, C_{q_1}, \dots, C_{q_s}, C_{q+1}, \dots, C_k$ $(k \geq 1,\ s \geq 0,\ 1 \leq q \leq k)$ be atom disjunctions, $G = \longleftarrow D_1, \dots, D_k$ be a goal, and $Im = H_2 \vee h_1 \vee \dots \vee h_r \longleftarrow b_1 \,\&\, \dots \,\&\, b_s$ $(r \geq 0)$ be an implication satisfying:

- $C_i \ = C^i \vee c_1^i \vee \dots \vee c_{u^i}^i \quad (u^i \geq 0)$ for $i \neq q$,

  $C_{q_i} = c_{q_1}^{\ i} \vee \dots \vee c_{q_{u_q}{}^i}^{\ i} \vee C_{q_2}{}^i \vee c_{q_1}{}_1^i \vee \dots \vee c_{q_1}{}_{v_q{}^i}^i \quad (u_q{}^i, v_q{}^i \geq 0)$;
- $vars(C_{q_i}) \cap vars(C_{q_j}) = \emptyset$ for $i \neq j$, $vars(C_i) \cap vars(C_j) = \emptyset$ for $i \neq j$, $i \neq q$, $j \neq q$, $vars(C_{q_i}) \cap vars(C_j) = \emptyset$ for $j \neq q$, $vars(C_{q_i}) \cap vars(G) = \emptyset$, $vars(C_i) \cap vars(G) = \emptyset$ for $i \neq q$, $vars(C_{q_i}) \cap vars(Im) = \emptyset$, $vars(C_i) \cap vars(Im) = \emptyset$ for $i \neq q$, $vars(G) \cap vars(Im) = \emptyset$.

Consider the following tuple of sets:

$$\begin{array}{l}
\left(\left\{\begin{array}{l} (d_{q_1}{}_1^0, \dots, d_{q_1}{}_r^0) \\ (\ h_1\ , \dots,\ h_r\ ) \end{array}\right\},\right. \\
\left\{\begin{array}{l} (\ b_1\ , \dots,\ b_1\ , \dots, b_s\ , \dots,\ b_s\ ) \\ (c_{q_1}^{\ 1}, \dots, c_{q_{u_q}{}^1}^{\ 1}, \dots, c_{q_1}^{\ s}, \dots, c_{q_{u_q}{}^s}^{\ s}) \end{array}\right\}, \left\{\begin{array}{l} (d_{q_1}{}_1^1, \dots, d_{q_1}{}_{v_q{}^1}^1, \dots, d_{q_1}{}_1^s, \dots, d_{q_1}{}_{v_q{}^s}^s) \\ (c_{q_1}{}_1^1, \dots, c_{q_1}{}_{v_q{}^1}^1, \dots, c_{q_1}{}_1^s, \dots, c_{q_1}{}_{v_q{}^s}^s) \end{array}\right\}, \\
\qquad \left.\left\{\begin{array}{l} (d_1^1, \dots, d_{u^1}^1, \dots, d_1^{q-1}, \dots, d_{u^{q-1}}^{q-1}, d_1^{q+1}, \dots, d_{u^{q+1}}^{q+1}, \dots, d_1^k, \dots, d_{u^k}^k) \\ (c_1^1, \dots, c_{u^1}^1, \dots, c_1^{q-1}, \dots, c_{u^{q-1}}^{q-1}, c_1^{q+1}, \dots, c_{u^{q+1}}^{q+1}, \dots, c_1^k, \dots, c_{u^k}^k) \end{array}\right\}\right)
\end{array} \tag{8}$$

where $d_j^i \in Atoms(D_i)$ for $i \neq q$, and $d_{q_1}{}_j^i \in Atoms(D_q)$ for $i = 0, \dots, s$.

Let

$$V = vars(G, Im, C_1, \ldots, C_{q-1}, C_{q_1}, \ldots, C_{q_s}, C_{q+1}, \ldots, C_k),$$

$$\begin{aligned}
V_1 &= vars(b_1, \ldots, b_s) \cup \textstyle\bigcup_{i=1}^{s} vars({c_q}_1^i, \ldots, {c_q}_{u_q{}^i}^i),\\
V_1' &= vars(Im) \cup \textstyle\bigcup_{i=1}^{s} vars(C_{q_i}),\\
V_2 &= vars({d_{q_1}}_1^0, \ldots, {d_{q_1}}_r^0) \cup vars(h_1, \ldots, h_r) \cup\\
&\quad \textstyle\bigcup_{i=1}^{s} vars({d_{q_1}}_1^i, \ldots, {d_{q_1}}_{v_q{}^i}^i) \cup \bigcup_{i=1}^{s} vars({c_{q_1}}_1^i, \ldots, {c_{q_1}}_{v_q{}^i}^i) \cup\\
&\quad \textstyle\bigcup_{\substack{i=1,\\ i\neq q}}^{k} vars(d_1^i, \ldots, d_{u^i}^i) \cup \bigcup_{\substack{i=1,\\ i\neq q}}^{k} vars(c_1^i, \ldots, c_{u^i}^i),\\
V_2' &= vars(G) \cup vars(H_2, h_1, \ldots, h_r) \cup \textstyle\bigcup_{i=1}^{s} vars({C_{q_2}}^i, {c_{q_1}}_1^i, \ldots, {c_{q_1}}_{v_q{}^i}^i) \cup\\
&\quad vars(C_1, \ldots, C_{q-1}, C_{q+1}, \ldots, C_k),
\end{aligned}$$

$$\begin{aligned}
V_3 &= vars({d_{q_1}}_1^0, \ldots, {d_{q_1}}_r^0) \cup vars(h_1, \ldots, h_r),\\
V_3' &= vars(G, Im),\\
V_4 &= vars(b_1, \ldots, b_s) \cup \textstyle\bigcup_{i=1}^{s} vars({c_q}_1^i, \ldots, {c_q}_{u_q{}^i}^i) \cup\\
&\quad \textstyle\bigcup_{i=1}^{s} vars({d_{q_1}}_1^i, \ldots, {d_{q_1}}_{v_q{}^i}^i) \cup \bigcup_{i=1}^{s} vars({c_{q_1}}_1^i, \ldots, {c_{q_1}}_{v_q{}^i}^i) \cup\\
&\quad \textstyle\bigcup_{\substack{i=1,\\ i\neq q}}^{k} vars(d_1^i, \ldots, d_{u^i}^i) \cup \bigcup_{\substack{i=1,\\ i\neq q}}^{k} vars(c_1^i, \ldots, c_{u^i}^i),\\
V_4' &= vars(b_1, \ldots, b_s) \cup \textstyle\bigcup_{i=1}^{s} vars(D_q) \cup \bigcup_{i=1}^{s} vars(C_{q_i}) \cup\\
&\quad vars(D_1, \ldots, D_{q-1}, D_{q+1}, \ldots, D_k) \cup vars(C_1, \ldots, C_{q-1}, C_{q+1}, \ldots, C_k).
\end{aligned}$$

There exist

- ϕ_1, $dom(\phi_1) = V_1$, an *mgu* for the set

$$\left\{ \begin{matrix} (\ b_1 \ ,\ldots,\ \ b_1 \ \ ,\ldots, b_s \ ,\ldots,\ \ b_s \ ) \\ ({c_q}_1^1,\ldots,{c_q}_{u_q{}^1}^1,\ldots,{c_q}_1^s,\ldots,{c_q}_{u_q{}^s}^s) \end{matrix} \right\}, \tag{9}$$

  ϕ_1 will be intended as an mgu of a $\mathcal{C}_P$ step
- ϕ_1', its regular extension to V,
- ϕ_2, $dom(\phi_2) = range(\phi_1'|_{V_2})$, an *mgu* for the tuple of sets

$$\begin{aligned}
\Bigg(&\left\{ \begin{matrix} ({d_{q_1}}_1^0,\ldots,{d_{q_1}}_r^0) \\ (\ h_1 \ ,\ldots,\ h_r \ ) \end{matrix} \right\} \phi_1', \\
&\qquad \left\{ \begin{matrix} ({d_{q_1}}_1^1,\ldots,{d_{q_1}}_{v_q{}^1}^1,\ldots,{d_{q_1}}_1^s,\ldots,{d_{q_1}}_{v_q{}^s}^s) \\ ({c_{q_1}}_1^1,\ldots,{c_{q_1}}_{v_q{}^1}^1,\ldots,{c_{q_1}}_1^s,\ldots,{c_{q_1}}_{v_q{}^s}^s) \end{matrix} \right\} \phi_1', \\
&\left\{ \begin{matrix} (d_1^1,\ldots,d_{u^1}^1,\ldots,d_1^{q-1},\ldots,d_{u^{q-1}}^{q-1},d_1^{q+1},\ldots,d_{u^{q+1}}^{q+1},\ldots,d_1^k,\ldots,d_{u^k}^k) \\ (c_1^1,\ldots,c_{u^1}^1,\ldots,c_1^{q-1},\ldots,c_{u^{q-1}}^{q-1},c_1^{q+1},\ldots,c_{u^{q+1}}^{q+1},\ldots,c_1^k,\ldots,c_{u^k}^k) \end{matrix} \right\} \phi_1' \Bigg),
\end{aligned} \tag{10}$$

– ϕ'_2, its regular extension to $range(\phi'_1)$;
if and only if there exist
– ϕ_3, $dom(\phi_3) = V_3$, an *mgu* for the set

$$\left\{ \begin{array}{l} ({d_{q_1}}^0_1, \dots, {d_{q_1}}^0_r) \\ (\ h_1\ , \dots, \ h_r\ ) \end{array} \right\}, \tag{11}$$

