Exploring the potentials of ordinal representations of uncertainty

Didier Dubois

IRIT, Universite Paul Sabatier

118 route de Narbonne

31062, Toulouse, cedex, France

dubois@irit.fr

Joint work with

S. Benferhat, H. Fargier, H. Prade

In the last 20 years, nonmonotonic reasoning has emphasized a modelling of uncertainty which is not numerical.

Basically relying on complete or partial (pre)ordering relations

-) between interpretations of a propositional or first order langage,

-) between propositional variables (circumscription)

-) between propositions or defaults

-) between sets of propositions.

Also integers are sometimes used

Questions:

What is “ordinal”??

Positioning w.r.t. probability?

What can we do with ordinal representations of uncertainty?

Inference? Diagnosis? Decision?

Can we perform *as well as / better than* with numbers?

expressivity of ordinal representations

limitations

A question addressed here in the framework of the KLM approach to NMR, and possibility theory

Working assumptions

a propositional language

A finite set S of interpretations (descriptions of possible worlds, of states of the world).

Identifying propositions and events

a set of events $\mathcal{E}$ equated with 2^S

Syntax-free environment.

Confidence relations $\geq_L$ on events

A, B, C

On $\mathcal{E}$ define relation $\geq_L$ attached to an agent

$A \geq_L B$ means

" agent has at least as much confidence in event A as in event B"

This relation describes an epistemic state that refers either to a particular situation (belief), or to a collection of situations (normality, typicality).

Usual/natural properties

reflexivity $A \geq_{\mathbf{L}} A$

complete $A \geq_{\mathbf{L}} B$ ou $B \geq_{\mathbf{L}} A$

transitivity $A \geq_{\mathbf{L}} B$ and $B \geq_{\mathbf{L}} C \Rightarrow A \geq_{\mathbf{L}} C$

Strict part:
$A >_{\mathbf{L}} B$ means $A \geq_{\mathbf{L}} B$ but not $B \geq_{\mathbf{L}} A$

non-trivialité $S >_{\mathbf{L}} \emptyset$

limit conditions $S \geq_{\mathbf{L}} A \geq_{\mathbf{L}} \emptyset$

Coherence with deduction :
if $A \subseteq B$ then $B \geq_{\mathbf{L}} A$

Some comments

- Completeness not compulsory: Partial preordering relations (Halpern)

- One may start from a strict confidence relation: $A >_{\mathbf{L}} B$ irreflexive and transitive.

Then $A \geq_{\mathbf{L}} B$ (= not ($B >_{\mathbf{L}} A$)) is not transitive

- **Additional property with respect to connectives**

Weak stability: If $(A \cup B) \cap C = \emptyset$ then

$$A >_{\mathbf{L}} B \text{ implies } A \cup C \geq_{\mathbf{L}} B \cup C$$

(equivalently, $A >_{\mathbf{L}} B$ implies $A \cap C \geq_{\mathbf{L}} B \cap C$)

Well-known examples

Comparative probability (De Finetti)

i) $\geq_{\mathbf{P}}$ is complete and transitive

ii) $S >_{\mathbf{P}} \emptyset$ (non-triviality),

iii) $\forall A$, $A \geq_{\mathbf{P}} \emptyset$ (consistency)

iv) If $A \cap (B \cup C) = \emptyset$ then:

$B \geq_{\mathbf{P}} C \Leftrightarrow A \cup B \geq_{\mathbf{P}} A \cup C$ (pre-additivity)

NOT characteristic of numerical probability

Necessity ordering

i) $\geq_N$ is complete and transitive

ii) $S >_N \emptyset$,

iii) $\forall A$, $S \geq_N A$

iv) Conjunctive stability

$$B \geq_N C \Rightarrow A \cap B \geq_N A \cap C$$

$B \geq_N C$ reads B is ***at least as certain*** as C.

This is known as epistemic entrenchment

(Gärdenfors)

Possibility ordering (Lewis, 1973)

i) $\geq_\Pi$ is complete and transitive

ii) $S >_\Pi \emptyset$,

iii) $\forall A, A \geq_\Pi \emptyset$

iv) disjunctive stability

$\forall A, B \geq_\Pi C \Rightarrow A \cup B \geq_\Pi A \cup C$

$B \geq_\Pi C$ reads B is at least as possible as C.

