1.3 Pairing Function

1.3.1 Modified Cantor pairing function. The *modified Cantor pairing function* is a p.r. function by the following explicit definition:

$$\langle x, y \rangle = \sum_{i=0}^{x+y} i + x + 1.$$

Figure 1.1 shows the initial segment of values of the pairing function in a tabular form.

Notational conventions. We will adopt the following conventions for the pairing function $\langle x, y \rangle$. We postulate that the pairing operator groups to the right, i.e. $\langle x, y, z \rangle$ abbreviates $\langle x, \langle y, z \rangle \rangle$. If $\vec{\tau}$ is an n-tuple of terms then the term $\langle \vec{\tau} \rangle$ stands for $\langle \tau_1, \ldots, \tau_n \rangle$ when $n \geq 1$ and 0 otherwise.

$\langle x, y \rangle$	0	1	2	3	4	5	6	$\cdots$
0	1	2	4	7	11	16	22	$\cdots$
1	3	5	8	12	17	23	30	$\cdots$
2	6	9	13	18	24	31	39	$\cdots$
3	10	14	19	25	32	40	49	$\cdots$
4	15	20	26	33	41	50	60	$\cdots$
5	21	27	34	42	51	61	72	$\cdots$
6	28	35	43	52	62	73	85	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Fig. 1.1. Modified Cantor pairing function

1.3.2 Basic properties of the pairing function. We have

$$\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \rightarrow x_1 = y_1 \land x_2 = y_2 \tag{1}$$

$$x < \langle x, y \rangle \land y < \langle x, y \rangle \tag{2}$$

$$x = 0 \lor \exists y \exists z \, x = \langle y, z \rangle. \tag{3}$$

Property (1) is called the *pairing property* and it says that the pairing function is an injection. Property (2) is needed for induction. From (2) we get that $0 \neq \langle x, y \rangle$ for every x and y. This means that 0 is not in the range of the pairing function and plays the role of the atom *nil* of LISP. From this and (3) we can see that the pairing function is onto the set $\mathrm{N} \setminus \{0\}$, i.e. that 0 is the only atom.

1.3.3 Ordering properties of the pairing function. We have

$$\langle x_1, x_2 \rangle \leq \langle y_1, y_2 \rangle \leftrightarrow x_1 + x_2 < y_1 + y_2 \lor x_1 + x_2 = y_1 + y_2 \land x_1 \leq y_1$$

$$\langle x_1, x_2 \rangle < \langle y_1, y_2 \rangle \leftrightarrow x_1 + x_2 < y_1 + y_2 \lor x_1 + x_2 = y_1 + y_2 \land x_1 < y_1.$$

1.3.4 Projection functions. From the basic properties of the pairing function we can see that every non-zero number x can be uniquely written in the form $x = \langle y, z\rangle$ for some y, z. The numbers y and z are called the *first* and the *second projection* of x, respectively.

The *first projection* function π_1 and *second projection* π_2 are unary functions satisfying

$$\pi_1(0) = 0 \qquad \pi_2(0) = 0$$
$$\pi_1\langle x, y\rangle = x \qquad \pi_2\langle x, y\rangle = y.$$

The projection functions are p.r. functions by bounded minimalization:

$$\pi_1(x) = \mu y < x[\exists z < x\, x = \langle y, z\rangle] \qquad \pi_2(x) = \mu z < x[\exists y < x\, x = \langle y, z\rangle].$$

1.3.5 Arithmetization of finite sequences. Every pairing function $\langle x, y\rangle$, which satisfies the properties 1.3.2(1)–(3), permits an extremely simple way of coding of finite sequences of natural numbers. We assign the code 0 to the empty sequence $\emptyset$. A non-empty sequence $x_1, \ldots, x_n$ is coded by the number

$$\langle x_1, x_2, \ldots, x_n, 0\rangle$$

as shown in Fig. 1.2. The reader will note that the assignment of codes is one to one: i.e. every finite sequence of natural numbers is coded by exactly one natural number, and vice versa, every natural number is the code of exactly one finite sequence of natural numbers.

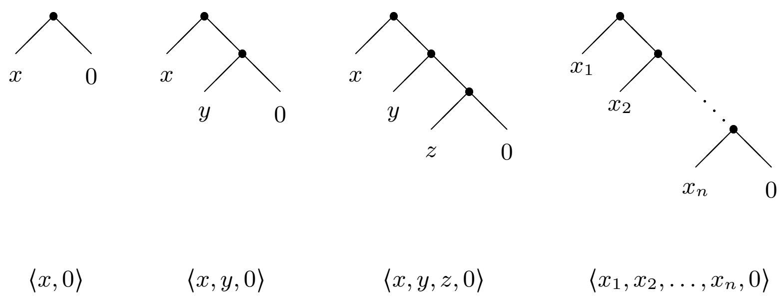

Fig. 1.2. Arithmetization of finite sequences

1.3.6 Length of sequences. The code $x = \langle x_1, x_2, \ldots, x_n, 0\rangle$ of the sequence $x_1, \ldots, x_n$ has the *length* n. The function $L(x)$ yielding the length of x is primitive recursive by the bounded minimalization:

$$L(x) = \mu n \le x[\pi_2^n(x) = 0].$$

The function satisfies

$$L(0) = 0$$
$$L\langle v, w\rangle = L(w) + 1.$$

1.3.7 Indexing function. The *indexing* function $(x)_i$ yields the $(i+1)$-st element of the sequence x, i.e.

$$(\langle x_0, \ldots, x_i, \ldots, x_{n-1}, 0\rangle)_i = x_i.$$

The function is defined explicitly by

$$(x)_i = \pi_1\, \pi_2^i(x)$$

as a primitive recursive function.

The recurrent properties of the indexing function are:

$$(\langle v, w\rangle)_0 = v$$
$$(\langle v, w\rangle)_{i+1} = (w)_i\,.$$