1.6 Recursion with Parameter Substitution

1.6.1 Substitution in parameters. Suppose that

$$\rho[\vec{y}], \tau[x, \vec{a}, \vec{y}], \vec{\sigma}_1[x, \vec{y}], \dots, \vec{\sigma}_k[x, \vec{y}]$$

are terms which do not apply f with all their free variables indicated. Consider the $(n+1)$-ary function f satisfying

$$\begin{aligned} f(0, \vec{y}) &= \rho[\vec{y}] \\ f(x+1, \vec{y}) &= \tau[x, f(x, \vec{\sigma}_1[x, \vec{y}]), \dots, f(x, \vec{\sigma}_k[x, \vec{y}]), \vec{y}]. \end{aligned}$$

We say that f is defined by *recursion with parameter substitution*.

At the end of this section we will show that primitive recursive functions are closed under recursion with parameter substitution (see Thm. 1.6.12). In fact, the claim will be proved for two instances of recursions with parameter substitution: for the case when $k = 1$ or $k = 2$. Our method method of proof will be perfectly general, however.

Recursion with Parameter Substitution: Case $k = 1$

1.6.2 Fixing notation. At the end of this subsection (see Thm. 1.6.5) we will show that primitive recursive functions are closed under recursion with parameter substitution for the case when $k = 1$. To simplify the discussion we shall consider definitions with one parameter substitution ($n = 1$):

$$\begin{aligned} f(0, y) &= g(y) \\ f(x+1, y) &= h\big(x, f(x, \sigma[x, y]), y\big). \end{aligned}$$

1.6.3 Auxiliary functions. The binary function $\mathbf{x}_i(x)$ satisfies

$$\mathbf{x}_0(x) = x \tag{1}$$
$$\mathbf{x}_{i+1}(x) = \mathbf{x}_i(x) \dot{-} 1 \tag{2}$$

and it is defined explicitly

$$\mathbf{x}_i(x) = x \dot{-} i$$

as a primitive recursive function.

The ternary function $\mathbf{y}_i(x, y)$ is defined by primitive recursion on i

$$\mathbf{y}_0(x, y) = y \tag{3}$$
$$\mathbf{y}_{i+1}(x, y) = \sigma[\mathbf{x}_{i+1}(x), \mathbf{y}_i(x, y)] \tag{4}$$

as a primitive recursive function.

1.6.4 Course of values function. The ternary function $\overline{f}_i(x, y)$ satisfies

$$i = x \to \overline{f}_i(x,y) = \big\langle g\big(\mathbf{y}_i(x,y)\big), 0\big\rangle \tag{1}$$
$$i < x \to \overline{f}_i(x,y) = \tau_<[\mathbf{x}_i(x) \dot{-} 1, \overline{f}_{i+1}(x,y), \mathbf{y}_i(x,y)], \tag{2}$$

where $\tau_<$ is the term

$$\tau_<[x, a, y] \equiv \big\langle h\big(x, \boldsymbol{\pi}_1(a), y\big), a\big\rangle.$$

The function is defined by backward recursion as a p.r. function by

$$\overline{f}_i(x,y) = \begin{cases} 0 & \text{if } i \geq x+1, \\ \tau_\leq[i, \overline{f}_{i+1}(x,y), x, y] & \text{if } i < x+1, \end{cases}$$

where $\tau_\leq$ is the term

$$\tau_\leq[i, a, x, y] \equiv D\Big(i =_* x, \big\langle g\big(\mathbf{y}_i(x,y)\big), 0\big\rangle, \tau_<[\mathbf{x}_i(x) \dot{-} 1, a, \mathbf{y}_i(x,y)]\Big).$$

We also have

$$i \leq x \to \overline{f}_i(x,y) = \overline{f}_0\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big). \tag{3}$$

