# Explicit clausal definitions

Lecture 11

# Examples of Discriminators built into CL

Discriminators **without patterns**:

- **negation**: $\mathbf{A} \mid \neg\mathbf{A}$
- **test on zero**: $\mathbf{s} = 0 \mid \mathbf{s} > 0$
- **trichotomy**: $\mathbf{s} < \mathbf{t} \mid \mathbf{s} = \mathbf{t} \mid \mathbf{s} > \mathbf{t}$

Discriminators **with patterns**:

- **let**: $\mathbf{s} = z$
- **binary**: $\mathbf{s} = z\mathbf{0} \land z = 0 \mid \mathbf{s} = z\mathbf{0} \land z > 0 \mid \mathbf{s} = z\mathbf{1}$
- **division by four**: $\mathbf{s} = 4{\cdot}z + v \land 0 \leq v \leq 3$

- **exactly** one alternative holds
- pattern variables **uniquely exist**

# Examples of Provable discriminators

- discr. on the **head of lists:** $Adj(\mathbf{s}) = 0 \mid \mathbf{s} = z; t$
- discr. on the **tail of lists:** $Adj(\mathbf{s}) = 0 \mid \mathbf{s} = t \boxplus (z; u) \land Adj(u) = 0$
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The head discrimination used in a clausal definition:

$$Rev(t) = t \qquad \leftarrow Adj(t)$$
$$Rev(x; t) = Rev(t) \boxplus (x; 0)$$

# Conditional discriminators:

**general division**: provided $\mathbf{t} > 0$ then $\mathbf{s} = \mathbf{t} \cdot z + v \land 0 \leq v < \mathbf{t}$
**special discrimination for** $g^*$: provided PA proves

$$g(x, n, a) = v\mathbf{1} \rightarrow \mu(v) < \mu(x)$$
$$2 \mid g(x, 0, a)$$

we have $g(\mathbf{s}, \mathbf{n}, \mathbf{a}) = v\mathbf{0} \mid g(\mathbf{s}, \mathbf{n}, \mathbf{a}) = v\mathbf{1} \land \mathbf{n} = m + 1$
This is used in the clauses for $g^*$:

$$g^*(x, n, a) = v \qquad \leftarrow g(x, n, a) = v\mathbf{0}$$
$$g^*(x, n+1, a) = g^*(x, n, a \boxplus (g^*(v, C, 0); 0)) \leftarrow g(x, n+1, a) = v\mathbf{1}$$

# General form of provable discriminators

We use **bold** variables $\mathbf{x}$ for possibly empty sequences of variables $x_1, \dots, x_n$,
we let $\exists \mathbf{x}\mathbf{D}$ to stand for $\exists x_1 \dots \exists x_n \mathbf{D}$ ($n$ can be empty), and write $\mathbf{x} = \mathbf{y}$ for $x_1 = y_1 \land \cdots x_n = y_n$.
Suppose that PA proves for $k \geq 1$:

$$\exists \mathbf{z}_1 \mathbf{D}_1[\mathbf{z}_1] \lor \exists \mathbf{z}_2 \mathbf{D}_2[\mathbf{z}_2] \lor \cdots \lor \exists \mathbf{z}_k \mathbf{D}_k[\mathbf{z}_k]$$

$$\mathbf{D}_i[\mathbf{z}_i] \to \neg \mathbf{D}_j[\mathbf{z}_j] \qquad \text{for all } 1 \leq i \neq j \leq k$$

$$\mathbf{D}_i[\mathbf{z}_i] \land \mathbf{D}_i[\mathbf{w}] \to \mathbf{z}_i = \mathbf{w} \qquad \text{for all } 1 \leq i \leq k$$

This means that **exactly** one $\mathbf{D}_i[\mathbf{z}]_i$ holds with **uniquely** determined patterns $\mathbf{z}_i$.
We can then use

$$\mathbf{D}_1[\mathbf{z}_1] \mid \mathbf{D}_2[\mathbf{z}_2] \mid \cdots \mid \mathbf{D}_k[\mathbf{z}_k]$$

as **provable discriminators** (we can even permit conditional discrimination).