Ordering used by Grove

The dual of necessity orderings:

$B \geq C$ is a possibility ordering iff $\neg C \geq \neg B$ is a necessity ordering.

Such orderings are almost disjoint families:

$\geq$ coincides with two of the above orderings if and only if it is *truth-like*

$\exists s \in S \; \forall A, B \; A \geq B$ iff $s \in A$

NO uncertainty

REPRESENTATION OF CONFIDENCE RELATION ON TOTALLY ORDERED SCALES (absolute representations)

Let L be a totally ordered scale with bottom 0 and top 1

(a finite chain; the set of integers; the unit interval)

A confidence function is a mapping g from events to L, such that

$g(\emptyset) = 0$, $g(S) = 1$

if $A \subseteq B$ then $g(B) \geq g(A)$

A confidence function g is a representation of a confidence ordering $\geq_{\mathbf{L}}$ *when*

$g(B) \geq g(A)$ iff $B \geq_{\mathbf{L}} A$

THE NATURAL BRIDGE BETWEEN ORDINAL AND LESS ORDINAL UNCERTAINTY THEORIES

REPRESENTATION OF COMPARATIVE PROBABILITY

On finite sets, any probability measure induces a comparative probability

The converse is false : some comparative probability orderings are not representable by probability functions. (Kraft et al.)

REPRESENTATION OF COMPARATIVE POSSIBILITY

characteristic axiom of possibility measures in the finite case

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B)) \quad \text{(maxitivity)}$$

On finite sets, any possibility ordering can be represented by and only by a possibility measure on L.

Unicity up to a monotone transformation.

REPRESENTATION OF COMPARATIVE NECESSITY

characteristic axiom of necessity measures in the finite case

$N(A \cap B) = \min(N(A), N(B))$ (minitivity)

On finite sets, any necessity ordering can be represented by and only by a necessity measure on a finite chain L.

Unicity up to a monotone transformation.

Duality Property: $N(A) = n(\Pi(\neg A))$ where n is an order-reversing map;

Like modal logic:

A is sure *if and only if*

"not A" is impossible.

Choosing between N or Π is a matter of convenience

THE SIMPLICITY OF POSSIBILITY THEORY

Possibility and necessity functions (orderings) are fully characterised by possibility distributions π (complete preorderings $\geq_\pi$) on the set of states S.
<u>Not true for comparative probabilitie</u>s!!!

A possibility structure is (S, $\geq_\pi$)

The set of states of affairs is partitioned **via π** into a totally ordered set of clusters of equally plausible states $S = E1 \cup E2 \cup \ldots \cup En+1$

E1 (normal worlds) $>_\pi$ E2 $>_\pi$
.... En+1 (virtually impossible worlds)
$B \geq_\Pi C$ iff $\max B \geq_\pi \max C$
$B \geq_N C$ iff $\max \neg C \geq_\pi \max \neg B$

Are confidence orderings and confidence functions on a finite chain equivalent representations ?

absolutely ordered scale L vs. ordering

NOT EXACTLY!!!

Relative expressive power of the set of copmplete preorderings on S and the set of mappings π from S to L.

• an ordering $\geq_\pi$ of states defined on S cannot express that some states are impossible.

One has TO ADD the condition $\emptyset \geq_\Pi \{s\}$ for concerned states.

With an absolute scale one just have to let

$$\pi(s) = 0$$

Hence it is not possible to express the deterministic situation $\pi(s) = 1$, for some unique s, and $\pi(s') = 0$, $\forall s' \neq s$ with $\geq_\pi$

• ordering relations are not commensurate: how to combine a confidence relation $\geq_\pi$ and a preference relation $\geq_P$ on S ?
Situation is easier if we have a common scale

• possibility orderings cannot be compared with regard to their internal conflict.