Proof. (1),(2): Directly from definition. (3): By backward induction on the difference $x \dot{-} i$ as $\forall y(3)$. So take any i, x, y such that $i \leq x$ and consider two cases. If $i = x$ then we have

$$\overline{f}_i(x,y) \overset{(1)}{=} \big\langle g\big(\mathbf{y}_i(x,y)\big), 0\big\rangle \overset{1.6.3(3)}{=} \big\langle g\big(\mathbf{y}_0(0, \mathbf{y}_i(x,y))\big), 0\big\rangle =$$
$$= \overline{f}_0\big(0, \mathbf{y}_i(x,y)\big) = \overline{f}_0\big(x \dot{-} i, \mathbf{y}_i(x,y)\big) = \overline{f}_0\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big).$$

If $i < x$ then first note that we have

$$\mathbf{x}_1(\mathbf{x}_i(x)) = \mathbf{x}_{i+1}(x) \tag{$\dagger_1$}$$
$$\mathbf{y}_1(\mathbf{x}_i(x), \mathbf{y}_i(x,y)) = \mathbf{y}_{i+1}(x,y). \tag{$\dagger_2$}$$

Indeed, we have

$$\mathbf{x}_1(\mathbf{x}_i(x)) \overset{1.6.3(2)}{=} \mathbf{x}_0(\mathbf{x}_i(x)) \dot{-} 1 \overset{1.6.3(1)}{=} \mathbf{x}_i(x) \dot{-} 1 \overset{1.6.3(2)}{=} \mathbf{x}_{i+1}(x)$$
$$\mathbf{y}_1(\mathbf{x}_i(x), \mathbf{y}_i(x,y)) \overset{1.6.3(4)}{=} \sigma[\mathbf{x}_1(\mathbf{x}_i(x)), \mathbf{y}_0(\mathbf{x}_i(x), \mathbf{y}_i(x,y))] \overset{(\dagger_1),1.6.3(3)}{=}$$
$$= \sigma[\mathbf{x}_{i+1}(x), \mathbf{y}_i(x,y)] \overset{1.6.3(4)}{=} \mathbf{y}_{i+1}(x,y).$$

We have $i + 1 \leq x$ and thus

$$\mathbf{x}_i(x) \dot{-} 1 \overset{1.6.3(2)}{=} \mathbf{x}_1(\mathbf{x}_i(x)) \overset{(\dagger_1)}{=} \mathbf{x}_{i+1}(x) = x \dot{-} (i+1) < x \dot{-} i.$$

Therefore

$$\overline{f}_1\big(\mathbf{x}_i(x),\mathbf{y}_i(x,y)\big) \overset{\text{IH}}{=} \overline{f}_0\big(\mathbf{x}_1(\mathbf{x}_i(x)),\mathbf{y}_1(\mathbf{x}_i(x),\mathbf{y}_i(x,y))\big) \overset{(\dagger_1),(\dagger_2)}{=}$$
$$= \overline{f}_0\big(\mathbf{x}_{i+1}(x),\mathbf{y}_{i+1}(x,y)\big).$$

Note that the induction hypothesis is applied with $\mathbf{y}_i(x,y)$ in place of y. This means that we have

$$\overline{f}_1\big(\mathbf{x}_i(x),\mathbf{y}_i(x,y)\big) = \overline{f}_0\big(\mathbf{x}_{i+1}(x),\mathbf{y}_{i+1}(x,y)\big)\ . \qquad (\dagger_3)$$