ϕ_3 will be intended as an mgu of a *ULSLD*-refutation step
– ϕ'_3, its regular extension to V,
– ϕ_4, $dom(\phi_4) = range(\phi'_3|_{V_4})$, an *mgu* for the tuple of sets

$$\left(\left\{ \begin{array}{l} (\ b_1\ , \dots, \ b_1\ , \dots, b_s\ , \dots, \ b_s\ ) \\ ({c_q}^1_1, \dots, {c_q}^1_{{u_q}^1}, \dots, {c_q}^s_1, \dots, {c_q}^s_{{u_q}^s}) \end{array} \right\} \phi'_3, \right.$$
$$\left\{ \begin{array}{l} ({d_{q_1}}^1_1, \dots, {d_{q_1}}^1_{{v_q}^1}, \dots, {d_{q_1}}^s_1, \dots, {d_{q_1}}^s_{{v_q}^s}) \\ ({c_{q_1}}^1_1, \dots, {c_{q_1}}^1_{{v_q}^1}, \dots, {c_{q_1}}^s_1, \dots, {c_{q_1}}^s_{{v_q}^s}) \end{array} \right\} \phi'_3,$$
$$\left. \left\{ \begin{array}{l} (d^1_1, \dots, d^1_{u^1}, \dots, d^{q-1}_1, \dots, d^{q-1}_{u^{q-1}}, d^{q+1}_1, \dots, d^{q+1}_{u^{q+1}}, \dots, d^k_1, \dots, d^k_{u^k}) \\ (c^1_1, \dots, c^1_{u^1}, \dots, c^{q-1}_1, \dots, c^{q-1}_{u^{q-1}}, c^{q+1}_1, \dots, c^{q+1}_{u^{q+1}}, \dots, c^k_1, \dots, c^k_{u^k}) \end{array} \right\} \phi'_3 \right), \tag{12}$$

– ϕ'_4, its regular extension to $range(\phi'_3)$;
such that

$$\phi'_1 \circ \phi'_2 = \phi'_3 \circ \phi'_4. \tag{13}$$

The proof is technical. The main idea is based on the following: Consider tuples

$$(A, B)$$
$$(C, D)$$

where A, B, C, D are subtuples such that the length of A equals the length of C, and the length of B equals the length of D.
If we first can unify, by an mgu ϕ_1, A, C and next we can unify, by an mgu ϕ_2, $B\phi_1$, $D\phi_1$;
then we can also unify, by an mgu ϕ_3, B, D and next we can unify, by an mgu ϕ_4, $A\phi_3$, $C\phi_3$ such that

$$\phi_1 \circ \phi_2 = \phi_3 \circ \phi_4 = mgu((A, B), (C, D))$$

.

We now prove the main theorem.

($\Longrightarrow$): At first, consider the following function: $Eval : \mathbf{X}^{<\omega}(\mathcal{C}^\omega_P) \longrightarrow \mathbf{X}^{<\omega}\,\omega$. Let $a = (a_1, \dots, a_n) \in \mathbf{X}^{<\omega}(\mathcal{C}^\omega_P)$. The function $Eval(a)$ firstly computes the tuple $(Least(a_1), \dots, Least(a_n))$ such that $Least(a_i)$ is the least number less than ω satisfying $a_i \in \mathcal{C}^{Least}_P(a_i)$. Denote by m the maximum of $Least(a_1), \dots, Least(a_n)$. Then the function yields the tuple $(o_1, \dots, o_m)$ where o_i means the number of occurrences of i in the tuple $(Least(a_1), \dots, Least(a_n))$.

The set $\mathbf{X}^{<\omega}\,\omega$ is well-ordered by the reverse lexicographic ordering $\prec$, defined as follows:

$$\begin{aligned}\forall\, t, t' \in \mathbf{X}^{<\omega}\,\omega\,(t \prec t' \longleftrightarrow \\ \exists\, l, l' \in \omega\,(t \in \omega^{l} \wedge t \in \omega^{l'} \wedge \\ (l < l' \vee \\ l = l' \wedge \exists\, 1 \le i \le l\,(t_i < t'_i \wedge (t_{i+1}, \dots, t_l) = \\ (t'_{i+1}, \dots, t'_l)\ )))).\end{aligned} \tag{14}$$

Assume that there exist

- graded disjunctions $(C_1, cv_1), \dots, (C_k, cv_k)$ where (C_i, cv_i) is a disjunctive factor of a variant (C_i^*, cv_i) of a graded disjunction $(C_i^{\#}, cv_i) \in \mathcal{C}_P^{\omega}$, $C_i = C^i \vee c_1^i \vee \cdots \vee c_{u^i}^i$ $(u^i \ge 0)$, and for $i \ne j$, $vars(C_i) \cap vars(C_j) = \emptyset$, $vars(C_i) \cap vars(G) = \emptyset$;
- an mgu θ for the tuples 6:

$$\begin{matrix}(c_1^1, \dots, c_{u^1}^1, \dots, c_1^k, \dots, c_{u^k}^k) \\ (d_1^1, \dots, d_{u^1}^1, \dots, d_1^k, \dots, d_{u^k}^k)\end{matrix}$$

  where $d_j^i \in Atoms(D_i)$ and $dom(\theta) = vars(6)$;
- and its regular extension θ' to $vars(G, C_1, \dots, C_k)$.

We prove the direction $\Longrightarrow$ by induction with the measure

$$Eval(((C_1^{\#}, cv_1), \dots, (C_k^{\#}, cv_k))) \in \mathbf{X}^{<\omega}\,\omega.$$

The base case. Let $Eval(((C_1^{\#}, cv_1), \dots, (C_k^{\#}, cv_k))) = ()$. By the definition of $Eval$, this yields $Max((Least((C_1^{\#}, cv_1)), \dots, Least((C_k^{\#}, cv_k)))) = 0$. However, for every $i = 1, \dots, k$, $Least((C_i^{\#}, cv_i)) > 0$ must hold because of $\mathcal{C}_P^0 = \emptyset$. Therefore, we get the case $k = 0$, which we have shown at the beginning of the proof.

The induction case. Assume that $Eval(((C_1^{\#}, cv_1), \dots, (C_k^{\#}, cv_k))) \succ ()$ and the statement holds $\forall t \in \mathbf{X}^{<\omega}(\mathcal{C}_P^{\omega})$ where $Eval(t) \prec Eval(((C_1^{\#}, cv_1), \dots, (C_k^{\#}, cv_k)))$. If $k = 0$ held, then $Eval(((C_1^{\#}, cv_1), \dots, (C_0^{\#}, cv_0))) = ()$, which would be a contradiction with $Eval(((C_1^{\#}, cv_1), \dots, (C_k^{\#}, cv_k))) \succ ()$. Hence, $k > 0$ and therefore $G \ne \square$. Let $R(G) = q$, $1 \le q \le k$. Denote by α_q $Least((C_q^{\#}, cv_q))$. So, $(C_q^{\#}, cv_q) \in \mathcal{C}_P^{\alpha_q}$. Without loss of generality and by the definition of $\mathcal{C}_P$, there exist

- $({c_q}_1^i \vee \cdots \vee {c_q}_{u_q{}^i}^i \vee {C_q}^i, {cv_q}_i)$, $i = 1, \dots, s$, disjunctive factors of some variants of graded disjunctions in $\mathcal{C}_P^{\alpha_q - 1}$;
- $(H \longleftarrow b_1 \,\&\cdots\&\, b_s, {cv_q}_0)$, an implication factor of a variant of a graded implication in P;
- $H \longleftarrow b_1 \,\&\cdots\&\, b_s$, ${c_q}_1^i \vee \cdots \vee {c_q}_{u_q{}^i}^i \vee {C_q}^i$, $i = 1 \dots s$, G, $C_1, \dots, C_{q-1}, C_{q+1}, \dots, C_k$ do not share variables in common;

- η is an *mgu* for the tuples

$$\begin{array}{c}(b_1 ,\dots, b_1 ,\dots, b_s ,\dots, b_s),\\ (c_{q}{}_1^1,\dots,c_{q}{}_{u_q{}^1}^1,\dots,c_{q}{}_1^s,\dots,c_{q}{}_{u_q{}^s}^s).\end{array} \tag{15}$$

$dom(\eta) = vars(b_1, \dots, b_s, c_{q}{}_1^1, \dots, c_{q}{}_{u_q{}^1}^1, \dots, c_{q}{}_1^s, \dots, c_{q}{}_{u_q{}^s}^s) = U$; and

- η' is its regular extension to $vars(H \longleftarrow b_1 \,\&\cdots\&\, b_s) \cup \bigcup_{i=1}^s vars(c_{q}{}_1^i \vee \cdots \vee c_{q}{}_{u_q{}^i}^i \vee C_q{}^i)$;

such that C_q^* is a disjunctive factor of $(H \vee \bigvee_{i=1}^s C_q{}^i)\eta'$, $cv_q = *_{i=0}^s cv_{q_i}$, and $(C_q^*, cv_q) \in \mathcal{C}_P^{\alpha_q}$.