If $\max_{s \in S} \pi(s) < 1$, π is said to be partially inconsistent. $\mathrm{Cons}(\pi) = \max_{s \in S} \pi(s)$

We may have $\mathrm{Cons}(\pi) < \mathrm{Cons}(\mu)$,
while $\geq_\pi = \geq_\pi$

• notions of quantity of information conveyed by the representation cannot be equivalently defined with absolute and relative representations

Comparing possibility orderings

$\geq_\pi = E1 \cup E2 \cup \ldots \cup En$

$\geq_\mu = F1 \cup F2 \cup \ldots \cup Fm$

specificity ordering : $\geq_\pi$ is said to be less **specific** than $\geq_\mu$ in the wide sense iff

$$\forall i = 1, \max(m, n),$$
$$F1 \cup F2 \cup \ldots \cup Fi \subseteq E1 \cup E2 \cup \ldots \cup Ei$$

(completing the coarsest partition with Ø)

Get less specific by moving states up in the plausibility ordering

- the least specific: $E1 = S$
- the most specific : $\forall i$, $Ei = \{s\}$ for some s:

linear possibility orderings (no ties).

Pseudo-deterministic possibility orderings are not maximally specific: $\geq_\pi = \{s\} \cup E2$

refinement: π is said to **refine** π' iff $\forall$ s_1, s_2 if $s_1 \geq_{\pi} s_2$ then $s_1 \geq_{\mu} s_2$.

Get more refined by splitting equivalence classes E_i

Like for specificity:

- the least refined: E1 = S
- the most specific ones: $\forall$i , Ei = {s} for some s: all linear possibility orderings (no ties).

With specificity comparison, there is an implicit assumption that E1 and F1 correspond to the same level.

Refinement is commensurateness-free, but some sets of possibility orderings have unique minimallly specific elements, but no minimally refined ones (in nonmonotonic reasoning).

Comparing L-valued possibility distributions

specificity ordering : π is said to be less **specific** than μ in the wide sense iff $\pi(s) \geq \mu(s), \forall s$

- if $\pi \geq \mu$ then : $\geq_\pi$ is less **specific** than $\geq_\mu$
- **if** : $\geq_\pi$ is less **specific** than $\geq_\mu$ then there exist $\pi \geq \mu$ representing these orderings on L (but not all representations are comparable)

- The least specific π: $\pi(s) = 1, \forall s$.
- The most specific $\pi(s) = 1$, for some unique s, and $\pi(s') = 0, \forall s' \neq s$

(linear possibility distributions are no longer the most specific ones)

INTERPRETING POSSIBILITY ORDERINGS

$s \geq_\pi s'$ means state s is more normal, typical plausible than s'

The most plausible states $S(\pi) = E1$

By default the truth lies in $S(\pi)$.

Properties: if $A >_N Ø$ then $Ø \geq_N \neg A$

Sure event : $A \geq_N S$

Unsure event: $Ø \geq_N \neg A$ and $Ø \geq_N A$

Impossible event A: $\neg A \geq_N S$ (iff $Ø \geq_\Pi A$)

Accepted events $A >_N Ø$

Accepted events are closed under deduction:

$A >_N Ø$ and $A \subseteq B$ imply $B >_N Ø$

$A >_N Ø$ and $B >_N Ø$ imply $A \cap B >_N Ø$

More generally, fixing $C >_N Ø$, $\{A \geq_N C\}$ forms a belief set for any C

$B \geq_\Pi C$ means " the most plausible situation where B occurs is at least as plausible as the most plausible situation where C occurs"
Comparing propositions on the basis of their most normal models.

$A >_N Ø$ means that A is true in all the normal states $S(\pi)$

In agreement with the idea of " reasoning as if the current situation were normal"
Jumping to conclusions obtained from the normality assumption.

DIFFERENT FROM PROBABILISTIC REASONING BASED ON AVERAGING

THE POSSIBILISTIC REPRESENTATION OF BELIEFS

The complete preordering of world states $\geq_\pi$ represents the epistemic state of an agent telling what is likely from what is less likely.