The induction step follows from

$$\overline{f}_i(x,y) \overset{(2)}{=} \tau_<[\mathbf{x}_i(x) \dot{-} 1, \overline{f}_{i+1}(x,y), \mathbf{y}_i(x,y)] \overset{\text{IH's}}{=}$$
$$\tau_<\big[\mathbf{x}_i(x) \dot{-} 1, \overline{f}_0\big(\mathbf{x}_{i+1}(x),\mathbf{y}_{i+1}(x,y)\big), \mathbf{y}_i(x,y)\big] \overset{(\dagger_3)}{=}$$
$$\tau_<\big[\mathbf{x}_i(x) \dot{-} 1, \overline{f}_1\big(\mathbf{x}_i(x),\mathbf{y}_i(x,y)\big), \mathbf{y}_i(x,y)\big] \overset{1.6.3(1)(3)}{=}$$
$$\tau_<\big[\mathbf{x}_0(\mathbf{x}_i(x)) \dot{-} 1, \overline{f}_1\big(\mathbf{x}_i(x),\mathbf{y}_i(x,y)\big), \mathbf{y}_0(\mathbf{x}_i(x),\mathbf{y}_i(x,y))\big] \overset{1.6.4(2)}{=}$$
$$\overline{f}_0\big(\mathbf{x}_i(x),\mathbf{y}_i(x,y)\big).$$
□

1.6.5 Theorem *Primitive recursive functions are closed under recursion with parameter substitution for the case* $k = 1$.

Proof. Let f be defined by the recursion with parameter substitution as in Par. 1.6.2 from p.r. functions. Let further $\overline{f}$ be its course of values function as in Par. 1.6.4. We claim that we have

$$f(x,y) = \pi_1 \overline{f}_0(x,y). \qquad (\dagger_1)$$

The function $\overline{f}$ is primitive recursive and so is f.

This is proved by induction on x as $\forall y(\dagger_1)$. In the base case we have

$$f(0,y) = g(y) = \pi_1\langle g(y),0\rangle \overset{1.6.3(3)}{=} \pi_1\big\langle g\big(\mathbf{y}_0(x,y)\big),0\big\rangle \overset{1.6.4(1)}{=} \pi_1\overline{f}_0(0,y).$$

In the induction step first note that we have

$$\mathbf{x}_1(x+1) = x \qquad (\dagger_2)$$
$$\mathbf{y}_1(x+1,y) = \sigma[x,y]. \qquad (\dagger_3)$$

Indeed, we have

$$\mathbf{x}_1(x+1) \overset{1.6.3(2)}{=} \mathbf{x}_0(x+1) \dot{-} 1 \overset{1.6.3(1)}{=} x+1 \dot{-} 1 = x$$
$$\mathbf{y}_1(x+1,y) \overset{1.6.3(4)}{=} \sigma[\mathbf{x}_1(x+1),\mathbf{y}_0(x+1,y)] \overset{(\dagger_2),1.6.3(3)}{=} \sigma[x,y].$$

Now we may continue

$$f(x+1,y) = h\big(x, f(x,\sigma[x,y]), y\big) \stackrel{\text{IH's}}{=}$$
$$h\big(x, \pi_1 \overline{f}_0(x,\sigma[x,y]), y\big) \stackrel{(\dagger_2),(\dagger_3)}{=}$$
$$h\big(x, \pi_1 \overline{f}_0\big(\mathbf{x}_1(x+1), \mathbf{y}_1(x+1,y)\big), y\big) \stackrel{1.6.4(3)}{=}$$
$$h\big(x, \pi_1 \overline{f}_1\big(x+1,y\big), y\big) \stackrel{1.6.3(1)(3)}{=}$$
$$h\big(\mathbf{x}_0(x+1) \dot{-} 1, \pi_1 \overline{f}_1(x+1,y), \mathbf{y}_0(x+1,y)\big) \stackrel{1.6.4(2)}{=}$$
$$\pi_1 \overline{f}_0(x+1,y). \qquad \square$$

Recursion with Parameter Substitution: Case $k = 2$

1.6.6 Fixing notation. At the end of this subsection (see Thm. 1.6.11) we will show that primitive recursive functions are closed under recursion with parameter substitution for the case when $k = 2$. To simplify the discussion we shall consider definitions with one parameter substitution ($n = 1$):

$$\begin{aligned} f(0,y) &= g(y) \\ f(x+1,y) &= h\big(x, f(x,\sigma_1[x,y]), f(x,\sigma_2[x,y]), y\big). \end{aligned}$$