# Clausal formulas

In the following we will write $\mathbf{A}[\mathbf{x}; v]$ for a formula with the **output** variable $v$ free and with other free variables among the **input** variables $\mathbf{x}$

A formula $\mathbf{A}[\mathbf{x}; v]$ is a **clausal formula** if $\mathbf{A}$ is either of a form

- $\mathbf{s}[\mathbf{x}] = v$ or
- $\exists \mathbf{z}_1(\mathbf{D}_1[\mathbf{x}, \mathbf{z}_1] \wedge \mathbf{A}_1[\mathbf{x}, \mathbf{z}_1; v]) \vee \cdots \vee \exists \mathbf{z}_k(\mathbf{D}_k[\mathbf{x}, \mathbf{z}_k] \wedge \mathbf{A}_k[\mathbf{x}, \mathbf{z}_k, v])$

where $\mathbf{D}_1, \ldots, \mathbf{D}_k$ is a provable discriminator and $\mathbf{A}_1, \ldots, \mathbf{A}_k$ are clausal formulas.

# Using clausal formulas in explicit definitions

In all clausal formulas $\mathbf{A}[\mathbf{x}; v]$ for every $\mathbf{x}$ the output variable is uniquely determined, i.e. PA proves:

$$\exists v \mathbf{A}[\mathbf{x}; v]$$

$$\mathbf{A}[\mathbf{x}; v] \land \mathbf{A}[\mathbf{x}; w] \rightarrow v = w$$

We can thus explictly introduce into PA a new function symbol $f$ by:

$$f(\mathbf{x}) = v \leftrightarrow \mathbf{A}[\mathbf{x}; v] \ ,$$

or in CL by $f(\mathbf{x}) = \mu_v[\mathbf{A}[\mathbf{x}; v]]$.
The above equivalence is actually equivalent in PA to

$$f(\mathbf{x}) = v \leftarrow \mathbf{A}[\mathbf{x}; v]$$

because if in the direction $(\rightarrow)$ $f(\mathbf{x}) = v$ holds then $\mathbf{A}[\mathbf{x}; w]$ for some $w$ by existence and $w = f(\mathbf{x})$ by $(\leftarrow)$.

# Unfolding the clausal formulas

We now assign to every formula $\mathbf{B}$, every clausal formula $\mathbf{A}[\mathbf{x}; v]$ and a new function symbol $f$ a **finite set** of **clauses** by the **unfolding** operator $U[f, \mathbf{B}, \mathbf{A}]$ such that:
• if $\mathbf{A} \equiv \mathbf{s}[\mathbf{x}] = v$ then $U[f, \mathbf{B}, \mathbf{A}] = \{f(\mathbf{x}) = v \leftarrow \mathbf{B} \wedge \mathbf{s}[\mathbf{x}] = v\}$
and if

$$\mathbf{A} \equiv \exists \mathbf{z}_1(\mathbf{D}_1[\mathbf{x}, \mathbf{z}_1] \wedge \mathbf{A}_1[\mathbf{x}, \mathbf{z}_1; v]) \vee \cdots \vee \exists \mathbf{z}_k(\mathbf{D}_k[\mathbf{x}, \mathbf{z}_k] \wedge \mathbf{A}_k[\mathbf{x}, \mathbf{z}_k, v])$$

then

$$U[f, \mathbf{B}, \mathbf{A}] = \cup_{1 \leq i \leq k} U[f, (\mathbf{B} \wedge \mathbf{D}_i[\mathbf{x}, \mathbf{z}_i]), \mathbf{A}_i[\mathbf{x}, \mathbf{z}_i, v]]$$

If $U[f, \top, \mathbf{A}[\mathbf{x}; v]] = \{\mathbf{C}_1, , \ldots \mathbf{C}_m\}$ then we have

$$\vdash f(\mathbf{x}) = b \leftarrow \mathbf{A}[\mathbf{x}; v] \text{ iff } \vdash \mathbf{C}_1 \text{ and } \ldots \text{and } \vdash \mathbf{C}_m$$