It shares the set of propositions into 3 subsets

Accepted beliefs *A*: $A >_N \emptyset$

Rejected beliefs: ***R*:**

A such that $S >_\Pi A$ (or $\neg A >_N \emptyset$)

Ignored beliefs: ***U*:** A such that neither $A >_N \emptyset$ nor $\neg A >_N \emptyset$ hold.

Like in classical logic:

A is deductively closed

incompleteness is captured

Unlike classical logic:

A is ranked in terms of certainty

R is ranked in terms of impossibility

CONDITIONAL POSSIBILITY ORDERINGS

A is at least as possible as B in the context C $>_\Pi \emptyset$ if and only if $A \cap C \geq_\Pi B \cap C$

A possibilistic ordering $\geq_\Pi^C$ restricted to the subsets of C

For possibility functions on L it corresponds to the following definition:

$$\Pi(A \mid C) = 1 \text{ if } A \neq \emptyset, \Pi(C) = \Pi(A \cap C) > 0$$
$$= \Pi(A \cap C) \text{ if } \Pi(A \cap C) < \Pi(C)$$

the greatest solution to the Bayesian-like equation

$\Pi(A \cap C) = \min(\Pi(A \mid C),\Pi(C))$

maximize $\Pi(A)$ = assume that every proposition that is not known as impossible is possibly true

CONDITIONAL NECESSITY ORDERINGS

A is at least as necessary as B in the context C $>_{\Pi} \emptyset$ if and only if $\neg A \geq_{\Pi}^{C} \neg B$

This is equivalent to $\neg C \cup A \geq_{N} \neg C \cup B$

For L-valued necessity functions on L, it corresponds to the following definition:

$N(A \mid C) = n(\Pi(A \mid C))$.

$N(A \mid C) > 0$ means "A is an accepted belief in the context C"

Basic Properties

- When $\Pi(C) > 0$,

$$N(A \mid C) > 0 \text{ iff } A \cap C >_{\Pi} \neg A \cap C$$

- The set $\mathcal{A}(C) = \{ A: N(A \mid C) > 0\}$ of conditionally accepted beliefs are deductively closed
- It may be the case that

$$N(A) > 0 \text{ and } N(\neg A \mid C) > 0$$

PLAUSIBLE INFERENCE WITH A POSSIBILITY DISTRIBUTION

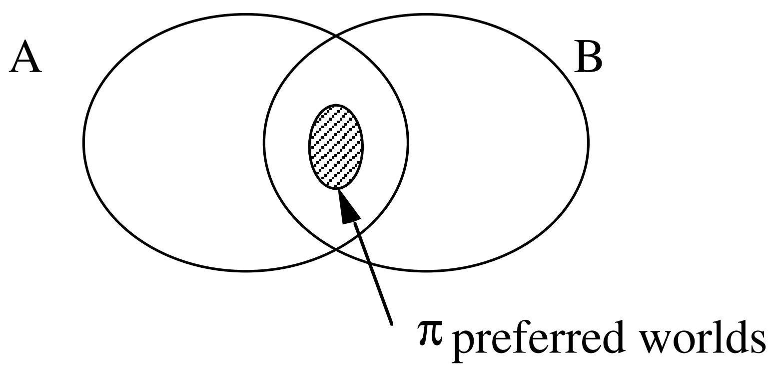
A possibility ordering $\geq_\Pi$ is said to iff non-dogmatic possibility iff A $>_\Pi$ Ø for all A ≠ Ø. ($\pi(s) > 0, \forall s$)

$$A \vDash_\pi B \text{ iff } A \cap B >_\Pi \neg A \cap B$$

$$(\text{iff } N(B|A) > 0)$$

Proposition: $A \vDash_\pi B$ iff B is true in the most plausible worlds where A is true:

$$\forall s \in B, \text{ if } \pi(s) = \Pi(B), \text{ then } s \in A$$



PROPERTIES OF $\models_\pi$

$A \models_\pi A$ if $A \neq \emptyset$ (*restricted reflexivity*)

$\emptyset \models_\pi A$ never holds (*nihil ex absurdo*)

if $A \neq \emptyset$, then $A \models_\pi \emptyset$ never holds
(*consistency preservation*)