1.6.7 Dyadic representation of natural numbers. The *dyadic successors* are unary p.r. functions $x\mathbf{1}$ and $x\mathbf{2}$ explicitly defined by

$$\begin{aligned} x\mathbf{1} &= 2x + 1 \\ x\mathbf{2} &= 2x + 2. \end{aligned}$$

It is not difficult to see that every natural number has a unique representation as a *dyadic numeral* which are terms built up from the constant 0 by applications of dyadic successors. Example:

$$\begin{aligned} 0 &= 0 & 0\mathbf{12} &= 2(2\times 0+1)+2 = 4 \\ 0\mathbf{1} &= 2\times 0+1 = 1 & 0\mathbf{21} &= 2(2\times 0+2)+1 = 5 \\ 0\mathbf{2} &= 2\times 0+2 = 2 & 0\mathbf{22} &= 2(2\times 0+2)+2 = 6 \\ 0\mathbf{11} &= 2(2\times 0+1)+1 = 3 & 0\mathbf{111} &= 2(2(2\times 0+1)+1)+1 = 7. \end{aligned}$$

1.6.8 Dyadic size. The unary *dyadic size* function $|x|_{\mathrm{d}}$ yields the number of dyadic successors in the dyadic numeral denoting the number x. The function satisfies the identities

$$|0|_{\mathrm{d}} = 0 \tag{1}$$
$$|x\mathbf{1}|_{\mathrm{d}} = |x|_{\mathrm{d}} + 1 \tag{2}$$
$$|x\mathbf{2}|_{\mathrm{d}} = |x|_{\mathrm{d}} + 1 \tag{3}$$

and it is defined by bounded minimalization

$$|x|_{\mathrm{d}} = \mu n \le x[x + 1 < 2^{n+1}]$$

as a primitive recursive function. We also have

$$|x|_{\mathrm{d}} \le n \leftrightarrow x + 1 < 2^{n+1}. \tag{4}$$

Proof. We have $n < 2^n$ and therefore $\exists x \le n\, x + 1 < 2^{n+1}$ since it suffices to take $x := n$. From this and the definition we obtain

$$x + 1 < 2^{|x|_{\mathrm{d}}+1} \tag{$\dagger_1$}$$

$$x + 1 < 2^{n+1} \to |x|_{\mathrm{d}} \le n. \tag{$\dagger_2$}$$

(1): We have $0 + 1 < 2 = 2^{0+1}$ and thus $|0|_{\mathrm{d}} \le 0$ by ($\dagger_2$); hence $|0|_{\mathrm{d}} = 0$.

(2): From ($\dagger_1$) we obtain

$$x\mathbf{1} + 1 = x + 1 + x + 1 < 2^{|x|_{\mathrm{d}}+1} + 2^{|x|_{\mathrm{d}}+1} = 22^{|x|_{\mathrm{d}}+1} = 2^{|x|_{\mathrm{d}}+1+1}$$

and thus $|x\mathbf{1}|_{\mathrm{d}} \le |x|_{\mathrm{d}} + 1$ by ($\dagger_2$). The reverse inequality is proved as follows. From ($\dagger_1$) again we obtain

$$2(x + 1) = x\mathbf{1} + 1 < 2^{|x\mathbf{1}|_{\mathrm{d}}+1} = 22^{|x\mathbf{1}|_{\mathrm{d}}}.$$

Hence $x + 1 < 2^{|x\mathbf{1}|_{\mathrm{d}}}$. It must be $|x\mathbf{1}|_{\mathrm{d}} \ne 0$ and therefore $x + 1 < 2^{|x\mathbf{1}|_{\mathrm{d}} \dot{-} 1+1}$. Now ($\dagger_2$) applies and we get $|x|_{\mathrm{d}} \le |x\mathbf{1}|_{\mathrm{d}} \dot{-} 1$, or equivalently $|x|_{\mathrm{d}} + 1 \le |x\mathbf{1}|_{\mathrm{d}}$.