Since $C_q = C^q \vee c_1^q \vee \cdots \vee c_{u^q}^q$ is a disjunctive factor of C_q^*, we have $Atoms(C_q) = Atoms((H \vee \bigvee_{i=1}^s C_q{}^i)\eta')$. Let $H_2 \vee h_1 \vee \cdots \vee h_r$ $(r \geq 0)$ be a disjunctive factor of H, and for $i = 1, \dots, s$, $C_{q_2}{}^i \vee c_{q_1}{}_1^i \vee \cdots \vee c_{q_1}{}_{v_q{}^i}^i$ $(v_q{}^i \geq 0)$ be a disjunctive factor of $C_q{}^i$ such that,

$$Atoms((H_2 \vee \bigvee_{i=1}^s C_{q_2}{}^i)\eta') = Atoms(C^q),$$

$$\{h_1, \dots, h_r\}\eta' \cup \bigcup_{i=1}^s \{c_{q_1}{}_1^i, \dots, c_{q_1}{}_{v_q{}^i}^i\}\eta' = \{c_1^q, \dots, c_{u^q}^q\},$$

$$\{d_{q_1}{}_1^0, \dots, d_{q_1}{}_r^0\} \cup \bigcup_{i=1}^s \{d_{q_1}{}_1^i, \dots, d_{q_1}{}_{v_q{}^i}^i\} = \{d_1^q, \dots, d_{u^q}^q\}.$$

It is a simple matter to see that the tuples

$$\begin{array}{c}(d_1^q,\dots,d_{u^q}^q)\\ (c_1^q,\dots,c_{u^q}^q)\end{array} \tag{16}$$

and the tuple of sets

$$\left(\left\{\begin{array}{l}(d_{q_1}{}_1^0,\dots,d_{q_1}{}_r^0)\\ (\ h_1\ ,\dots,\ h_r\ )\eta'\end{array}\right\}, \left\{\begin{array}{l}(d_{q_1}{}_1^1,\dots,d_{q_1}{}_{v_q{}^1}^1,\dots,d_{q_1}{}_1^s,\dots,d_{q_1}{}_{v_q{}^s}^s)\\ (c_{q_1}{}_1^1,\dots,c_{q_1}{}_{v_q{}^1}^1,\dots,c_{q_1}{}_1^s,\dots,c_{q_1}{}_{v_q{}^s}^s)\eta'\end{array}\right\}\right) \tag{17}$$

have the same unifiers.

Since $H_2 \vee h_1 \vee \cdots \vee h_r$ is a disjunctive factor of H, and for $i = 1, \dots, s$, $C_{q_2}{}^i \vee c_{q_1}{}_1^i \vee \cdots \vee c_{q_1}{}_{v_q{}^i}^i$ is a disjunctive factor of $C_q{}^i$, we get that $H_2 \vee h_1 \vee \cdots \vee h_r \longleftarrow b_1 \,\&\cdots\&\, b_s$ is an implication factor of $H \longleftarrow b_1 \,\&\cdots\&\, b_s$, and for $i = 1, \dots, s$, $c_{q}{}_1^i \vee \cdots \vee c_{q}{}_{u_q{}^i}^i \vee C_{q_2}{}^i \vee c_{q_1}{}_1^i \vee \cdots \vee c_{q_1}{}_{v_q{}^i}^i$ is a disjunctive factor of $c_{q}{}_1^i \vee \cdots \vee c_{q}{}_{u_q{}^i}^i \vee C_q{}^i$. This yields

- $(C_{q_i}, cv_{q_i}) = (c_{q}{}_1^i \vee \cdots \vee c_{q}{}_{u_q{}^i}^i \vee C_{q_2}{}^i \vee c_{q_1}{}_1^i \vee \cdots \vee c_{q_1}{}_{v_q{}^i}^i, cv_{q_i})$, $i = 1, \dots, s$, are disjunctive factors of some variants of graded disjunctions $(C_{q_i}^{\#}, cv_{q_i}) \in \mathcal{C}_P^{\alpha_q - 1}$, respectively; and

- $(Im, cv_{q_0}) = (H_2 \vee h_1 \vee \cdots \vee h_r \longleftarrow b_1 \,\&\, \cdots \,\&\, b_s, cv_{q_0})$ is an implication factor of a variant of a graded implication in P.

By the definition of $\mathcal{C}_P$, every disjunctive factor of

$$((H_2 \vee h_1 \vee \cdots \vee h_r \vee \bigvee_{i=1}^{s} (C_{q_2}{}^i \vee c_{q_1}{}_1^i \vee \cdots \vee c_{q_1}{}_{v_q{}^i}^i))\eta', cv_q)$$

is in $\mathcal{C}_P^{\alpha_q}$. However,

$$Atoms((H \vee \bigvee_{i=1}^{s} C_q{}^i)\eta') =$$
$$Atoms((H_2 \vee h_1 \vee \cdots \vee h_r \vee \bigvee_{i=1}^{s} (C_{q_2}{}^i \vee c_{q_1}{}_1^i \vee \cdots \vee c_{q_1}{}_{v_q{}^i}^i))\eta').$$

Hence, C_q^* is a disjunctive factor of $(H_2 \vee h_1 \vee \cdots \vee h_r \vee \bigvee_{i=1}^{s}(C_{q_2}{}^i \vee c_{q_1}{}_1^i \vee \cdots \vee c_{q_1}{}_{v_q{}^i}^i))\eta'$ and this implies again $(C_q^*, cv_q) \in \mathcal{C}_P^{\alpha_q}$.

We are in position to put:

- $\phi_1 = \eta$, $dom(\eta) = V_1$, ϕ_1 is an mgu for the tuple of sets 9;
- $\phi_1' = \eta''$ where η'' is a regular extension of η' and also η to V.

Let us make the following denotation:

$$U_1 = vars(h_1, \ldots, h_r) \cup \textstyle\bigcup_{i=1}^{s} vars(c_{q_1}{}_1^i, \ldots, c_{q_1}{}_{v_q{}^i}^i),$$
$$U_2 = vars(d_{q_1}{}_1^0, \ldots, d_{q_1}{}_r^0) \cup \textstyle\bigcup_{i=1}^{s} vars(d_{q_1}{}_1^i, \ldots, d_{q_1}{}_{v_q{}^i}^i) \cup$$
$$\textstyle\bigcup_{\substack{i=1,\\ i \neq q}}^{k} vars(d_1^i, \ldots, d_{u^i}^i) \cup \bigcup_{\substack{i=1,\\ i \neq q}}^{k} vars(c_1^i, \ldots, c_{u^i}^i),$$
$$U_1' = vars(H_2, h_1, \ldots, h_r) \cup \textstyle\bigcup_{i=1}^{s} vars(C_{q_2}{}^i, c_{q_1}{}_1^i, \ldots, c_{q_1}{}_{v_q{}^i}^i),$$
$$U_2' = vars(G, C_1, \ldots, C_{q-1}, C_{q+1}, \ldots, C_k).$$

We can correctly define the following substitutions:

- $\phi_2^* = (id|_{range(\eta''|_{U_1'})} \cup \eta''|_{U_2'}{}^{-1}) \circ \theta'$, $dom(\phi_2^*) = range(\eta''|_{U_1'}) \cup range(\eta''|_{U_2'})$, notice that $range(id|_{range(\eta''|_{U_1'})} \cup \eta''|_{U_2'}{}^{-1}) = vars(C^q) \cup U_2' = dom(\theta')$;
- $\phi_2 = (id|_{range(\eta''|_{U_1})} \cup \eta''|_{U_2}{}^{-1}) \circ \theta$,
$dom(\phi_2) = range(\eta''|_{U_1}) \cup range(\eta''|_{U_2}) = range(\eta''|_{V_2})$, notice that $range(id|_{range(\eta''|_{U_1})} \cup \eta''|_{U_2}{}^{-1}) = vars(c_1^q, \ldots, c_{u^q}^q) \cup U_2 = dom(\theta)$, ϕ_2 is an mgu for the tuple of sets 10;
- ϕ_2' as a regular extension of ϕ_2^* to $range(\eta'')$.

By the statement 13, there exist

– ϕ_3, $dom(\phi_3) = V_3$, an *mgu* for the set 11

$$\left\{ \begin{matrix} (d_{q_1}{}^0_1, \dots, d_{q_1}{}^0_r) \\ (\ h_1\ , \dots,\ h_r\ ) \end{matrix} \right\},$$

– ϕ'_3, its regular extension to V,
– ϕ_4, $dom(\phi_4) = range(\phi'_3|_{V_4})$, an *mgu* for the tuple of sets 12

$$\left(\left\{ \begin{matrix} (\ b_1\ , \dots,\ b_1\ , \dots, b_s\ , \dots,\ b_s\ ) \\ (c_{q_1^1}, \dots, c_{q_{u_q}{}^1}^1, \dots, c_{q_1^s}, \dots, c_{q_{u_q}{}^s}^s) \end{matrix} \right\} \phi'_3, \right.$$
$$\left\{ \begin{matrix} (d_{q_1}{}^1_1, \dots, d_{q_1}{}^1_{v_q{}^1}, \dots, d_{q_1}{}^s_1, \dots, d_{q_1}{}^s_{v_q{}^s}) \\ (c_{q_1}{}^1_1, \dots, c_{q_1}{}^1_{v_q{}^1}, \dots, c_{q_1}{}^s_1, \dots, c_{q_1}{}^s_{v_q{}^s}) \end{matrix} \right\} \phi'_3,$$
$$\left. \left\{ \begin{matrix} (d^1_1, \dots, d^1_{u^1}, \dots, d^{q-1}_1, \dots, d^{q-1}_{u^{q-1}}, d^{q+1}_1, \dots, d^{q+1}_{u^{q+1}}, \dots, d^k_1, \dots, d^k_{u^k}) \\ (c^1_1, \dots, c^1_{u^1}, \dots, c^{q-1}_1, \dots, c^{q-1}_{u^{q-1}}, c^{q+1}_1, \dots, c^{q+1}_{u^{q+1}}, \dots, c^k_1, \dots, c^k_{u^k}) \end{matrix} \right\} \phi'_3 \right),$$

– ϕ'_4, its regular extension to $range(\phi'_3)$,

such that

$$\phi'_1 \circ \phi'_2 = \phi'_3 \circ \phi'_4.$$

We can execute the following unlimited derivation step:

$$\longleftarrow D_1, \dots, D_k \xmapsto{H_2\phi'_3|_{V'_3} \mid (Im, cv_{q_0}),\, \phi'_3|_{V'_3},\, R}$$
$$\longleftarrow (D_1, \dots, D_{q-1}, D_q \vee b_1, \dots, D_q \vee b_s, D_{q+1}, \dots, D_k)\phi'_3|_{V'_3}. \quad (18)$$

Denote by $G' \longleftarrow (D_1, \dots, D_{q-1}, D_q \vee b_1, \dots, D_q \vee b_s, D_{q+1}, \dots, D_k)\phi'_3|_{V'_3}$.