If A ‟ B and $C \models_\pi A$ then $C \models_\pi B$
(*right weakening rule* RW)

Cumulativity:

If $A \models_\pi B$ and $A \models_\pi C$ then $A \cap C \models_\pi B$
(*cautious monotony* RM)

If $A \models_\pi B$ and $A \cap B \models_\pi C$ then $A \models_\pi C$
(*cut, weak transitivity*)

If $A \models_\pi B$ and $A \models_\pi \neg C$ do not hold,
then $A \cap C \models_\pi B$
(rational monotony RM)

If $A \cap B \vDash_\pi C$ then $B \vDash_\pi \neg A \cup C$

(half of deduction theorem)

If $A \vDash_\pi B$; $A \vDash_\pi C$ then $A \vDash_\pi B \cap C$

(Right AND)

If $A \vDash_\pi C$; $B \vDash_\pi C$ then $A \cup B \vDash_\pi C$

(Left OR)

The set $\{B: A \vDash_\pi B\}$ is **deductively closed**

The set **of default rules** $\{A \vDash_\pi B\}$ **form what KLM call a rational extension**

In fact it has been proved that

any consequence relation $\mid\!\sim$ ***satisfying the above properties is representable by possibilistic inference,***

and induces a complete plausibility preordering on S.

REPRESENTATION THEOREM FOR POSSIBILISTIC ENTAILMENT

Let $\mid\!\sim$ be a consequence relation on $2^{\Omega} \times 2^{\Omega}$

Define an induced partial relation on subsets as $A > B$ iff $A \cup B \mid\!\sim \neg B$

Lemma : given a qualitative possibility ordering, π, $A >_{\Pi} B$ iff $A \cup B \models_{\pi} \neg B$

Theorem: The consequence relation $\mid\!\sim$ satisfies

- restricted reflexivity
- consistency preservation
- right weakening
- **rational monotony**
- Right AND and Left OR

if and only if $A > B$ is the strict part of a possibilistic ordering of events

Universal possibilistic entailment

Definition: $A \rightarrow B$ is a *universal possibilistic consequence* of Δ, denoted by $\Delta \models_{\forall \Pi} A \rightarrow B$,

iff

for each π in $\Pi(\Delta)$, $A \models_{\pi} B$.

Proposition (Dubois & Prade, 1995)

The universal entailment $\models_{\forall \Pi}$ is equivalent to System P.

(= the intersection of rational extensions of Δ induced by each π in $\Pi(\Delta)$

is the preferential extension

$\Delta^P = \{ A \rightarrow B , \Delta \models_{\forall \Pi} A \rightarrow B \}$

Lehmann &Magidor, 1992)

Probabilistic interpretation

Principle (Adams, Pearl)
" Generally, if A then B " is interpreted as

$$P(B|A) \geq 1\text{-}\varepsilon$$

where ε is a positive infinitesimal real number.

Let $\Delta=\{ A_i \rightarrow B_i \,/ i=1,\ldots,n\}$ be a set of default rules,and $\Delta\varepsilon = \{ \mathcal{P} / \mathcal{P}(B_i|A_i) \geq 1\text{-}\varepsilon$, for each $A_i \rightarrow B_i \in \Delta \}$

Definition: $\Delta \vdash_\varepsilon A \rightarrow B$
iff $\forall \mathcal{P} \in \Delta\varepsilon$, $\mathcal{P}(B|A) \geq 1\text{-}O(\varepsilon)$

Theorem (Lehmann and Magidor)
$\vdash_\varepsilon$ **is equivalent to System P inference**

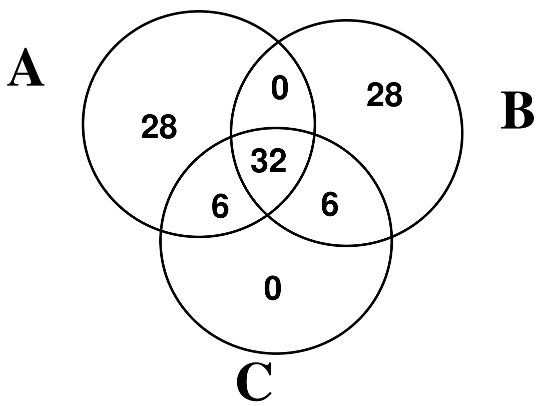
Do we need infinitesimal probabilities?