(3): This is proved similarly.

(4): If $|x|_{\mathrm{d}} \le n$ then $x + 1 < 2^{|x|_{\mathrm{d}}+1} \le 2^{n+1}$ by ($\dagger_1$). The reverse direction is, in fact, the property ($\dagger_2$). □

1.6.9 Auxiliary functions. The binary function $\mathbf{x}_i(x)$ satisfies

$$\mathbf{x}_0(x) = x \tag{1}$$

$$\mathbf{x}_{i\mathbf{1}}(x) = \mathbf{x}_i(x) \dot{-} 1 \tag{2}$$

$$\mathbf{x}_{i\mathbf{2}}(x) = \mathbf{x}_i(x) \dot{-} 1 \tag{3}$$

and it is defined explicitly

$$\mathbf{x}_i(x) = x \dot{-} |i|_{\mathrm{d}}$$

as a primitive recursive function.

The ternary function $\mathbf{y}_i(x, y)$ satisfies

$$\mathbf{y}_0(x, y) = y \tag{4}$$

$$\mathbf{y}_{i\mathbf{1}}(x, y) = \sigma_1[\mathbf{x}_{i\mathbf{1}}(x), \mathbf{y}_i(x, y)] \tag{5}$$

$$\mathbf{y}_{i\mathbf{2}}(x, y) = \sigma_2[\mathbf{x}_{i\mathbf{2}}(x), \mathbf{y}_i(x, y)] \tag{6}$$

and it is defined by course of values recursion on i

$$\mathbf{y}_0(x, y) = y$$

$$\mathbf{y}_{i+1}(x, y) = D\big((i + 1) \bmod 2, \sigma_1[\mathbf{x}_{i+1}(x), \mathbf{y}_{i\div 2}(x, y)], \sigma_2[\mathbf{x}_{i+1}(x), \mathbf{y}_{i\div 2}(x, y)]\big)$$

as a primitive recursive function.

Proof. (1): By 1.6.8(1) we have $\mathbf{x}_0(x) = x \dot{-} |0|_\mathrm{d} = x \dot{-} 0 = x$. (2): It follows from

$$\mathbf{x}_{i\mathbf{1}}(x) = x \dot{-} |i\mathbf{1}|_\mathrm{d} \stackrel{1.6.8(2)}{=} x \dot{-} (|i|_\mathrm{d} + 1) = x \dot{-} |i|_\mathrm{d} \dot{-} 1 = \mathbf{x}_i(x) \dot{-} 1.$$

(3): This is proved similarly.

(4): From definition. (5): It follows from

$$\mathbf{y}_{i\mathbf{1}}(x,y) = \mathbf{y}_{2i+1}(x,y) = \sigma_1[\mathbf{x}_{2i+1}(x), \mathbf{y}_{2i\div 2}(x,y)] = \sigma_1[\mathbf{x}_{i\mathbf{1}}(x), \mathbf{y}_i(x,y)].$$

(6): This is proved similarly. □

1.6.10 Course of values function. The ternary function $\overline{f}_i(x,y)$ satisfies

$$|i|_\mathrm{d} = x \to \overline{f}_i(x,y) = \big\langle g\big(\mathbf{y}_i(x,y)\big), 0, 0\big\rangle \qquad (1)$$

$$|i|_\mathrm{d} < x \to \overline{f}_i(x,y) = \tau_<[\mathbf{x}_i(x) \dot{-} 1, \overline{f}_{i\mathbf{1}}(x,y), \overline{f}_{i\mathbf{2}}(x,y), \mathbf{y}_i(x,y)], \qquad (2)$$

where $\tau_<$ is the term

$$\tau_<[x, a_1, a_2, y] \equiv \big\langle h\big(x, \boldsymbol{\pi}_1(a_1), \boldsymbol{\pi}_1(a_2), y\big), a_1, a_2\big\rangle.$$