We can summarise that there exist

1) the graded disjunctions

$$(C_1, cv_1), \dots, (C_{q-1}, cv_{q-1}), (C_{q_1}, cv_{q_1}), \dots,$$
$$(C_{q_s}, cv_{q_s}), (C_{q+1}, cv_{q-1}), \dots, (C_k, cv_k)$$

$(k \geq 1,\ s \geq 0,\ 1 \leq q \leq k)$ such that (C_i, cv_i) is a disjunctive factor of some variant of the graded disjunction $(C_i^{\#}, cv_i) \in \mathcal{C}_P^{\omega}$ for $i = 1, \dots, k$, $i \neq q$, (C_{q_i}, cv_{q_i}) is a disjunctive factor of some variant of the graded disjunction $(C_{q_i}^{\#}, cv_{q_i}) \in \mathcal{C}_P^{\alpha_q - 1}$ for $i = 1, \dots, s$, and C_{q_i}, C_j for $j \neq q$, G, *Im* do not share variables in common.
2) an *mgu* ϕ_4 for the tuples 12 where $d^i_j\phi'_3|_{V'_3} \in Atoms(D_i\phi'_3|_{V'_3})$ for $i = 1, \dots, k$, $i \neq q$, $b_i\phi'_3|_{V'_3}$, $d_{q_1}{}^i_j\phi'_3|_{V'_3} \in Atoms((D_q \vee b_i)\phi'_3|_{V'_3})$ for $i = 1, \dots, s$, $dom(\phi_4) = vars(9) = range(\phi'_3|_{V_4})$;
3) and its regular extension $\phi^*_4 = \phi'_4|_{range(\phi'_3|_{V'_4})}$, $dom(\phi^*_4) = range(\phi'_3|_{V'_4}) = vars(G', (C_1, \dots, C_{q-1}, C_{q_1}, \dots, C_{q_s}, C_{q+1}, \dots, C_k)\phi'_3)$; because of $V_4 \subseteq V'_4 \subseteq V = dom(\phi'_3)$ and ϕ'_4 is a regular extension of ϕ_4 to $range(\phi'_3)$.

Let us make the following denotation:

$$U_d = vars(b_1,\dots,b_s) \cup \bigcup_{i=1}^{s} D_q \cup vars(D_1,\dots,D_{q-1},D_{q+1},\dots,D_k),$$
$$U_c = vars(C_1,\dots,C_{q-1},C_{q_1},\dots,C_{q_s},C_{q+1},\dots,C_k).$$

As $dom(\phi_3) = V_3 \subseteq V_3'$, $V_3' \cap U_c = \emptyset$, and ϕ_3' is a regular extension of ϕ_3 to $V \supseteq U_c$, $\phi_3'|_{U_c}$ is a variable renaming. Therefore from the point 1), we get that there exist

1') the graded disjunctions

$$(C_1\phi_3', cv_1),\dots,(C_{q-1}\phi_3', cv_{q-1}),(C_{q_1}\phi_3', cv_{q_1}),\dots,$$
$$(C_{q_s}\phi_3', cv_{q_s}),(C_{q+1}\phi_3', cv_{q+1}),\dots,(C_k\phi_3', cv_k)$$

($k \geq 1$, $s \geq 0$, $1 \leq q \leq k$) such that $(C_i\phi_3', cv_i)$ is a disjunctive factor of some variant of the graded disjunction $(C_i^{\#}, cv_i) \in \mathcal{C}_P^{\omega}$ for $i = 1,\dots,k$, $i \neq q$, $(C_{q_i}\phi_3', cv_{q_i})$ is a disjunctive factor of some variant of the graded disjunction $(C_{q_i}^{\#}, cv_{q_i}) \in \mathcal{C}_P^{\alpha_q - 1}$ for $i = 1,\dots,s$, and $C_{q_i}\phi_3'$, $C_j\phi_3'$ for $j \neq q$, G' do not share variables in common.

Note that $Least((C_{q_i}^{\#}, cv_{q_i})) \leq \alpha_q - 1 < \alpha_q = Least((C_q^{\#}, cv_q))$.
By the definition of $\prec$,

$$Eval(((C_1^{\#}, cv_1),\dots,(C_k^{\#}, cv_k)))) \succ$$
$$Eval(((C_1^{\#}, cv_1),\dots,(C_{q-1}^{\#}, cv_{q-1}),(C_{q_1}^{\#}, cv_{q_1}),\dots,$$
$$(C_{q_s}^{\#}, cv_{q_s}),(C_{q+1}^{\#}, cv_{q+1}),\dots,(C_k^{\#}, cv_k)))).$$

Hence, we can make use of induction hypothesis: there exists a *ULSLD*-refutation

$$G' \models\!\!\!\xrightarrow[\;]{\substack{(C^1\phi_3'\phi_4^*,\dots,C^{q-1}\phi_3'\phi_4^*,C_{q_2}{}^1\phi_3'\phi_4^*,\dots,C_{q_2}{}^s\phi_3'\phi_4^*,C^{q+1}\phi_3'\phi_4^*,\dots,C^k\phi_3'\phi_4^*)\\ (cv_1,\dots,cv_{q-1},cv_{q_1},\dots,cv_{q_s},cv_{q+1},\dots,cv_k)}\Big|\, P,\delta,R} \Box \tag{19}$$

such that $\delta = \phi_4^*|_{vars(G')}$.

Combining 18 and 19, we get a *ULSLD*-refutation

$$G \models\!\!\!\xrightarrow[\;]{\substack{(C^1\phi_3'\phi_4^*,\dots,C^{q-1}\phi_3'\phi_4^*,rd_q,C^{q+1}\phi_3'\phi_4^*,\dots,C^k\phi_3'\phi_4^*)\\ (cv_1,\dots,cv_{q-1},cv_q,cv_{q+1},\dots,cv_k)}\Big|\, P,\phi_3'|_{vars(G)}\circ\delta',R} \Box \tag{20}$$

where $\delta' = \phi_4'|_{range(\phi_3'|_{V_3'})}$, $dom(\delta') = range(\phi_3'|_{V_3'})$, and rd_q is a disjunctive factor of $H_2\phi_3'|_{V_3'}\delta' \vee C_{q_2}{}^1\phi_3'\phi_4^* \vee \dots \vee C_{q_2}{}^s\phi_3'\phi_4^*$. Notice that $cv_q = *_{i=0}^{s}\, cv_{q_i}$.

We let $\vartheta = (\phi_3'|_{V_3'})|_{vars(G)} \circ \delta'$. By the statement 13, we obtain:

$$\vartheta = \theta'|_{vars(G)}. \tag{21}$$

For $i \neq q$,

$$C^i\phi'_3\phi^*_4 = C^i\theta'. \tag{22}$$

$$Atoms(H_2\phi'_3|_{V'_3}\delta' \vee C_{q_2}{}^1\phi'_3\phi^*_4 \vee \cdots \vee C_{q_2}{}^s\phi'_3\phi^*_4) = Atoms(C^q\theta'). \tag{23}$$

So, $C^q\theta'$ is a disjunctive factor of $H_2\phi'_3|_{V'_3}\delta' \vee C_{q_2}{}^1\phi'_3\phi^*_4 \vee \cdots \vee C_{q_2}{}^s\phi'_3\phi^*_4$. We can put $rd_q = C^q\theta'$.

Using the above derivations, we can rewrite the *ULSLD*-refutation 20 as follows:

$$G \mathrel{|}\!\xRightarrow{\begin{smallmatrix}(C^1\theta',\ldots,C^q\theta',\ldots,C^k\theta') \\ (\ cv_1\ ,\ldots,\ cv_q\ ,\ldots,\ cv_k\ )\end{smallmatrix}\ \Big|\ P,\vartheta,R} \square \tag{24}$$

where $\vartheta = \theta'|_{vars(G)}$.

Thus the implication $\Longrightarrow$ is proved.

($\Longleftarrow$) Assume that there exists a *ULSLD*-refutation

$$G \mathrel{|}\!\xRightarrow{\begin{smallmatrix}(rd_1,\ldots,rd_k) \\ (dv_1,\ldots,dv_k)\end{smallmatrix}\ \Big|\ P,\vartheta,R} \square. \tag{25}$$

We proceed by induction on the length n of the refutation.

The base case $n = 0$. In this case, we have $G = \square$ and the implication holds, as shown at the beginning of the proof.

The induction case. Suppose that $n > 0$ and the statement holds for $n - 1$. In this case, $G \neq \square$ because we may execute unlimited derivation steps only for non-empty goals. We can rewrite the entire refutation as follows:

$$\begin{aligned}&\longleftarrow D_1,\ldots,D_k \mathrel{|}\!\xrightarrow{H_2\phi^*_3\ |\ (Im,cv_{q_0}),\ \phi^*_3,\ R} \\ &\qquad\qquad \longleftarrow (D_1,\ldots,D_{q-1},D_q\,\dot{\vee}\,b_1,\ldots,D_q\,\dot{\vee}\,b_s,D_{q+1},\ldots,D_k)\phi^*_3\end{aligned} \tag{26}$$

where

- $R(G) = q$, $1 \leq q \leq k$;
- $(Im, cv_{q_0}) = (H_2 \vee h_1 \vee \cdots \vee h_r \longleftarrow b_1 \,\&\cdots\&\, b_s, cv_{q_0})$ $(r, s \geq 0)$ is an implication factor of a variant of a graded implication in P such that $vars(G) \cap vars(Im) = \emptyset$;
- ϕ_3, $dom(\phi_3) = V_3$, is an *mgu* for the set 11:
$$\left\{\begin{matrix}(d_{q_1}{}_1^0,\ldots,d_{q_1}{}_r^0) \\ (\ h_1\ ,\ldots,\ h_r\ )\end{matrix}\right\},$$
$d_{q_1}{}_i^0 \in Atoms(D_q)$;
- ϕ^*_3 is a regular extension of ϕ_3 to V'_3.

