Coding of n-tuples and of Finite Sequences in Natural Numbers

Motivation: Pairs of functions.

Consider the pairs of functions:

$$y > 0 \rightarrow x = (x \div y) \cdot y + (x \bmod y)$$
$$n \leq |a|_d \rightarrow a = Dr(n,a) * Tk(n,a)$$

$$
\begin{aligned}
x \div y &= 0 && \leftarrow y > 0 \wedge x < y \\
x \div y &= (x \dot{-} y) \div y + 1 && \leftarrow y > 0 \wedge x \geq y \\
x \bmod y &= x && \leftarrow y > 0 \wedge x < y \\
x \bmod y &= (x \dot{-} y) \bmod y && \leftarrow y > 0 \wedge x \geq y
\end{aligned}
$$

$$
\begin{aligned}
Dr(0,a) &= a & Tk(0,a) &= 0 \\
Dr(n+1,0) &= 0 & Tk(n+1,0) &= 0 \\
Dr(n+1,a\mathbf{1}) &= Dr(n,a) & Tk(n+1,a\mathbf{1}) &= Tk(n,a)\mathbf{1} \\
Dr(n+1,a\mathbf{2}) &= Dr(n,a) & Tk(n+1,a\mathbf{2}) &= Tk(n,a)\mathbf{2}
\end{aligned}
$$

In Pascal or C we would have two functions yielding **pairs**, i.e **records** (**structures**), $\langle x \div y, x \bmod y\rangle$ and $\langle Dr(n,a), Tk(n,a)\rangle$ respectively.

Pairs in natural numbers

Suppose that we could define a binary function $\langle x, y\rangle$ **over** $\mathbb{N}$ with the **pairing property**:

$$\langle x_1, y_1\rangle = \langle x_2, y_2\rangle \rightarrow x_1 = x_2 \wedge y_1 = y_2$$

which says that the function is an **injection into** $\mathbb{N}$. We could then define:

$$\begin{aligned}
&Dm(x,y){=}\langle 0,x\rangle && \leftarrow y{>}0 \wedge x{<}y\\
&Dm(x,y){=}\langle q{+}1,r\rangle && \leftarrow\ y{>}0 \wedge x{\geq}y \wedge\\
& && \qquad Dm(x \dot{-} y, y){=}\langle q,r\rangle
\end{aligned}$$

$$\begin{aligned}
&Split(0,a) = \langle a, 0\rangle\\
&Split(n{+}1,0) = \langle 0, 0\rangle\\
&Split(n{+}1,a\mathbf{1}) = \langle d, t\mathbf{1}\rangle \leftarrow Split(n,a) = \langle d,t\rangle\\
&Split(n{+}1,a\mathbf{2}) = \langle d, t\mathbf{2}\rangle \leftarrow Split(n,a) = \langle d,t\rangle
\end{aligned}$$

With properties $Dm(x,y) = \langle x{\div}y, x \bmod y\rangle$ and $Split(n,a) = \langle Drop(n,a), Take(n,a)\rangle$

Modified Cantor's Pairing function

$Tr(n) = 1 + \sum_{i=0}^{n} i$ is called the **triangular** function. If we define:

$$\langle x, y \rangle = Tr(x + y) + x$$

then the function satisfies the pairing property:

$\langle x, y \rangle$	0	1	2	3	4	5	6	$\cdots$
0	1	2	4	7	11	16	22	$\cdots$
1	3	5	8	12	17	23	30	$\cdots$
2	6	9	13	18	24	31	39	$\cdots$
3	10	14	19	25	32	40	49	$\cdots$
4	15	20	26	33	41	50	60	$\cdots$
5	21	27	34	42	51	61	72	$\cdots$
6	28	35	43	52	62	73	85	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

We write $x; y$ instead of $\langle x, y \rangle$ and abbreviate $a; (b; c)$, i.e. $\langle a, \langle b, c \rangle \rangle$, to $a; b; c$