The function is defined by backward recursion as a p.r. function by

$$\overline{f}_i(x,y) \begin{cases} 0 & \text{if } i \geq 2^{x+1} \dot{-} 1, \\ \tau_\leq[i, \overline{f}_{i\mathbf{1}}(x,y), \overline{f}_{i\mathbf{2}}(x,y), x, y] & \text{if } i < 2^{x+1} \dot{-} 1, \end{cases}$$

where $\tau_\leq$ is the term

$$\tau_\leq[i, a_1, a_2, x, y] \equiv D\Big(|i|_\mathrm{d} =_* x, \big\langle g\big(\mathbf{y}_i(x,y)\big), 0, 0\big\rangle, \tau_<[\mathbf{x}_i(x) \dot{-} 1, a_1, a_2, \mathbf{y}_i(x,y)]\Big).$$

We also have

$$|i|_\mathrm{d} \leq x \to \overline{f}_i(x,y) = \overline{f}_0\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big). \qquad (3)$$

Proof. (1),(2): It follows from the definition by noting that we have

$$i < 2^{x+1} \dot{-} 1 \Leftrightarrow i + 1 < 2^{x+1} \stackrel{1.6.8(4)}{\Leftrightarrow} |i|_\mathrm{d} \leq x.$$

(3): By backward induction on the difference $x \dot{-} |i|_\mathrm{d}$ as $\forall y(3)$. So take any i, x, y such that $|i|_\mathrm{d} \leq x$ and consider two cases. If $|i|_\mathrm{d} = x$ then we have

$$\overline{f}_i(x,y) \stackrel{(1)}{=} \big\langle g\big(\mathbf{y}_i(x,y)\big), 0, 0\big\rangle \stackrel{1.6.9(4)}{=} \big\langle g\big(\mathbf{y}_0(0, \mathbf{y}_i(x,y))\big), 0, 0\big\rangle =$$
$$= \overline{f}_0\big(0, \mathbf{y}_i(x,y)\big) = \overline{f}_0\big(x \dot{-} |i|_\mathrm{d}, \mathbf{y}_i(x,y)\big) = \overline{f}_0\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big).$$

If $|i|_\mathrm{d} < x$ then first note that we have

$$\mathbf{x}_{0\mathbf{1}}(\mathbf{x}_i(x)) = \mathbf{x}_{i\mathbf{1}}(x) \wedge \mathbf{x}_{0\mathbf{2}}(\mathbf{x}_i(x)) = \mathbf{x}_{i\mathbf{2}}(x) \qquad (\dagger_1)$$
$$\mathbf{y}_{0\mathbf{1}}(\mathbf{x}_i(x), \mathbf{y}_i(x,y)) = \mathbf{y}_{i\mathbf{1}}(x,y) \wedge \mathbf{y}_{0\mathbf{2}}(\mathbf{x}_i(x), \mathbf{y}_i(x,y)) = \mathbf{y}_{i\mathbf{2}}(x,y). \qquad (\dagger_2)$$

For instance, we have

$$\mathbf{x}_{0\mathbf{1}}(\mathbf{x}_i(x)) \overset{1.6.9(2)}{=} \mathbf{x}_0(\mathbf{x}_i(x)) \dot{-} 1 \overset{1.6.9(1)}{=} \mathbf{x}_i(x) \dot{-} 1 \overset{1.6.9(2)}{=} \mathbf{x}_{i\mathbf{1}}(x)$$
$$\mathbf{y}_{0\mathbf{1}}(\mathbf{x}_i(x), \mathbf{y}_i(x,y)) \overset{1.6.9(5)}{=} \sigma_1[\mathbf{x}_{0\mathbf{1}}(\mathbf{x}_i(x)), \mathbf{y}_0(\mathbf{x}_i(x), \mathbf{y}_i(x,y))] \overset{(\dagger_1),1.6.9(4)}{=}$$
$$= \sigma_1[\mathbf{x}_{i\mathbf{1}}(x), \mathbf{y}_i(x,y)] \overset{1.6.9(5)}{=} \mathbf{y}_{i\mathbf{1}}(x,y).$$