Denote by $G' \longleftarrow (D_1,\ldots,D_{q-1},D_q \vee b_1,\ldots,D_q \vee b_s,D_{q+1},\ldots,D_k)\phi^*_3$. Then the tail of the refutation 25 is of the form:

$$G' \mathrel{|}\!\xRightarrow{\begin{smallmatrix}(rd_1,\ldots,rd_{q-1},rd_{q_1},\ldots,rd_{q_s},rd_{q+1},\ldots,rd_k) \\ (dv_1,\ldots,dv_{q-1},dv_{q_1},\ldots,dv_{q_s},dv_{q+1},\ldots,dv_k)\end{smallmatrix}\ \Big|\ P,\delta,R} \square \tag{27}$$

so that $\vartheta = \phi_3^*|_{vars(G)} \circ \delta'$, rd_q is a disjunctive factor of $H_2\phi_3^*\delta' \vee rd_{q_1} \vee \cdots \vee rd_{q_s}$ where δ', $dom(\delta') = range(\phi_3^*) \supseteq vars(G') = dom(\delta)$, is a regular extension of δ satisfying

$$range(\delta'|_{range(\phi_3^*)-vars(G')}) \cap \\ vars(rd_1, \ldots, rd_{q-1}, rd_{q_1}, \ldots, rd_{q_s}, rd_{q+1}, \ldots, rd_k) = \emptyset, \quad (28)$$

and $dv_q = cv_{q_0} * *_{i=1}^{s} dv_{q_i}$.

Since the length of the tail 27 is $n-1$, by induction hypothesis, there exist

- graded disjunctions

$$(C_1', cv_1), \ldots, (C_{q-1}', cv_{q-1}), ({C_{q}}_1', cv_{q_1}), \ldots, \\ ({C_{q}}_s', cv_{q_s}), (C_{q+1}', cv_{q-1}), \ldots, (C_k', cv_k)$$

  such that (C_i', cv_i) is a disjunctive factor of some variant of a graded disjunction $(C_i^{\#}, cv_i) \in \mathcal{C}_P^{\omega}$ for $i = 1, \ldots, k$, $i \neq q$, $({C_q}_i', cv_{q_i})$ is a disjunctive factor of some variant of a graded disjunction $({C_q}_i^{\#}, cv_{q_i}) \in \mathcal{C}_P^{\omega}$ for $i = 1, \ldots, s$, and ${C_q}_i'$, C_j' for $j \neq q$, G' do not share variables in common;
- $C_i' \; = C^{i'} \vee c_1^{i\,\prime} \vee \cdots \vee c_{u^i}^{i\,\prime} \quad (u^i \geq 0)$ for $i \neq q$,

  ${C_q}_i' = {c_q}_1^{i\,\prime} \vee \cdots \vee {c_q}_{{u_q}^i}^{i\,\prime} \vee {C_{q_2}}^{i'} \vee {c_{q_1}}_1^{i\,\prime} \vee \cdots \vee {c_{q_1}}_{{v_q}^i}^{i\,\prime} \quad ({u_q}^i, {v_q}^i \geq 0)$;
- an mgu $\phi_4^{\#}$ for the tuple of sets

$$\left(\left\{ \begin{matrix} (\; b_1 \;, \ldots, \;\; b_1 \;\;, \ldots, b_s \;, \ldots, \;\; b_s \;\;)\phi_3^* \\ ({c_q}_1^{1\,\prime}, \ldots, {c_q}_{{u_q}^1}^{1\;\;\prime}, \ldots, {c_q}_1^{s\,\prime}, \ldots, {c_q}_{{u_q}^s}^{s\;\;\prime}) \end{matrix} \right\}, \right. \\ \left\{ \begin{matrix} ({d_{q_1}}_1^{1}, \ldots, {d_{q_1}}_{{v_q}^1}^{1}, \ldots, {d_{q_1}}_1^{s}, \ldots, {d_{q_1}}_{{v_q}^s}^{s})\phi_3^* \\ ({c_{q_1}}_1^{1\,\prime}, \ldots, {c_{q_1}}_{{v_q}^1}^{1\;\;\prime}, \ldots, {c_{q_1}}_1^{s\,\prime}, \ldots, {c_{q_1}}_{{v_q}^s}^{s\;\;\prime}) \end{matrix} \right\}, \\ \left. \left\{ \begin{matrix} (d_1^1, \ldots, d_{u^1}^1, \ldots, d_1^{q-1}, \ldots, d_{u^{q-1}}^{q-1}, d_1^{q+1}, \ldots, d_{u^{q+1}}^{q+1}, \ldots, d_1^k, \ldots, d_{u^k}^k)\phi_3^* \\ (c_1^{1\,\prime}, \ldots, c_{u^1}^{1\;\;\prime}, \ldots, c_1^{q-1\,\prime}, \ldots, c_{u^{q-1}}^{q-1\;\;\prime}, c_1^{q+1\,\prime}, \ldots, c_{u^{q+1}}^{q+1\;\;\prime}, \ldots, c_1^{k\,\prime}, \ldots, c_{u^k}^{k\;\;\prime}) \end{matrix} \right\} \right) \quad (29)$$

  where $dom(\phi_4^{\#}) = vars(29)$, $d_j^i \in Atoms(D_i)$ for $i \neq q$, and ${d_{q_1}}_j^{i} \in Atoms(D_q)$ for $i = 1, \ldots, s$;
- and its regular extension $\phi_4^{\#'}$ to

$$vars(G', C_1', \ldots, C_{q-1}', {C_q}_1', \ldots, {C_q}_s', C_{q+1}', \ldots, C_k');$$

so that $\delta = \phi_4^{\#'}|_{vars(G')}$, $rd_i = C^{i'}\phi_4^{\#'}$ for $i \neq q$, $rd_{q_i} = {C_{q_2}}^{i'}\phi_4^{\#'}$, and $dv_i = cv_i$ for $i \neq q$, $dv_{q_i} = cv_{q_i}$.

Let ρ_i, $i \neq q$, ρ_{q_i}, $i = 1, \ldots, s$, be variable renamings such that $dom(\rho_i) = vars(C_i')$ for $i \neq q$, $dom(\rho_{q_i}) = vars({C_q}_i')$, $vars({C_q}_i'\rho_{q_i}) \cap vars({C_q}_j'\rho_{q_j}) = \emptyset$ for $i \neq j$, $vars(C_i'\rho_i) \cap vars(C_j'\rho_j) = \emptyset$ for $i \neq j$, $i \neq q$, $j \neq q$, $vars({C_q}_i'\rho_{q_i}) \cap vars(C_j'\rho_j) = \emptyset$ for $j \neq q$, $vars({C_q}_i'\rho_{q_i}) \cap vars(G) = \emptyset$, $vars(C_i'\rho_i) \cap vars(G) = \emptyset$

for $i \neq q$, $vars({C_{q}}_{i}'\rho_{q_i}) \cap vars(Im) = \emptyset$, $vars(C_i'\rho_i) \cap vars(Im) = \emptyset$ for $i \neq q$. Denote by C_i $C_i'\rho_i$ for $i \neq q$, and by C_{q_i} ${C_{q}}_{i}'\rho_{q_i}$ for $i = 1, \dots, s$.

It is easy to see that (C_i, cv_i) is a disjunctive factor of some variant of the graded disjunction $(C_i^{\#}, cv_i) \in \mathcal{C}_P^{\omega}$ for $i = 1, \dots, k$, $i \neq q$, and (C_{q_i}, cv_{q_i}) is a disjunctive factor of some variant of the graded disjunction $({C_{q}}_{i}^{\#}, cv_{q_i}) \in \mathcal{C}_P^{\omega}$ for $i = 1, \dots, s$.

Consider a regular extension ϕ_3' of ϕ_3^* to $V \supseteq V_3' = dom(\phi_3^*) \supseteq V_3 = dom(\phi_3)$. ϕ_3' is also a regular extension of ϕ_3 because ϕ_3^* is a regular extension of ϕ_3. Let us make the following denotation:

$$u_d = vars(b_1, \dots, b_s) \cup \bigcup_{i=1}^{s} vars({d_{q_1}}_1^i, \dots, {d_{q_1}}_{v_q{}^i}^i) \cup \bigcup_{\substack{i=1,\\ i \neq q}}^{k} vars(d_1^i, \dots, d_{u^i}^i),$$

$$u_c = \bigcup_{i=1}^{s} vars({c_q}_1^i, \dots, {c_q}_{u_q{}^i}^i) \cup \bigcup_{i=1}^{s} vars({c_{q_1}}_1^i, \dots, {c_{q_1}}_{v_q{}^i}^i) \cup \bigcup_{\substack{i=1,\\ i \neq q}}^{k} vars(c_1^i, \dots, c_{u^i}^i),$$

$$u_c' = \bigcup_{i=1}^{s} vars({c_q}_1^{i\prime}, \dots, {c_q}_{u_q{}^i}^{i\prime}) \cup \bigcup_{i=1}^{s} vars({c_{q_1}}_1^{i\prime}, \dots, {c_{q_1}}_{v_q{}^i}^{i\prime}) \cup \bigcup_{\substack{i=1,\\ i \neq q}}^{k} vars(c_1^{i\prime}, \dots, c_{u^i}^{i\prime}),$$

$$u_C' = vars(C^{1\prime}, \dots, C^{q-1\prime}, {C_{q_2}}^{1\prime}, \dots, {C_{q_2}}^{s\prime}, C^{q+1\prime}, \dots, C^{k\prime}),$$

$$U_c' = vars(C_1', \dots, C_{q-1}', {C_q}_1', \dots, {C_q}_s', C_{q+1}', \dots, C_k').$$

