1.4 Course of Values Recursion

1.4.1 Course of values recursive definitions. Suppose that

$$\rho[\vec{y}], \tau[x,\vec{a},\vec{y}], \sigma_1[x,\vec{y}], \sigma_2[x,\vec{y}], \ldots, \sigma_k[x,\vec{y}]$$

are terms which do not apply f with all their free variables indicated s.t.

$$\sigma_1[x,\vec{y}] \le x \quad \sigma_2[x,\vec{y}] \le x \quad \ldots \quad \sigma_k[x,\vec{y}] \le x.$$

Consider the $(n+1)$-ary function f satisfying

$$\begin{aligned} f(0,\vec{y}) &= \rho[\vec{y}] \\ f(x+1,\vec{y}) &= \tau[x, f(\sigma_1[x,\vec{y}],\vec{y}), \ldots, f(\sigma_k[x,\vec{y}],\vec{y}), \vec{y}]. \end{aligned}$$

We say that f is defined by *course of values recursion*.

We keep the notation introduced in this paragraph fixed until the end of this section where we prove in Thm. 1.4.4 that p.r. functions are closed under course of values recursive definitions.

1.4.2 The outline of the proof. We wish to introduce as primitive recursive the course of values function $\overline{f}(x,\vec{y})$ which yields the course of values sequence for $f(x,\vec{y})$, i.e. we would like to have

$$\overline{f}(x,\vec{y}) = \big\langle f(x,\vec{y}), f(x-1,\vec{y}), \ldots, f(2,\vec{y}), f(1,\vec{y}), f(0,\vec{y}), 0\big\rangle.$$

Note that then the following holds for every $z \le x$:

$$f(z,\vec{y}) = \big(\overline{f}(x,\vec{y})\big)_{x \dot{-} z}.$$

Thus the function f can be defined explicitly by

$$f(x,\vec{y}) = \big(\overline{f}(x,\vec{y})\big)_0.$$

1.4.3 Course of values function. We define the $(n+1)$-ary course of value function $\overline{f}(x,\vec{y})$ as primitive recursive by the primitive recursive definition:

$$\begin{aligned} \overline{f}(0,\vec{y}) &= \langle \rho[\vec{y}], 0\rangle \\ \overline{f}(x+1,\vec{y}) &= \Big\langle \tau\Big[x, \big(\overline{f}(x,\vec{y})\big)_{x \dot{-} \sigma_1[x,\vec{y}]}, \ldots, \big(\overline{f}(x,\vec{y})\big)_{x \dot{-} \sigma_k[x,\vec{y}]}, \vec{y}\Big], \overline{f}(x,\vec{y})\Big\rangle. \end{aligned}$$

The following holds for every $i = 1, \ldots, k$:

$$\big(\overline{f}(\sigma_i[x,\vec{y}],\vec{y})\big)_0 = \big(\overline{f}(x,\vec{y})\big)_{x \dot{-} \sigma_i[x,\vec{y}]}. \tag{1}$$

Proof. First note that we have

$$\big(\overline{f}(x_1+x_2,\vec{y})\big)_{x_2} = \big(\overline{f}(x_1,\vec{y})\big)_0. \tag{$\dagger_1$}$$

This is proved by induction on x_2. The base case is obvious and the induction step follows from

$$\big(\overline{f}(x_1+x_2+1,\vec{y})\big)_{x_2+1} = \big(\langle\tau[\dots],\overline{f}(x_1+x_2,\vec{y})\rangle\big)_{x_2+1} = \\ = \big(\overline{f}(x_1+x_2,\vec{y})\big)_{x_2} \overset{\text{IH}}{=} \big(\overline{f}(x_1,\vec{y})\big)_0.$$

Now we ready to prove (1). For every $i = 1,\dots,k$ we have

$$\sigma_i[x,\vec{y}] + (x \dot{-} \sigma_i[x,\vec{y}]) = x \tag{$\dagger_2$}$$

since $\sigma_i[x,\vec{y}] \le x$. Consequently

$$\big(\overline{f}(\sigma_i[x,\vec{y}],\vec{y})\big)_0 \overset{(\dagger_1)}{=} \big(\overline{f}(\sigma_i[x,\vec{y}] + (x \dot{-} \sigma_i[x,\vec{y}]),\vec{y})\big)_{x \dot{-} \sigma_i[x,\vec{y}]} \overset{(\dagger_2)}{=} \\ = \big(\overline{f}(x,\vec{y})\big)_{x \dot{-} \sigma_i[x,\vec{y}]}.$$

□

1.4.4 Theorem *Primitive recursive functions are closed under course of values recursion.*

Proof. Let f be defined by the course of values recursion as in Par. 1.4.1 from p.r. functions. Let further $\overline{f}$ be its course of values function as in Par. 1.4.3. We claim that we have

$$f(x,\vec{y}) = \big(\overline{f}(x,\vec{y})\big)_0.$$

The function $\overline{f}$ is primitive recursive and so is f.

The property is proved by complete induction on x.There are two cases to consider. If $x = 0$ then we have

$$f(0,\vec{y}) = \rho[\vec{y}] = \big(\langle\rho[\vec{y}],0\rangle\big)_0 = \big(\overline{f}(0,\vec{y})\big)_0.$$

If $x = z+1$ for some z then $\sigma_i[z,\vec{y}] \le z < z+1$ and thus

$$f(z+1,\vec{y}) = \tau[z,\dots f(\sigma_i[z,\vec{y}],\vec{y})\dots,\vec{y}] \overset{\text{IHs}}{=} \\ = \tau\big[z,\dots\big(\overline{f}(\sigma_i[z,\vec{y}],\vec{y})\big)_0\dots,\vec{y}\big] \overset{1.4.3(1)}{=} \\ = \tau\big[z,\dots\big(\overline{f}(z,\vec{y})\big)_{z \dot{-} \sigma_i[z,\vec{y}]}\dots,\vec{y}\big] = \big(\overline{f}(z+1,\vec{y})\big)_0.$$

□