We have $|i\mathbf{1}|_{\mathrm{d}} = |i|_{\mathrm{d}} + 1 \le x$ by 1.6.8(2) and thus

$$\mathbf{x}_i(x) \dot{-} |0\mathbf{1}|_{\mathrm{d}} \overset{1.6.9(2)}{=} \mathbf{x}_{0\mathbf{1}}(\mathbf{x}_i(x)) \overset{(\dagger_1)}{=} \mathbf{x}_{i\mathbf{1}}(x) = x \dot{-} |i\mathbf{1}|_{\mathrm{d}} < x \dot{-} |i|_{\mathrm{d}}\,.$$

Therefore

$$\overline{f}_{0\mathbf{1}}\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big) \overset{\mathrm{IH}}{=} \overline{f}_0\big(\mathbf{x}_{0\mathbf{1}}(\mathbf{x}_i(x)), \mathbf{y}_{0\mathbf{1}}(\mathbf{x}_i(x), \mathbf{y}_i(x,y))\big) \overset{(\dagger_1),(\dagger_2)}{=}$$
$$= \overline{f}_0\big(\mathbf{x}_{i\mathbf{1}}(x), \mathbf{y}_{i\mathbf{1}}(x,y)\big).$$

Note that the induction hypothesis is applied with $\mathbf{y}_i(x,y)$ in place of y. This means that we have

$$\overline{f}_{0\mathbf{1}}\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big) = \overline{f}_0\big(\mathbf{x}_{i\mathbf{1}}(x), \mathbf{y}_{i\mathbf{1}}(x,y)\big) \qquad (\dagger_3)$$

and by a similar argument also

$$\overline{f}_{0\mathbf{2}}\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big) = \overline{f}_0\big(\mathbf{x}_{i\mathbf{2}}(x), \mathbf{y}_{i\mathbf{2}}(x,y)\big). \qquad (\dagger_4)$$

The induction step follows from

$$\overline{f}_i(x,y) \overset{(2)}{=} \tau_<[\mathbf{x}_i(x) \dot{-} 1, \overline{f}_{i\mathbf{1}}(x,y), \overline{f}_{i\mathbf{2}}(x,y), \mathbf{y}_i(x,y)] \overset{\mathrm{IH's}}{=}$$
$$\tau_<\big[\mathbf{x}_i(x) \dot{-} 1, \overline{f}_0\big(\mathbf{x}_{i\mathbf{1}}(x), \mathbf{y}_{i\mathbf{1}}(x,y)\big), \overline{f}_0\big(\mathbf{x}_{i\mathbf{2}}(x), \mathbf{y}_{i\mathbf{2}}(x,y)\big), \mathbf{y}_i(x,y)\big] \overset{(\dagger_3),(\dagger_4)}{=}$$
$$\tau_<\big[\mathbf{x}_i(x) \dot{-} 1, \overline{f}_{0\mathbf{1}}\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big), \overline{f}_{0\mathbf{2}}\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big), \mathbf{y}_i(x,y)\big] \overset{1.6.9(1)(4)}{=}$$
$$\tau_<\big[\mathbf{x}_0(\mathbf{x}_i(x)) \dot{-} 1, \overline{f}_{0\mathbf{1}}\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big), \overline{f}_{0\mathbf{2}}\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big), \mathbf{y}_0(\mathbf{x}_i(x), \mathbf{y}_i(x,y))\big] \overset{1.6.10(2)}{=}$$
$$\overline{f}_0\big(\mathbf{x}_i(x), \mathbf{y}_i(x,y)\big).$$ □