It is clear that $u_d \subseteq U_d \subseteq V_3'$, $u_c \subseteq U_c$, $u_d \cup u_c = V_4$, $U_d \cup U_c = V_4'$, and $u_c' \cup u_C' = U_c'$. Note that $U_c \cap V_3' = \emptyset$; hence, $\phi_3'|_{U_c}$ is a variable renaming such that $range(\phi_3'|_{U_c}) \cap range(\phi_3^*) = \emptyset$. Notice that $range(\phi_3^*|_{U_d}) = vars(G')$. Consider a variable renaming $\rho^{\#}$, $dom(\rho^{\#}) = range(\phi_4^{\#\prime}) - range(\phi_4^{\#\prime}|_{range(\phi_3^*|_{U_d}) \cup u_C'})$, such that

$$range(\rho^{\#}) \cap range(\phi_4^{\#\prime}|_{range(\phi_3^*|_{U_d}) \cup u_C'}) = \emptyset \tag{30}$$

and

$$range(\rho^{\#}) \cap range(\delta'|_{range(\phi_3^*) - range(\phi_3^*|_{U_d})}) = \emptyset. \tag{31}$$

Let $\rho_{\delta'} = id|_{range(\phi_4^{\#\prime}|_{range(\phi_3^*|_{U_d}) \cup u_C'})} \cup \rho^{\#}$, $dom(\rho_{\delta'}) = range(\phi_4^{\#\prime})$. From 30, we have $\rho_{\delta'}$ is also a variable renaming. We can correctly define the following substitutions:

- $\phi_4^* = (id|_{range(\phi_3^*|_{U_d})} \cup \phi_3'|_{U_c}{}^{-1} \circ (\bigcup_{i=1}^{s} \rho_{q_i}{}^{-1} \cup \bigcup_{\substack{i=1,\\ i \neq q}}^{k} \rho_i{}^{-1})) \circ \phi_4^{\#\prime} \circ \rho_{\delta'}$,
  $dom(\phi_4^*) = range(\phi_3'|_{V_4'})$, ϕ_4^* is a regular extension of ϕ_4;
- $\phi_4 = (id|_{range(\phi_3^*|_{u_d})} \cup \phi_3'|_{u_c}{}^{-1} \circ (\bigcup_{i=1}^{s} \rho_{q_i}{}^{-1}|_{u_c} \cup \bigcup_{\substack{i=1,\\ i \neq q}}^{k} \rho_i{}^{-1}|_{u_c})) \circ \phi_4^{\#} \circ \rho_{\delta'}$,
  $dom(\phi_4) = range(\phi_3'|_{V_4})$, ϕ_4 is an mgu for the tuple of sets 12;
- $\phi_4' = \phi_4^* \cup \delta'|_{range(\phi_3^*) - range(\phi_3^*|_{U_d})}$.
  ϕ_4' is a regular extension of ϕ_4^* to $range(\phi_3')$.

By the statement 13, there exist

– ϕ_1, $dom(\phi_1) = V_1$, an *mgu* for the set 9

$$\left\{ \begin{matrix} (\ b_1\ ,\ldots,\ \ b_1\ \ ,\ldots,b_s\ ,\ldots,\ \ b_s\ \ ) \\ (c_{q_1}{}^1,\ldots,c_{q_{u_q}{}^1}{}^1,\ldots,c_{q_1}{}^s,\ldots,c_{q_{u_q}{}^s}{}^s) \end{matrix} \right\},$$

– ϕ_1', its regular extension to V,

– ϕ_2, $dom(\phi_2) = range(\phi_1'|_{V_2})$, an *mgu* for the tuple of sets 10

$$\left(\left\{ \begin{matrix} (d_{q_1}{}_1^0,\ldots,d_{q_1}{}_r^0) \\ (\ h_1\ ,\ldots,\ h_r\ ) \end{matrix} \right\} \phi_1', \right.$$
$$\left\{ \begin{matrix} (d_{q_1}{}_1^1,\ldots,d_{q_1}{}_{v_q{}^1}^1,\ldots,d_{q_1}{}_1^s,\ldots,d_{q_1}{}_{v_q{}^s}^s) \\ (c_{q_1}{}_1^1,\ldots,c_{q_1}{}_{v_q{}^1}^1,\ldots,c_{q_1}{}_1^s,\ldots,c_{q_1}{}_{v_q{}^s}^s) \end{matrix} \right\} \phi_1',$$
$$\left. \left\{ \begin{matrix} (d_1^1,\ldots,d_{u^1}^1,\ldots,d_1^{q-1},\ldots,d_{u^{q-1}}^{q-1},d_1^{q+1},\ldots,d_{u^{q+1}}^{q+1},\ldots,d_1^k,\ldots,d_{u^k}^k) \\ (c_1^1,\ldots,c_{u^1}^1,\ldots,c_1^{q-1},\ldots,c_{u^{q-1}}^{q-1},c_1^{q+1},\ldots,c_{u^{q+1}}^{q+1},\ldots,c_1^k,\ldots,c_{u^k}^k) \end{matrix} \right\} \phi_1' \right),$$

– ϕ_2', its regular extension to $range(\phi_1')$,

such that

$$\phi_1' \circ \phi_2' = \phi_3' \circ \phi_4'.$$

Denote by ϕ_1^* the substitution $\phi_1'|_{V_1'}$. Then ϕ_1' is a regular extension of ϕ_1^* and ϕ_1^* is a regular extension of ϕ_1 because ϕ_1' is a regular extension of ϕ_1 and $V_1 = dom(\phi_1) \subseteq V_1' = dom(\phi_1^*) \subseteq V = dom(\phi_1')$.

We know that

$$(C_1, cv_1), \ldots, (C_{q-1}, cv_{q-1}), (C_{q_1}, cv_{q_1}), \ldots,$$
$$(C_{q_s}, cv_{q_s}), (C_{q+1}, cv_{q+1}), \ldots, (C_k, cv_k)$$

are such that (C_i, cv_i) is a disjunctive factor of some variant of the graded disjunction $(C_i^{\#}, cv_i) \in \mathcal{C}_P^{\omega}$ for $i = 1, \ldots, k$, $i \neq q$, (C_{q_i}, cv_{q_i}) is a disjunctive factor of some variant of the graded disjunction $(C_{q_i}^{\#}, cv_{q_i}) \in \mathcal{C}_P^{\omega}$ for $i = 1, \ldots, s$, and C_{q_i}, C_j for $j \neq q$, G, *Im* do not share variables in common. This yields

$$V_1' \cap U_2' = \emptyset;$$

hence, $\phi_1'|_{U_2'}$ is a variable renaming such that

$$range(\phi_1'|_{V_1'}) \cap range(\phi_1'|_{U_2'}) = \emptyset. \tag{32}$$

Consider a variable renaming $\rho = \phi_1'|_{U_2'}{}^{-1}$, $dom(\rho) = range(\phi_1'|_{U_2'})$, and its regular extension ρ' to $range(\phi_1') = range(\phi_1'|_{V_1'}) \cup range(\phi_1'|_{U_2'})$.

In view of 32, we get

$$range(\rho'|_{range(\phi_1'|_{V_1'})}) \cap U_2' = \emptyset. \tag{33}$$

It is a simple matter to see that $\phi_1 \circ \rho'$ is an mgu for the set 9, $\phi_1^* \circ \rho'$ is a regular extension of $\phi_1 \circ \rho'$, and $\phi_1' \circ \rho'$ is a regular extension of $\phi_1^* \circ \rho'$ and of $\phi_1 \circ \rho'$.

By the definition of $\mathcal{C}_P$, every disjunctive factor of

$$((H_2 \vee h_1 \vee \cdots \vee h_r \vee \bigvee_{i=1}^{s} ({C_{q_2}}^i \vee {c_{q_1}}_1^i \vee \cdots \vee {c_{q_1}}_{{v_q}^i}^i))(\phi_1^* \circ \rho'), *_{i=0}^{s}\, cv_{q_i})$$

is in $\mathcal{C}_P^{\omega}$. Let $(C_q^{\#}, \bigcap_{i=0}^{s} cv_{q_i})$ be one of them.

We can define the following substitutions:

- $\theta = \rho'|_{range(\phi_1'|_{V_2})}{}^{-1} \circ \phi_2$, $dom(\theta) = range(\rho'|_{range(\phi_1'|_{V_2})}) = vars(34)$, θ is an mgu for the tuple of sets 34;
- $\theta' = \rho'|_{range(\phi_1'|_{V_2'})}{}^{-1} \circ \phi_2'$, $dom(\theta') = range(\rho'|_{range(\phi_1'|_{V_2'})}) =$
$vars(G, C_1, \dots, C_{q-1}, C_q^{\#}, C_{q+1}, \dots, C_k)$, θ' is a regular extension of θ.

$$\left(\left\{\begin{matrix}({d_{q_1}}_1^0, \dots, {d_{q_1}}_r^0) \\ (\ h_1\ , \dots,\ h_r\ )(\phi_1^* \circ \rho')\end{matrix}\right\},\right.$$
$$\left\{\begin{matrix}({d_{q_1}}_1^1, \dots, {d_{q_1}}_{{v_q}^1}^1, \dots, {d_{q_1}}_1^s, \dots, {d_{q_1}}_{{v_q}^s}^s) \\ ({c_{q_1}}_1^1, \dots, {c_{q_1}}_{{v_q}^1}^1, \dots, {c_{q_1}}_1^s, \dots, {c_{q_1}}_{{v_q}^s}^s)(\phi_1^* \circ \rho')\end{matrix}\right\},$$
$$\left.\left\{\begin{matrix}(d_1^1, \dots, d_{u^1}^1, \dots, d_1^{q-1}, \dots, d_{u^{q-1}}^{q-1}, d_1^{q+1}, \dots, d_{u^{q+1}}^{q+1}, \dots, d_1^k, \dots, d_{u^k}^k) \\ (c_1^1, \dots, c_{u^1}^1, \dots, c_1^{q-1}, \dots, c_{u^{q-1}}^{q-1}, c_1^{q+1}, \dots, c_{u^{q+1}}^{q+1}, \dots, c_1^k, \dots, c_{u^k}^k)\end{matrix}\right\}\right). \tag{34}$$

