1.2 Explicit Definitions

1.2.1 Constant functions are primitive recursive. We first show, by induction on m, that every unary constant function $\mathrm{C}_m(x) = m$ is primitive recursive. In the base case we have $\mathrm{C}_0 = \mathrm{Z}$ is one of the basic p.r. functions. In the induction step we assume that C_m is primitive recursive by IH and define C_{m+1} as primitive recursive by unary composition:

$$\mathrm{C}_{m+1}(x) = \mathrm{S}\,\mathrm{C}_m(x).$$

The n-ary constant function $\mathrm{C}^n_m(\vec{x}) = m$ is obtained as primitive recursive by the following composition:

$$\mathrm{C}^n_m(x_1,\dots,x_n) = \mathrm{C}_m\,\mathrm{I}^n_1(x_1,\dots,x_n).$$

1.2.2 Explicit definitions of functions. Every explicit definition

$$f(x_1,\dots,x_n) = \tau[x_1,\dots,x_n]$$

can be viewed as a function operator which takes all functions applied in the term τ and returns as a result the function f satisfying the identity. We suppose here that the term τ does not apply the symbol f and that all its free variables are among the indicated ones.

1.2.3 Theorem *Primitive recursive functions are closed under explicit definitions.*

Proof. By induction on the structure of terms τ we prove that primitive recursive functions are closed under explicit definitions of n-ary functions:

$$f(\vec{x}) = \tau[\vec{x}].$$

If $\tau \equiv x_i$ then the function f is the n-ary identity function I^n_i which is one of the basic primitive recursive functions.

If $\tau \equiv m$ then the function f is the n-ary constant function C^n_m which is primitive recursive by Par. 1.2.1.

If $\tau \equiv h(\rho_1,\dots,\rho_m)$, where h is an m-ary primitive recursive function, then the n-ary functions $g_1,\dots,g_m$ defined explicitly by

$$g_1(\vec{x}) = \rho_1[\vec{x}] \qquad \dots \qquad g_m(\vec{x}) = \rho_m[\vec{x}]$$

are primitive recursive by IH. The function f is obtained as primitive recursive by the following composition

$$f(\vec{x}) = h\big(g_1(\vec{x}),\dots,g_m(\vec{x})\big). \qquad \square$$