1.6.11 Theorem *Primitive recursive functions are closed under recursion with parameter substitution for the case $k = 2$.*

Proof. Let f be defined by the recursion with parameter substitution as in Par. 1.6.6 from p.r. functions. Let further $\overline{f}$ be its course of values function as in Par. 1.6.10. We claim that we have

$$f(x,y) = \pi_1 \overline{f}_0(x,y). \tag{$\dagger_1$}$$

The function $\overline{f}$ is primitive recursive and so is f.

This is proved by induction on x as $\forall y(\dagger_1)$. In the base case we have

$$f(0,y) = g(y) = \pi_1\langle g(y),0,0\rangle \overset{1.6.9(4)}{=} \pi_1\big\langle g\big(\mathbf{y}_0(x,y)\big),0,0\big\rangle \overset{1.6.10(1)}{=} \pi_1\overline{f}_0(0,y).$$

In the induction step first note that we have

$$\mathbf{x}_{0\mathbf{1}}(x+1) = x \wedge \mathbf{x}_{0\mathbf{2}}(x+1) = x \tag{$\dagger_2$}$$
$$\mathbf{y}_{0\mathbf{1}}(x+1,y) = \sigma_1[x,y] \wedge \mathbf{y}_{0\mathbf{2}}(x+1,y) = \sigma_2[x,y]. \tag{$\dagger_3$}$$

Indeed, we have

$$\mathbf{x}_{0\mathbf{1}}(x+1) \overset{1.6.9(2)}{=} \mathbf{x}_0(x+1) \dot{-} 1 \overset{1.6.9(1)}{=} x+1 \dot{-} 1 = x$$
$$\mathbf{y}_{0\mathbf{1}}(x+1,y) \overset{1.6.9(5)}{=} \sigma_1[\mathbf{x}_{0\mathbf{1}}(x+1),\mathbf{y}_0(x+1,y)] \overset{(\dagger_2),1.6.9(4)}{=} \sigma_1[x,y].$$

Other conjuncts are proved similarly. Now we may continue

$$f(x+1,y) = h\big(x, f(x,\sigma_1[x,y]), f(x,\sigma_2[x,y]), y\big) \overset{\text{IH's}}{=}$$
$$h\big(x, \pi_1\overline{f}_0(x,\sigma_1[x,y]), \pi_1\overline{f}_0(x,\sigma_2[x,y]), y\big) \overset{(\dagger_2),(\dagger_3)}{=}$$
$$h\big(x, \pi_1\overline{f}_0\big(\mathbf{x}_{0\mathbf{1}}(x+1),\mathbf{y}_{0\mathbf{1}}(x+1,y)\big), \pi_1\overline{f}_0\big(\mathbf{x}_{0\mathbf{2}}(x+1),\mathbf{y}_{0\mathbf{2}}(x+1,y)\big), y\big) \overset{1.6.10(3)}{=}$$
$$h\big(x, \pi_1\overline{f}_{0\mathbf{1}}\big(x+1,y\big), \pi_1\overline{f}_{0\mathbf{2}}\big(x+1,y\big), y\big) \overset{1.6.9(1)(4)}{=}$$
$$h\big(\mathbf{x}_0(x+1) \dot{-} 1, \pi_1\overline{f}_{0\mathbf{1}}(x+1,y), \pi_1\overline{f}_{0\mathbf{2}}(x+1,y), \mathbf{y}_0(x+1,y)\big) \overset{1.6.10(2)}{=}$$
$$\pi_1\overline{f}_0(x+1,y).$$
□

Recursion with Parameter Substitution: General Case

1.6.12 Theorem *Primitive recursive functions are closed under recursion with parameter substitution.*

Proof. By inspection of the proof of Thms. 1.6.5 and 1.6.11. □