Consider the following equations:

$$\vartheta = \theta'|_{vars(G)}. \tag{35}$$

$$Atoms(rd_q) = Atoms((H_2(\phi_1^* \circ \rho') \vee {C_{q_2}}^1(\phi_1^* \circ \rho') \vee \cdots \vee {C_{q_2}}^s(\phi_1^* \circ \rho'))\theta'). \tag{36}$$

For $i \neq q$,

$$rd_i = C^i \theta'. \tag{37}$$

We can summarise:

1) There exist the graded disjunctions

$$(C_1, cv_1), \dots, (C_{q-1}, cv_{q-1}), (C_{q+1}, cv_{q+1}), \dots, (C_k, cv_k)$$

$(k \geq 1,\ s \geq 0,\ 1 \leq q \leq k)$ such that (C_i, cv_i), $i \neq q$, is a disjunctive factor of a variant of the graded disjunction $(C_i^{\#}, cv_i) \in \mathcal{C}_P^{\omega}$, $C_i = C^i \vee c_1^i \vee \cdots \vee c_{u^i}^i$ $(u^i \geq 0)$, and for $i \neq j$, $i \neq q$, $j \neq q$, $vars(C_i) \cap vars(C_j) = \emptyset$, $vars(C_i) \cap vars(G) = \emptyset$, and

$$C^i \theta' = rd_i,$$
$$cv_i = dv_i.$$

2) We know that $(C_q^{\#}, *_{i=0}^{s} cv_{q_i}) \in \mathcal{C}_P^{\omega}$ is a disjunctive factor of

$$((H_2 \vee h_1 \vee \cdots \vee h_r \vee \bigvee_{i=1}^{s} (C_{q_2}{}^{i} \vee c_{q_1}{}_{1}^{i} \vee \cdots \vee c_{q_1}{}_{v_q{}^i}^{i}))(\phi_1^* \circ \rho'), *_{i=0}^{s} cv_{q_i})$$

and $Atoms(rd_q) = Atoms((H_2(\phi_1^* \circ \rho') \vee C_{q_2}{}^{1}(\phi_1^* \circ \rho') \vee \cdots \vee C_{q_2}{}^{s}(\phi_1^* \circ \rho'))\theta')$. So, there exists a disjunctive factor C^q of $H_2(\phi_1^* \circ \rho') \vee C_{q_2}{}^{1}(\phi_1^* \circ \rho') \vee \cdots \vee C_{q_2}{}^{s}(\phi_1^* \circ \rho')$ such that

$$C^q\theta' = rd_q.$$

Hence, $C_q = C^q \vee (h_1 \vee \cdots \vee h_r \vee \bigvee_{i=1}^{s}(c_{q_1}{}_{1}^{i} \vee \cdots \vee c_{q_1}{}_{v_q{}^i}^{i}))(\phi_1^* \circ \rho')$ is a disjunctive factor of
$(H_2 \vee h_1 \vee \cdots \vee h_r \vee \bigvee_{i=1}^{s}(C_{q_2}{}^{i} \vee c_{q_1}{}_{1}^{i} \vee \cdots \vee c_{q_1}{}_{v_q{}^i}^{i}))(\phi_1^* \circ \rho')$ and also of $C_q^{\#}$. Note that

$$\begin{aligned} &vars(C_q) = vars(C_q^{\#}) = \\ &\quad vars((H_2 \vee h_1 \vee \cdots \vee h_r \vee \bigvee_{i=1}^{s} (C_{q_2}{}^{i} \vee c_{q_1}{}_{1}^{i} \vee \cdots \vee c_{q_1}{}_{v_q{}^i}^{i}))(\phi_1^* \circ \rho')) \subseteq \\ &\qquad\qquad range(\rho'|_{range(\phi_1'|_{V_1'})}). \end{aligned}$$

However from 33

$$range(\rho'|_{range(\phi_1'|_{V_1'})}) \cap U_2' = \emptyset,$$

we get that

$$vars(C_q) \cap U_2' = \emptyset.$$

By the definition of the refutation 25 and the induction hypothesis,

$$*_{i=0}^{s} cv_{q_i} = cv_{q_0} * *_{i=1}^{s} dv_{q_i} = dv_q.$$

3) There exists θ, $dom(\theta) = vars(34)$, an mgu for the tuple of sets 34:

$$\left(\left\{ \begin{matrix} (d_{q_1}{}_{1}^{0}, \ldots, d_{q_1}{}_{r}^{0}) \\ (\ h_1\ , \ldots,\ h_r\ )(\phi_1^* \circ \rho') \end{matrix} \right\}, \left\{ \begin{matrix} (d_{q_1}{}_{1}^{1}, \ldots, d_{q_1}{}_{v_q{}^1}^{1}, \ldots, d_{q_1}{}_{1}^{s}, \ldots, d_{q_1}{}_{v_q{}^s}^{s}) \\ (c_{q_1}{}_{1}^{1}, \ldots, c_{q_1}{}_{v_q{}^1}^{1}, \ldots, c_{q_1}{}_{1}^{s}, \ldots, c_{q_1}{}_{v_q{}^s}^{s})(\phi_1^* \circ \rho') \end{matrix} \right\}, \left\{ \begin{matrix} (d_1^1, \ldots, d_{u^1}^1, \ldots, d_1^{q-1}, \ldots, d_{u^{q-1}}^{q-1}, d_1^{q+1}, \ldots, d_{u^{q+1}}^{q+1}, \ldots, d_1^k, \ldots, d_{u^k}^k) \\ (c_1^1, \ldots, c_{u^1}^1, \ldots, c_1^{q-1}, \ldots, c_{u^{q-1}}^{q-1}, c_1^{q+1}, \ldots, c_{u^{q+1}}^{q+1}, \ldots, c_1^k, \ldots, c_{u^k}^k) \end{matrix} \right\} \right)$$

where $d_j^i \in Atoms(D_i)$ for $i \neq q$, and $d_{q_1}{}_{j}^{i} \in Atoms(D_q)$ for $i = 1, \ldots, s$.

4) There exists θ', $dom(\theta') = vars(G, C_1, \ldots, C_{q-1}, C_q^{\#}, C_{q+1}, \ldots, C_k) = vars(G, C_1, \ldots, C_{q-1}, C_q, C_{q+1}, \ldots, C_k)$, a regular extension of θ such that

$$\theta'|_{vars(G)} = \vartheta.$$

Hence, the implication $\Longleftarrow$ is proved.
Thus the proof is completed. $\square$

The procedural semantics can be proposed by means of *ULSLD*-refutation:

Definition 6 (Procedural semantics). *Let P be a fuzzy disjunctive program of $\mathcal{L}$. $\mathcal{PS}(P) = \{(D, c_d) \,|\, (D, c_d)$ is a graded disjunction of $\mathcal{L}$,*

$$c_d \leq \bigcup \{c'_d \,|\, \longleftarrow D \left| \xrightarrow[\quad]{\substack{(\square) \\ (c'_d)} \left| \, P, id|_{vars(D)}, R \right.} \right. \square \text{ for some } R\}.$$

Using Theorem 2, we conclude:

Corollary 2. *Let P be a fuzzy disjunctive program of $\mathcal{L}$.*

$$\mathcal{DS}(P) = \mathcal{FS}(P) = \mathcal{PS}(P).$$

Thereby we have reached the equivalence between the semantics.

4 Experimental results

Fuzzy disjunctive programs with *ULSLD*-refutation can be exploited as a suitable representation and inference tool in robotic applications. We have implemented such a knowledge base into a robotic mouse constructed at the University of Trenčín (Češka, Jánošík, Zervan), see Fig. 2. [3] The mouse is able to explore

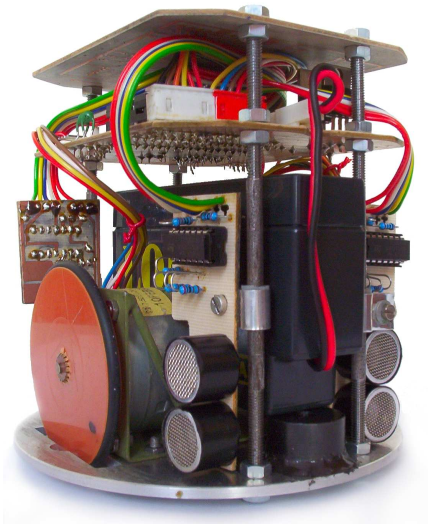

Fig. 2. Robotic mouse.

in surface mazes, build up the inner model of a maze, and search the shortist and fastist paths between given points. Its perception is based on an ultrasound sonar system. The 'heart' of the mouse consists of a *DSP* processor with a 16MB external memory. The diameter of the circle ground-plane is only 12cm, which allows the mouse to pass through narrow maze corridors. The height and weight are about 20cm and half a kilogramme, respectively.

Since our mouse is a mobile robot, its power supply is an accumulator. One task of the robot power-managment is that of estimating the supply voltage after some working time. Without loss of generality, we can assume that the supply voltage ranges within the unit interval $[0, 1]$. The empty accumulator has the voltage 0, while the charged one the voltage 1 (in a suitable measuring unit). Denote the capacity of the accumulator by C, the supply voltage by U, the supply current by I, the total resistance by R, the charge in the accumulator by Q, and the working time by t. The voltage U can be expressed by means of Q and C:

$$U = \frac{Q}{C}.$$

Consequently we obtain the first derivation $\frac{\Delta U}{\Delta t}$:

$$\frac{\Delta U}{\Delta t} = \frac{\Delta \frac{Q}{C}}{\Delta t} = \frac{1}{C}\frac{\Delta Q}{\Delta t} = -\frac{I}{C} = -\frac{U}{RC}.$$

So, we have the differential equation:

$$\frac{\Delta U}{\Delta t} = -\frac{U}{RC}.$$

The solution is of the form $U = a^t$:

$$a^t \ln a = -\frac{a^t}{RC}; \text{ hence, } a = \frac{1}{e^{\frac{1}{RC}}}.$$

Since $R > 0$, $C > 0$, and $e > 1$, we get $e^{\frac{1}{RC}} > 1$. This yields $0 < a < 1$. Thus for $t \geq 0$, $0 < a^t \leq 1$ and $\lim_{t\to\infty} a^t = 0$.