If $A \rightarrow B$ is interpreted as :
"the majority of α's are β"
(i.e. $\forall A \rightarrow B \in \Delta$, $P(B|A) > 0.5$)
Then
Cautious monotony and Or are not always satisfied

Counter-example for CM:

"51% of students are male", i.e., $P(m|s) > 0.5$
"51% of all students will pass the exam", i.e., $P(g|s) > 0.5$,
but:
"2% of male student pass the exam", i.e., $P(g|s \wedge m) < 0.5$

Counter-example for OR:



$P(C|A)=38/66>0.5$, & $P(C|B)=38/66 > 0.5$, but
$P(C|A \vee B)=44/100 < 0.5$

Do we need all of $\Pi(\Delta)$ to represent system P?

Definition:

$\pi = \{E_0, \ldots, E_n\}$ is said to be linear if each E_i contains a single interpretation .

$\geq_\pi$ *is a total ranking of interpretations*

Notation: $\Pi_L(\Delta) = \{\pi \in \Pi(\Delta) \text{ and } \pi \text{ is linear}\}$

Main results

1. If π is linear, then $\forall A,B$:

either $A \models_\pi B$ or $A \models_\pi \neg B$

(conditional completeness)

2. For any $\pi \in \Pi(\Delta)$, there exists $\Sigma \subseteq \Pi_L(\Delta)$ such that: $A \models_\pi B$ iff $\forall \pi' \in \Sigma \quad A \models_{\pi'} B$.

3. **Total ranking semantics of system P**

$\Delta \vdash_p A \rightarrow B$ iff $\forall \pi \in \Pi_L(\Delta)$, $A \models_\pi B$.

Atomic Bound Probability Systems

Definition (Snow)**:** A probability distribution P is **atomic bound (big-stepped)** if:

- for each s, $P(s)>0$
- P induces a total ranking $>$ on Ω

(as $P(s) > P(s')$ where $P(s) < P(s')$ or $P(s') < P(s)$)

- $P(s) > \sum_{s':\, s > s'} P(s')$

Each A possesses a "usual" model (it occurs more probably that the disjunction of other models of A)

Example : a/ 0.6, b/0.3, c/0.06/, d/0.03, e/0.01

Proposition (Snow):
Let π_be a linear possibility distribution.
Then any atomic bound probability distribution P where $> = >_\pi$ is such that:

$$A \vDash_\pi B \quad \text{iff} \quad P(B|A)>1/2.$$

Standard probabilistic semantics for System P

<u>**Theorem**</u>

$\mathbb{P}_L(\Delta) = \{P \,/\, P(B_i|A_i)>1/2$ for each $A_i \rightarrow B_i$ in Δ and P is big-stepped$\}$, then:

Define: $A \mid\!\sim_{Pab} B$ as

$$\forall P \in \mathbb{P}_L(\Delta),\ P(B|A)>1/2,$$

then:

$\mid\!\sim_{Pab}$ **is equivalent to System P inference**

Corollary:

Each possibility distribution can be represented by an atomic bound system **namely**

$\forall\, \pi\, \exists$ a family $\mathbb{P}$ of big-stepped probability distributions such that:

$A \models_\pi B$ iff $\forall P \in \mathbb{P}$, $P(B|A)>1/2$,

Existing semantics of system P

-**Preferential semantics** of KLM:
({states} +partial ordering)
function from states to interpretations

- Infinitesimal **Probabilistic semantics**
Defaults are infinitesimal conditional probabilities (Adams, Pearl)

-**Three-valued logic**
The logic of conditional objects (De Finetti, Dubois & Prade)

- **Possibilistic semantics (**families of total rankings of states)
- Standard **Probabilistic semantics**
- using big-stepped probabilities
- **partial confidence orderings (Friedman and Halpern)**

LIMITATIONS OF ORDINAL REPRESENTATIONS OF UNCERTAINTY

Lottery paradox

1,000,000 lottery tickets are sold

There is one winner ticket

Prob (player i loses| all tickets sold) = 0.99 999

Prob (all players lose | all tickets sold) = 0

If *A is accepted in context C =*

P(A | C) > a fixed close to 1

and system P is applied:

(all tickets sold $\mid\!\sim$ player i loses) *for all players*

implies

all tickets sold $\mid\!\sim$ all players lose

Ordinal reasoning fails!!!

