7.2.17 Theorem (Rice-Shapiro). *For every $n \geq 1$ and every class $\mathcal{F}$ of n-ary partially computable functions, if the problem $\varphi_e^{(n)} \in \mathcal{F}$ is semidecidable then the following holds for every n-ary partially computable function f:*

$$f \in \mathcal{F} \text{ iff there is a finite } \theta \subseteq f \text{ such that } \theta \in \mathcal{F}.$$

Proof. Take any n-ary partially computable function $f \in \mathcal{F}$. Suppose by contradiction that for every finite $\theta \subseteq f$ we have $\theta \notin \mathcal{F}$. Consider the $(n+3)$-ary partially computable function g defined by

$$g(e, x, t, \vec{y}) \simeq \begin{cases} \perp & \text{the computation of } \varphi_e(x) \text{ converges in} \leq t \text{ steps,} \\ f(\vec{y}) & \text{otherwise.} \end{cases}$$

By the *s-m-n* theorem there is a primitive recursive function $k(e, x)$ such that

$$\varphi_{k(e,x)}^{(n)}(\vec{y}) \simeq g(e, x, y_1, \vec{y}).$$

It is easy to see that $\varphi_{k(e,x)}^{(n)} \subseteq f$ and

$$W_e(x) \to \varphi_{k(e,x)}^{(n)} \text{ is finite}$$
$$\neg W_e(x) \to \varphi_{k(e,x)}^{(n)} = f.$$

This yields

$$\neg W_e(x) \leftrightarrow \varphi_{k(e,x)}^{(n)} \in \mathcal{F}^{(n)}$$

and thus the predicate $\neg W_e(x)$ is semicomputable. Contradiction.

In the proof of the reverse implication, assume by contradiction that there is a finite $\theta \in \mathcal{F}$ such that $\theta \subseteq f$ for an n-ary partially computable function $f \notin \mathcal{F}$. Consider the $(n+2)$-ary partially computable function g defined by

$$g(e, x, \vec{y}) \simeq \begin{cases} f(\vec{y}) & \text{if } \theta(\vec{y})\downarrow \text{ or } \varphi_e(x)\downarrow, \\ \perp & \text{otherwise.} \end{cases}$$

By the *s-m-n* theorem there is a primitive recursive function $k(e, x)$ such that

$$\varphi_{k(e,x)}^{(n)}(\vec{y}) \simeq g(e, x, \vec{y}).$$

It is easy to see that we have

$$W_e(x) \to \varphi_{k(e,x)}^{(n)} = f$$
$$\neg W_e(x) \to \varphi_{k(e,x)}^{(n)} = \theta.$$

Consequently

$$\neg W_e(x) \leftrightarrow \varphi_{k(e,x)}^{(n)} \in \mathcal{F}^{(n)}$$

and thus the predicate $\neg W_e(x)$ is semicomputable. Contradiction. □

Exercise. Prove Rice's theorem from the Rice-Shapiro theorem. (*Hint.* Consider the cases $\emptyset^{(n)} \in \mathcal{F}^{(n)}$ and $\emptyset^{(n)} \notin \mathcal{F}^{(n)}$ separately.).