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Nested iteration

Lecture 10

The schema of nested iteration

For every three-place function g, unary **measure** function μ, and a constant C giving a **recursion count** introduced into PA such that

$$\vdash g(x,n,a) = v\mathbf{1} \rightarrow \mu(v) < \mu(x)$$
$$\vdash 2 \mid g(x,0,a)$$

we wish to introduce a three-place **nested iteration** function g^* such that:

$$\vdash g^*(x,n,a) = v \leftarrow g(x,n,a) = v\mathbf{0}$$
$$\vdash g^*(x,n,a) = y \leftarrow g(x,n,a) = v\mathbf{1} \land n = m+1 \land g^*(v,C,0) = w \land g^*(x,m,a \boxplus (w;0)) = y$$

The measure of this recursion is $\mu(x){\cdot}C + n$ because $\mu(x){\cdot}C + n > \mu(x){\cdot}C + m$ (for the outer recursion) and $\mu(x){\cdot}C + n > (\mu(v)+1){\cdot}C = \mu(v){\cdot}C + C$ (for the inner recursion).

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Example: Reduction of Fibonacci sequence to nested iteration

$F_0 = F_1 = 1$ and $F_{x+2} = F_x + F_{x+1}$. For this **explicitly** define $C = 2$, $\mu(x) = x$, and

$$g(x,n,a) = \begin{cases} (x \dot{-} 2)\mathbf{1} & \text{if } x \geq 2 \wedge n = 2 \\ (x \dot{-} 1)\mathbf{1} & \text{if } x \geq 2 \wedge n = 1 \wedge a = v;b \\ (v + w)\mathbf{0} & \text{if } x \geq 2 \wedge a = v;w;b \\ 1\mathbf{0} & \text{otherwise} \end{cases}$$

Since PA proves $2 \mid g(x,0,a)$ and $g(x,n,a) = v\mathbf{1} \rightarrow v < x$ we can use the schema of iteration:

$$\vdash g^*(x,n,a) = v \leftarrow g(x,n,a) = v\mathbf{0}$$
$$\vdash g^*(x,n,a) = y \leftarrow g(x,n,a) = v\mathbf{1} \wedge n = m+1 \wedge g^*(v,C,0) = w \wedge g^*(x,m,a \boxplus (w;0)) = y$$

We can now explicitly define $F_x = g^*(x,C,0)$ and prove in PA the **recurrences** for F.

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Arithmetization of computation trees for g^*

We will code derivations of identities $g^*(\underline{x},\underline{n},\underline{a}) = \underline{y}$ where $\underline{i}$ abbreviates $S^i(0)$.
The nodes in trees satisfy **local conditions**:

$$\frac{g^*(\underline{x},\underline{n},\underline{a}) = \underline{y}}{0 \mid 0} \qquad \text{if } g(x,n,a) = y\mathbf{0}$$

$$\frac{g^*(\underline{x},\underline{n+1},\underline{a}) = \underline{y}}{g^*(\underline{v},C,0) = \underline{w} \mid g^*(\underline{x},\underline{n},\underline{a \boxplus (w;0)}) = \underline{y}} \quad \text{if } g(x,n+1,a) = v\mathbf{1}$$

We **arithmetize** $g^*(x,n,a) = y$ as $Lb(x,n,a,y)$ where $Lb(x,n,a,y) = x;n;a;y$ and abbreviate this to $(\mathbf{g}^*(x,n,a) {=}^{\bullet} y)$.

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The predicate Ct

Computation trees are **flattened** to lists containing $(\mathbf{g}^*(x,n,a){=}^{\bullet}y)$ such that for $t = (\mathbf{g}^*(x,n,a){=}^{\bullet}y); s$ the list s contains the sons (if any).

$$Lcond(x,n,a,y,t) \leftrightarrow \exists v(g(x,n,a) = v\mathbf{0} \land v = y \lor \exists m \exists w(n = m+1 \land g(x,n,a) = v\mathbf{1} \land (\mathbf{g}^*(v,C,0){=}^{\bullet}w)\ \varepsilon\ t \land (\mathbf{g}^*(x,m,a \boxplus (w;0)){=}^{\bullet}y)\ \varepsilon\ t))$$

$$Ct(s) \leftrightarrow \forall x \forall n \forall a \forall y \forall t((\mathbf{g}^*(x,n,a){=}^{\bullet}y); t \sqsubseteq s \rightarrow Lcond(x,n,a,y,t))$$

We then prove

$$Ct(s) \land t \sqsubseteq s \rightarrow Ct(t)$$
$$Adj(s) = 0 \rightarrow Ct(s)$$
$$\forall x \forall n \forall a \forall y\ b \neq (\mathbf{g}^*(x,n,a){=}^{\bullet}y) \rightarrow Ct(b;s) \leftrightarrow Ct(s)$$
$$Ct((\mathbf{g}^*(x,n,a){=}^{\bullet}y);s) \leftrightarrow Lcond((x,n,a,y,s) \land Ct(s)$$
$$Ct(s) \land Ct(t) \rightarrow Ct(s \boxplus t)$$

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Graph of nested iteration function

We introduce a four-place predicate $\mathbf{g}^*(x, n, a) \doteq y$, which will be the **graph** of g^*:

$$\mathbf{g}^*(x, n, a) \doteq y \leftrightarrow \exists t\, Ct((\mathbf{g}^*(x, n, a) =^{\bullet} y); t)\ . \qquad (1)$$

We have the following **recurrences**:

$$\vdash g(x, n, a) = v\mathbf{0} \rightarrow \mathbf{g}^*(x, n, a) \doteq y \leftrightarrow y = v$$

$$\vdash g(x, n+1, a) = v\mathbf{1} \rightarrow \mathbf{g}^*(x, n+1, a) \doteq y \leftrightarrow$$
$$\exists w(\mathbf{g}^*(v, C, 0) \doteq w \land \mathbf{g}^*(x, n, a \boxplus (w; 0)) \doteq y)$$

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Introducttion of nested iteration function

By **measure induction** with $\mu(x){\cdot}C + n$ we prove the **existence** and **uniqueness** which assert that $\mathbf{g}^*(x, n, a) \doteq y$ is indeed a **graph**:

$$\vdash \exists y \mathbf{g}^*(x, n, a) \doteq y$$
$$\vdash \mathbf{g}^*(x, n, a) \doteq y_1 \wedge \mathbf{g}^*(x, n, a) \doteq y_2 \to y_1 = y_2$$

We can thus introduce g^* by **minimization**:

$$g^*(x, n, a) = \mu_y[\mathbf{g}^*(x, n, a) \doteq y]$$

and prove the desired **recurrences**.