Reason: uniform probabilities, nothing typical

HOWEVER THE PARADOX DOES NOT WORK WITH BIG-STEPPED PROBABILITIES since then
{A, P(A | C) > 0.5} remains consistent and deductively closed

BIG-STEPPED PROBABILITIES ACCOUNT FOR RANDOM PHENOMENA WHERE TYPICAL PHENOMENA ARE OBSERVED

They seem to be rare, since the constraints bearing on them are very strong

PROBLEM: given a sample space Ω and a probability measure P, find a ***linguistically meaningful partition S on*** *Ω (= a language)*, ***informative enough, such that the coarsening of P in S is a big-stepped probability.***
Toward learning default rules from data???

LIMITATION OF QUALITATIVE CONDITIONING : *The absolute value of $N(B \mid A)$ is not meaningful on a finite scale L*

Theorem: if $N(A) > 0$ then

either $N(A|B) \geq N(A)$ or $N(A|B) = 0$

The acceptance of A cannot be attenuated.

When B becomes known, A is either at least as sure as it was or is canceled or refuted by B

Hence certainty reinforcement is not meaningful in the finite ordinal setting

This problem does not occur with quantitative conditional possibility

$$\Pi(A|_* B) = \Pi(A \cap B)/\Pi(B)$$

Moreover quantitative event independence is symmetric

$$\Pi(A|_* B) = \Pi(A) \text{ iff } \Pi(A \cap B) = \Pi(A) \cdot \Pi(B)$$

DIAGNOSIS IN THE ORDINAL SETTING

h = hypothesis; e = evidence

A priori knowledge; $(\Pi(h), \Pi(\neg h))$

Likelihoods: $\Pi(e \mid h)$, $\Pi(e \mid \neg h)$

We assume

$\min(\Pi(h), \Pi(\neg h)) > 0$, $\max(\Pi(h), \Pi(\neg h)) = 1$

Qualitative Bayes Theorem

$\min(\Pi(h \mid e), \Pi(e)) = \min(\Pi(e \mid h), \Pi(h))$;

$\min(\Pi(\neg h \mid e), \Pi(e)) = \min(\Pi(e \mid \neg h), \Pi(\neg h))$.

where $\Pi(e) = \max(\Pi(e \wedge h), \Pi(e \wedge \neg h))$

Trying to compute posteriors

$\Pi(h \mid e)$, $\Pi(\neg h \mid e)$

$\Pi(h \mid e) = \min(\Pi(e \mid h), \Pi(h))$

if $\Pi(e \wedge h) < \Pi(e \wedge \neg h)$

$= 1$ otherwise

General remarks

Assume e is observed:

1) Confirmation of hypothesis h requires
• $\Pi(e \mid h) > \Pi(e \mid \neg h)$,
and also
• $\min(\Pi(h),\Pi(\neg h)) > \Pi(e \mid \neg h)$,
no strong a priori opinion is held in favor or against h
Otherwise the prior possibilities are unchanged

2) Convergence of the process is in one step.

3) No attenuation of belief is possible: h is either confirmed, or forgotten, or ¬h is confirmed

CANNOT MATCH NUMERICAL BAYESIAN DIAGNOSIS

Axiomatization of ordinal decision in uncertainty

S and X are finite

A **strict** preference $\succ$ on acts $\{f : S \to X\}$

WS1 Strict preference on acts is transitive and irreflexive.

