1.1 Primitive Recursive Functions

1.1.1 Basic primitive recursive functions. The *zero* function Z is such that $\mathrm{Z}(x) = 0$; the *successor* function S satisfies the equation $\mathrm{S}(x) = x + 1$. For every $n \geq 1$ and $1 \leq i \leq n$, the n-ary *identity* function I_i^n yields its i-th argument, i.e. we have

$$\mathrm{I}_i^n(x_1, \ldots, x_n) = x_i.$$

We usually write shortly I instead of I_1^1 and we have $\mathrm{I}(x) = x$.

1.1.2 Composition. For every $m \geq 1$ and $n \geq 1$, the operator of *composition* takes an m-ary function h and m n-ary functions $g_1, \ldots, g_m$ and yields an n-ary function f satisfying:

$$f(\vec{x}) = h\big(g_1(\vec{x}), \ldots, g_m(\vec{x})\big).$$

1.1.3 Primitive recursion. For every $n \geq 1$, the operator of *primitive recursion* takes an n-ary function g and an $(n{+}2)$-ary function h and yields an $(n{+}1)$-ary function f such that

$$\begin{aligned} f(0, \vec{y}) &= g(\vec{y}) \\ f\big(\mathrm{S}(x), \vec{y}\big) &= h\big(x, f(x, \vec{y}), \vec{y}\big). \end{aligned}$$

The first argument is the *recursive argument* whereas the remaining arguments are *parameters*. Note that the definition has at least one parameter.

1.1.4 Primitive recursive functions. A sequence of functions $f_1, \ldots, f_n$ is called a *primitive recursive derivation* of a function f if

(i) the f is the last element of the sequence, i.e. $f = f_n$;
(ii) for every i such that $1 \leq i \leq n$, each function f_i is either one of the basic primitive recursive functions or is obtained from some of the previous functions $f_1, \ldots, f_{i-1}$ by composition or primitive recursion.

A function is primitive recursive if it has a primitive recursive derivation. A predicate is primitive recursive if its characteristic function is.

1.1.5 Example. Consider the ternary function $h(x, z, y) = z + 1$. We have

$$h(x, z, y) = z + 1 = \mathrm{S}(z) = \mathrm{S}\,\mathrm{I}_2^3(x, z, y).$$

Hence the following composition of the initial p.r. functions S and I_2^3:

$$h(x, z, y) = \mathrm{S}\,\mathrm{I}_2^3(x, z, y)$$

constitues a p.r. derivation of h as a p.r. function.

1.1.6 Addition is primitive recursive. Below is a p.r. derivation of addition as a p.r. function:

$$\begin{aligned} 0 + y &= \mathrm{I}(y) \\ \mathrm{S}(x) + y &= h(x, x + y, y). \end{aligned}$$

Here the auxiliary p.r. function h is from the previous example. The derivation is based on the following property of addition:

$$\begin{aligned} 0 + y &= y \\ x + 1 + y &= x + y + 1. \end{aligned}$$

Exercises

1.1.7 Exercise. Find a p.r. derivation of the multiplication function $x \times y$.
Hint. Recall that

$$\begin{aligned} 0 \times y &= 0 \\ (x + 1) \times y &= x \times y + y. \end{aligned}$$

Solution.

$$\begin{aligned} h(x, z, y) &= \mathrm{I}^3_2(x, z, y) + \mathrm{I}^3_3(x, z, y) \\ 0 \times y &= \mathrm{Z}(y) \\ \mathrm{S}(x) \times y &= h(x, x \times y, y). \end{aligned}$$

Note that the auxiliary function satisfies the identity

$$h(x, z, y) = z + y.$$

1.1.8 Exercise. Find a p.r. derivation of the exponentiation function x^y.
Hint. Recall that

$$\begin{aligned} x^0 &= 1 \\ x^{y+1} &= x \times x^y. \end{aligned}$$

Rewrite the recurrences for the binary function $f(y, x) = x^y$.
Solution.

$$
\begin{aligned}
\mathrm{C}_1(x) &= \mathrm{S}\,\mathrm{Z}(x) \\
h(y,z,x) &= \mathrm{I}^3_3(y,z,x) \times \mathrm{I}^3_2(y,z,x) \\
f(0,x) &= \mathrm{C}_1(x) \\
f(\mathrm{S}(y),x) &= h(y,f(y,x),x) \\
x^y &= f(I^2_2(x,y), I^2_1(x,y)).
\end{aligned}
$$

Note that the auxiliary functions satisfy the identities

$$
\begin{aligned}
\mathrm{C}_1(x) &= 1 \\
h(y,z,x) &= x \times z.
\end{aligned}
$$

1.1.9 Exercise. Find a p.r. derivation of the factorial function $n!$.
Hint. Recall that

$$
\begin{aligned}
0! &= 1 \\
(n+1)! &= (n+1) \times n!.
\end{aligned}
$$

Rewrite the recurrences for the binary function $f(n,m) = n!$ with the dummy second argument m.

1.1.10 Exercise. Find a 'short' p.r. derivation of the unary constant function $\mathrm{C}_{256}(x) = 256$.