Let us consider the maze of a grid form in Fig. 3. In the figure certainly percepted (by the sonar system) barrier edges between edge adjacent squares are outlined by bold lines. In knowledge base, such the barrier edges are represented by facts of the form: $b(X_1, Y_1, X_2, Y_2)$ where (X_1, Y_1) and (X_2, Y_2) are the coordinates of edge adjacent squares. However, owing to low visibility or high interference, our sonar system can obtain also incomplete information about barriers. Consider three adjacent squares S_1, S_2, and S_3 where S_1, S_2 are edge adjacent, so are S_1, S_3, and the squares coincide in one common corner (e.g. the squares $(4, 0)$, $(5, 0)$, and $(4, 1)$). We know that there is at least one barrier either between S_1, S_2 or S_1, S_3. Yet, that is all what we know about it. This kind of incomplete information is naturally represented by the disjunction $b(X_1, Y_1, X_2, Y_2) \vee b(X_1, Y_1, X_3, Y_3)$ where (X_1, Y_1), (X_2, Y_2), and (X_3, Y_3) are

[3] More information can be found at `http://www.ii.fmph.uniba.sk/~guller/robots`.

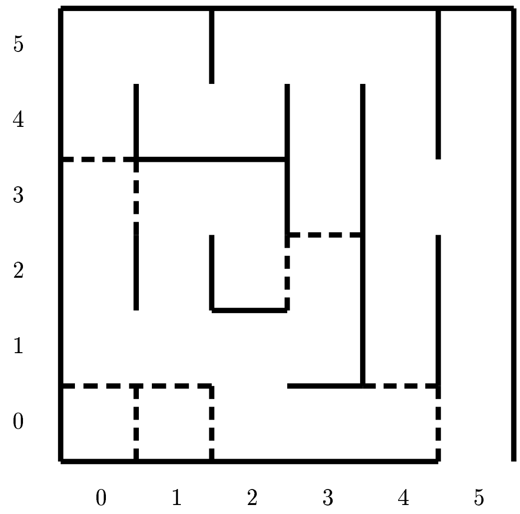

Fig. 3. Grid maze.

the coordinates of S_1, S_2, and S_3, respectively. Such the uncertain barriers are depicted by dashed lines.

As mentioned above, our task is to estimate the remaining supply voltage after some working time of the robot. The motion of our robot is carried out by two synchronised step motors, the biggest power consumers. If the robot is motionless, the no-load consumption is negligible. So, really good approximation of the working time will be the count of executed motor steps. For a given distance, we have the fixed number of motor steps. Hence, the removal from one square to another edge adjacent square takes some fixed number of steps, say t_1. Let t_0 be the number already executed steps. As we have derived above, the remaining voltage after t_0 executed steps $U_0 = a^{t_0}$ and then the voltage after an additional removal to another edge adjacent square will be $U_2 = a^{t_0+t_1} = a^{t_0} \cdot a^{t_1} = U_0 \cdot U_1$ where we put $U_1 = a^{t_1}$. This leads us to make use of the product residuated lattice $([0,1], \leq, \max, \min, \cdot, \Rightarrow, 0, 1)$ with the residuum

$$a \Rightarrow b = \begin{cases} b/a, & a > b; \\ 1, & a \leq b. \end{cases}$$

In our maze representation, facts will be graded by the voltage factors U_1:

$$\begin{array}{ll} (b(2,0,3,0), 0.9) & (b(0,1,0,2), 0.9) \\ (b(0,0,-1,0), 0) & (b(0,0,0,-1), 0) \\ (b(4,0,5,0) \vee b(4,0,4,1), 0.9) & (b(3,2,2,2) \vee b(3,2,3,3), 0.9) \end{array}$$

The predicate *connected*$(X1, Y1, X2, Y2)$ computes the remaining supply voltage for the path from $(X1, Y1)$ to $(X2, Y2)$ [4] on the basis of the predicate

[4] Provided that the robot starts with the charged accumulator.

$con(X1, Y1, X2, Y2)$ which does the same for the edge or corner adjacent squares $(X1, Y1)$ and $(X2, Y2)$. $(ad(X1, Y1, X2, Y2), 1)$ means that $(X1, Y1)$, $(X2, Y2)$ are edge adjacent.

$(con(X1, Y1, X2, Y2) \longleftarrow ad(X1, Y1, X2, Y2) \,\&\, b(X1, Y1, X2, Y2), 1)$
$(con(X1, Y1, X2, Y2) \longleftarrow ad(X1, Y1, X1, Y2) \,\&\, ad(X1, Y2, X2, Y2) \,\&\, b(X1, Y1, X2, Y1), 0.9^2)$
$(con(X1, Y1, X2, Y2) \longleftarrow ad(X1, Y1, X1, Y2) \,\&\, ad(X1, Y2, X2, Y2) \,\&\, b(X1, Y1, X1, Y2), 0.9^2)$

$(connected(X1, Y1, X2, Y2) \longleftarrow con(X1, Y1, X2, Y2), 1)$
$(connected(X1, Y1, X2, Y2) \longleftarrow con(X1, Y1, X3, Y3) \,\&\, connected(X3, Y3, X2, Y2), 1)$

Consider the following derivation:

$G_0 = \longleftarrow con(4, 0, 5, 1)$
$\square$ $\mid mgu = \{x_1/4, y_1/0, x_2/5, y_2/1\}$
$(con(x_1, y_1, x_2, y_2) \longleftarrow ad(x_1, y_1, x_1, y_2) \,\&\, ad(x_1, y_2, x_2, y_2) \,\&\, b(x_1, y_1, x_2, y_1), 0.9^2)$

$G_1 = \longleftarrow con(4, 0, 5, 1) \vee ad(4, 0, 4, 1), con(4, 0, 5, 1) \vee ad(4, 1, 5, 1), con(4, 0, 5, 1) \vee b(4, 0, 5, 0)$
$\square$ $\mid mgu = \emptyset$
$(ad(4, 0, 4, 1) \longleftarrow, 1)$

$G_2 = \longleftarrow con(4, 0, 5, 1) \vee ad(4, 1, 5, 1), con(4, 0, 5, 1) \vee b(4, 0, 5, 0)$
$\square$ $\mid mgu = \emptyset$
$(ad(4, 1, 5, 1) \longleftarrow, 1)$

$G_3 = \longleftarrow con(4, 0, 5, 1) \vee b(4, 0, 5, 0)$
$\square$ $\mid mgu = \{x_1/4, y_1/0, x_2/5, y_2/1\}$
$(con(x_1, y_1, x_2, y_2) \longleftarrow ad(x_1, y_1, x_1, y_2) \,\&\, ad(x_1, y_2, x_2, y_2) \,\&\, b(x_1, y_1, x_1, y_2), 0.9^2)$

$G_4 = \longleftarrow con(4, 0, 5, 1) \vee b(4, 0, 5, 0) \vee ad(4, 0, 4, 1), con(4, 0, 5, 1) \vee b(4, 0, 5, 0) \vee ad(4, 1, 5, 1),$
$con(4, 0, 5, 1) \vee b(4, 0, 5, 0) \vee b(4, 0, 4, 1)$
$\square$ $\mid mgu = \emptyset$
$(ad(4, 0, 4, 1) \longleftarrow, 1)$

$G_5 = \longleftarrow con(4, 0, 5, 1) \vee b(4, 0, 5, 0) \vee ad(4, 1, 5, 1), con(4, 0, 5, 1) \vee b(4, 0, 5, 0) \vee b(4, 0, 4, 1)$
$\square$ $\mid mgu = \emptyset$
$(ad(4, 1, 5, 1) \longleftarrow, 1)$

$G_6 = \longleftarrow con(4, 0, 5, 1) \vee b(4, 0, 5, 0) \vee b(4, 0, 4, 1)$
$\square$ $\mid mgu = \emptyset$
$(b(4, 0, 5, 0) \vee b(4, 0, 4, 1) \longleftarrow, 0.9)$

$G_7 = \square$

We compute the refutation answer for $\longleftarrow con(4, 0, 5, 1)$:

goal	remainder	truth vector
G_7	$()$	$()$
G_6	$(\square)$	(0.9)
G_5	$(\square, \square)$	$(1, 0.9)$
G_4	$(\square, \square, \square)$	$(1, 1, 0.9)$
G_3	$(\square)$	(0.9^3)
G_2	$(\square, \square)$	$(1, 0.9^3)$
G_1	$(\square, \square, \square)$	$(1, 1, 0.9^3\})$
G_0	$(\square)$	(0.9^5)

goal	θ	ϑ^*	ϑ
G_7			$\emptyset$
G_6	$\emptyset$	$\emptyset$	$\emptyset$
G_5	$\emptyset$	$\emptyset$	$\emptyset$
G_4	$\emptyset$	$\emptyset$	$\emptyset$
G_3	$\{x_1/4, y_1/0, x_2/5, y_2/1\}$	$\emptyset$	$\emptyset$
G_2	$\emptyset$	$\emptyset$	$\emptyset$
G_1	$\emptyset$	$\emptyset$	$\emptyset$
G_0	$\{x_1/4, y_1/0, x_2/5, y_2/1\}$	$\emptyset$	$\emptyset$

Thus the remaining voltage is 0.9^5.

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