Indifference not transitive

Pure Savage axioms

S2 $fAh \succcurlyeq gAh \Leftrightarrow fAh' \succcurlyeq gAh'$

S3 if $A \subseteq S$, $A \neq \emptyset$ then $(f^x \geq f^y)_A \Leftrightarrow f^x \succcurlyeq f^y$)

S4 $\forall\, x \geq_P y$, $xAy \succcurlyeq xBy$

S5 $\exists\, x , x' \in X$, $f^x > f^{x'}$

Additional Savagean axiom

WSTP: if $fAg \succcurlyeq g$ and $gAf \succcurlyeq g$ then $f \succcurlyeq g$

(*due to WS1*)

ORDINALITY

Two pairs of acts (f, g) and (f', g') will be called *statewise order-equivalent* if and only if $\forall s \in S$, $f(s) \geq_P g(s)$ if and only if $f'(s) \geq_P g'(s)$.

This is denoted $(f, g) \equiv (f', g')$.

What matters for preference between two acts is the relative positions of consequences of acts for each state, not the consequences themselves, nor the positions of these acts relative to other acts.

***Ordinal Invariance* axiom** (Fargier and Perny, 1999),

OI: $\forall f, f' g, g'$,

if $(f, g) \equiv (f', g')$ then $(f \succcurlyeq g \Leftrightarrow f' \succcurlyeq g')$.

Properties

Proposition : Under assumptions that OI holds, and that relation $\succcurlyeq$ on acts is reflexive, it follows that S2, and S4 hold too.

Proposition : Under the assumption of ordinal invariance, if the weak preference on acts is reflexive and the induced weak preference on consequences is complete, the only possible decision rule is

Likely dominance rule.

If $\succ$ projects to $(2^S, \geq_L)$ and $(X, \geq_P)$ then

$$f \succcurlyeq g \text{ iff } [f \geq_P g] \geq_L [g \geq_P f]$$

Proposition : Under WS1, S3, S'5 and OI,

A, B, C disjoint,

$A \cup C >_L B$ and $B \cup C >_L A$ imply

$$C >_L A \cup B$$

This is the basic property of partial confidence orderings underlying system P, after Friedman and Halpern.

THEOREM

If a strict preference $\succ$ on acts satisfies axioms WS1, S3, S'5 WSTP, and OI, then

-) $(X, \geq_P)$ is a complete preorder

-) There exists a family $\mathcal{F}$ of complete preorders $\geq_\pi$ on states (preorders of plausibility) such that

$A >_L B$ iff $\forall \geq_\pi \in \mathcal{F}$,

$$\max A \cap \bar{B} >_\pi \max B \cap \bar{A}$$

UNCERTAINTY IS POSSIBILISTIC

DECISION RULES for a single relation $\geq_\pi$

If all states are equivalent:

UNANIMITY RULE

$f \succ g$ iff $f \geq_P g$ and $\exists s, f(s) >_P g(s)$

If total order on states $s_1 >_\pi s_2 >_\pi \dots >_\pi s_n$

$f \succ g$ iff $\exists k$ such that $f(s_i) \sim_P g(s_i), \forall i < k$

and $f(s_k) >_P g(s_k)$.

(DICTATORSHIP of the most plausible discriminant state)

General case:

a partition of S : $S_1 >_\pi S_2 >_\pi \dots >_\pi S_n$

$f \succ g$ iff $\exists k$ such that

$f \sim_P g$ on S_i, $\forall i < k$ and $f >_P g$ on S_k

(OLIGARCHY of the most plausible states)

These DECISION RULES ARE TOO ADVENTUROUS OU TOO LITTLE DECISIVE

CONCLUSION:

Purely ordinal theories of uncertainty are much more expressive that classical logic:

They capture a form of nonmonotonic inference that is an abstraction of probabilistic reasoning.

•Computational complexity of possibilistic inference **is not so high.**
Can be computerized by means of prioritized propositional bases using possibilistic logic, with a sound and complere refutation technique.

•Potential for learning default rules from big stepped probabilities

BUT they may be insufficient for other tasks

Diagnosis : lack of graceful behavior when iterating Bayesian-like approaches

Decision under uncertainty: paradoxes and impossibility theorems from social choice are met again.
A way out: qualitative decision with commensurate scales for unceratinty and preference (Dubois Prade Sabbadin)