Coding of n-tuples and of Finite Sequences in Natural Numbers (Optimal Pairing Function)

Problems with Cantor's Pairing

The numbers $|x|_j$, $|x|_d$, and $|x|_b$ should be of the same **order**, i.e to differ only by constants.

This is not the case: For pair size n the minimal and maximal numbers min and max such that $n = |\text{min}|_j = |\text{max}|_j$ have $|\text{min}|_d$ and $|\text{max}|_d$ as listed:

p. size	$\|\text{min}\|_d$	$\|\text{max}\|_d$
1	1	1
2	1	2
3	2	3
4	3	6
5	3	11
6	4	21
7	5	41
8	6	82
9	7	163

Optimal pairing function

Solution define a **pairing** function x, y which keeps the numbers with the same pair size together, for instance, by **lexicographically ordering** the pairs:

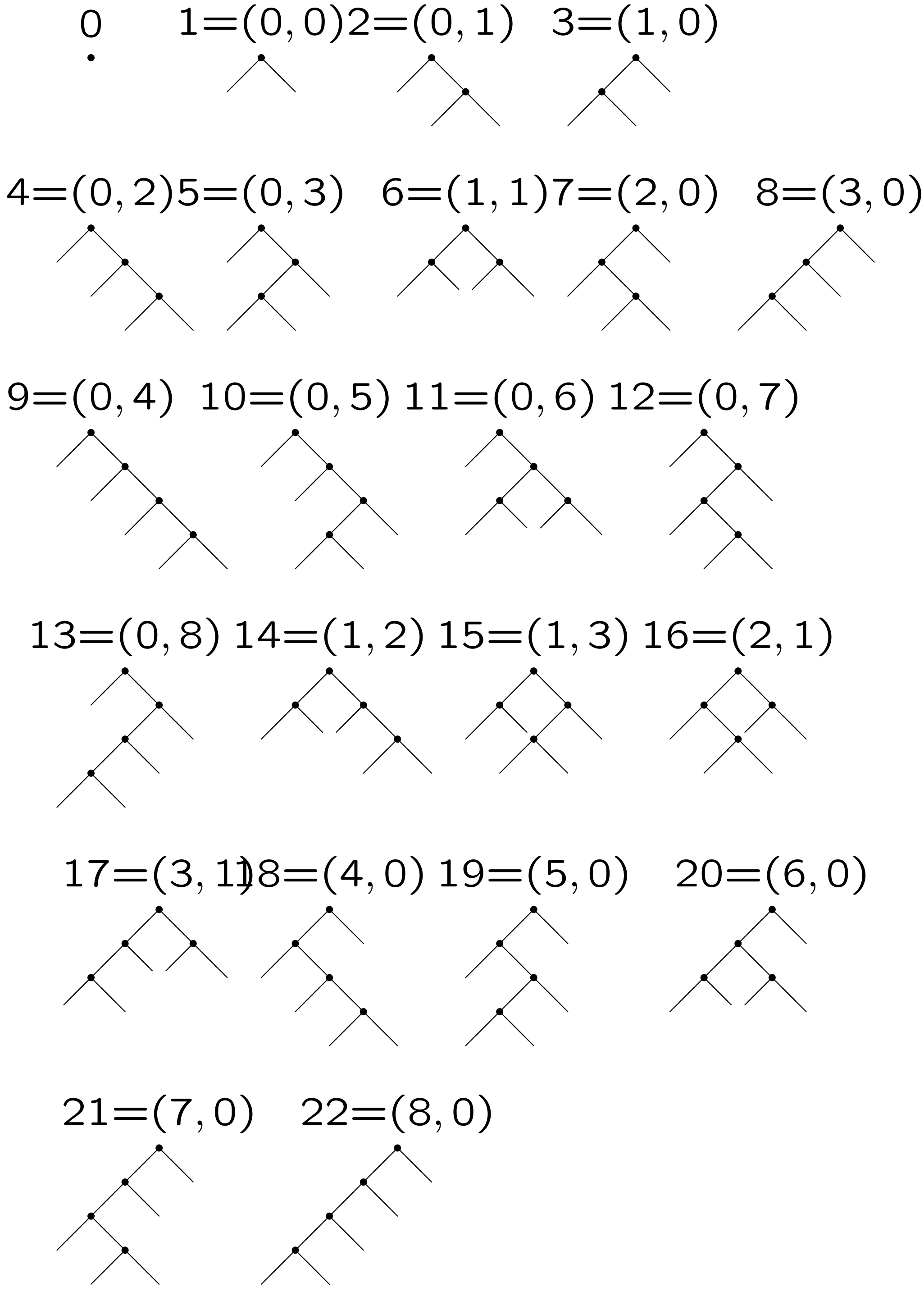
$$
\begin{aligned}
&0 < x, y \\
&x_1, y_1 < x_2, y_2 \leftrightarrow |x_1, y_1|_p < |x_2, y_2|_p \vee \\
&\qquad\qquad\qquad\qquad |x_1, y_1|_p = |x_2, y_2|_p \wedge \\
&\qquad\qquad\qquad\qquad (x_1 < x_2 \vee x_1 = x_2 \wedge y_1 < y_2)
\end{aligned}
$$

where $|0|_p = 0$ and $|(x, y)|_p = |x|_p + |y|_p + 1$
The function also satisfies:

$$(x, y) = (v, w) \rightarrow x = v \wedge y = w$$

$$v < (v, w) \wedge w < (v, w)$$

$$x = 0 \vee \exists v \exists w \, x = (v, w)$$


0
1=(0,0)
2=(0,1)
3=(1,0)
4=(0,2)
5=(0,3)
6=(1,1)
7=(2,0)
8=(3,0)
9=(0,4)
10=(0,5)
11=(0,6)
12=(0,7)
13=(0,8)
14=(1,2)
15=(1,3)
16=(2,1)
17=(3,1)
18=(4,0)
19=(5,0)
20=(6,0)
21=(7,0)
22=(8,0)

Pairing Discrimination

The basis for the **pairing** discrimination is the following property:

$$x = 0 \vee \exists! v \exists! w\, x = v, w$$

Concatenation and pair size can be thus programmed as:

$$\begin{aligned} &x \oplus y = y && \leftarrow x = 0 \\ &x \oplus y = v, w \oplus y && \leftarrow x = v, w \\ &|0|_p = 0 \\ &|x, y|_p = |x|_p + |y|_p + 1 \end{aligned}$$

These are **built** into CL and we write $x{+}{+}y$ and $|x|$.

Convention: Suppose that f is **ternary**. Then $f(3, 4, 5, 6, 7)$ is abbreviation for $f(3, 4, (5, 6, 7))$.