We then have

$$x; y = \frac{(x + y) \cdot (x + y + 1)}{2} + x + 1$$

We clearly have $Tr(0) = 1$ and $Tr(n+1) = Tr(n) + n + 1$.
Also $0; y = Tr(y)$:
We wish to define a pair of **projection** functions Hd and Tl such that

$$z > 0 \to z = Hd(z); Tl(z)$$

For that we need the **inverse** Dg of Tr satisfying:

$$z > 0 \to Tr\,Dg(z) \leq z < Tr(Dg(z) + 1)$$

i.e. for positive z the number $Dg(z) = d$ is the **diagonal** with z, i.e. the **smallest** d s.t. $z < Tr(d+1)$.
Since $x; y = Tr(x+y) + x = z > 0$, we have $x + y = Dg(z)$ and so $z = Tr\,Dg(z) + x$
For $z > 0$ we wish $Hd(z) = x$ and $Tl(z) = y$
We thus define:

$$Hd(z) = z \dot{-} Tr\,Dg(z) \qquad Tl(z) = Dg(z) \dot{-} Hd(z)$$

The diagonal function

Monadic recursion (implied by the least d such that . . . from the previous slide) leads to a sub-optimal definition of Dg. For a better definition we note that $2{\cdot}Tr(n) = n^2 + n + 2$. Thus for $z > 0$ and $Dg(z) = d$ from

$$Tr(d) \le z < Tr(d+1) = Tr(d) + d + 1$$

we get:

$$(2{\cdot}d + 1)^2 + 7 = 8{\cdot}Tr(d) \le 8{\cdot}z \le 8{\cdot}(Tr(d) + d) = \\ 4{\cdot}d^2 + 12{\cdot}d + 8 < (2{\cdot}d + 3)^2 + 7\ .$$

Hence

$$2{\cdot}d + 1 \le [\sqrt{8{\cdot}z \dot{-} 7}] < 2{\cdot}d + 3$$

and so:

$$Dg(z) = d = ([\sqrt{8{\cdot}z \dot{-} 7}] \dot{-} 1) \div 2$$

All we need now is an optimal definition of $[\sqrt{n}]$, say, by 4-adic discrimination.

Additional properties of pairing

We have $x < x;y$ and $y < x;y$ and also $0 \neq x;y$. Moreover

$$z = 0 \vee \exists! x \exists! y\, z = x;y$$

Thus 0 is the only **atom**, i.e. not in the range of the pairing function.

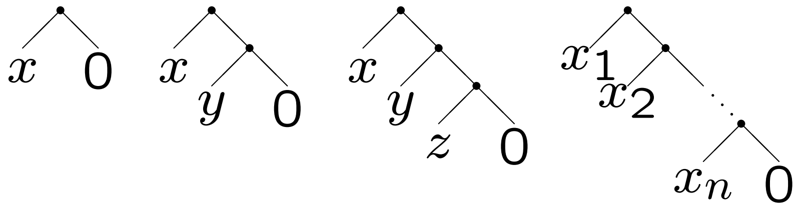
Every number $z \in \mathbb{N}$ is thus either 0 or **uniquely** $z = x;y$ (with $x, y < z$) and so every number in $\mathbb{N}$ can be uniquely written as a **pair numeral**, i.e. a term composed from 0 by $a;b$.

The **pair size** $|z|_j$ of z satisfies

$$|0|_j = 0 \qquad |x;y|_j = |x|_j + |y|_j + 1$$

and it **counts** the number of **pairings** in the numeral for z.

Coding of finite sequences over $\mathbb{N}$



These represent the numbers 0, $x;0$, $x;y;0$, $x;y;z;0$, and
$x_1;x_2;\cdots;x_n;0$ respectively.
The numbers, called **lists**, **code** the finite sequences ϵ (the **empty** sequence), x (**singleton** seq.), $x\,y$ (two element seq.) $x\,y\,z$ (three element seq.), and $x_1\,x_2\,\cdots\,x_n$ (n element seq.).
There is **one to one** correspondence between $\mathbb{N}$ and **lists**.
Concatenation of lists:

$$(x_1;\ldots;x_n;0) \boxplus (y_1;\ldots;y_m;0) = x_1;\ldots;x_n;y_1;\ldots;y_m;0$$

Concatenation satisfies the **recurrences**:

$$0 \boxplus y = y \qquad (v;w) \boxplus y = v;(w \boxplus y)$$