1.8 Recursion with Measure

1.8.1 Introduction. For efficient computation computer programming requires definitions of functions with almost arbitrary recursion. Since we do not wish such extensions to be inconsistent we restrict ourselves to *regular* recursive definitions. The condition of regularity for the recursive definition of a form $f(\vec{x}) = \tau[f;\vec{x}]$ means that there must be a *measure* $\mu[\vec{x}]$ in which the recursion goes down; i.e. we have $\mu[\vec{\rho}] < \mu[\vec{x}]$ for every recursive application $f(\vec{\rho})$ in τ. Regular recursive definitions are discussed in the next section. In this section we consider a slightly restrictive form of regular recursion.

Let $\tau[f;\vec{x}]$ be a term with all free variables indicated, and let $\mu[\vec{x}]$ be a measure. We call

$$f(\vec{x}) = \tau[\dot{\lambda}\vec{y}.D(\mu[\vec{y}] <_* \mu[\vec{x}], f(\vec{y}), 0); \vec{x}] \tag{1}$$

the definition by *(course of values) recursion with measure* μ. Note that every recursive application in (1) is surrounded by a *guard* guaranteeing the decrease of recursive arguments in the measure μ. This means that every recursive application $f(\vec{\rho})$ in τ is replaced by the term

$$D(\mu[\vec{\rho}] <_* \mu[\vec{x}], f(\vec{\rho}), 0).$$

In the sequel we will use the notation $\tau\big[[f]^{\mu}_{\vec{x}};\vec{x}\big]$ (or even $\tau\big[[f];\vec{x}\big]$) as an abbreviation for the term on the right-hand side of the identity (1).

1.8.2 The principle of measure induction. For every formula $\varphi[\vec{x}]$ and term $\mu[\vec{x}]$, the formula of *induction on* $\vec{x}$ *with measure* $\mu[\vec{x}]$ *for* φ is the following one:

$$\forall\vec{x}\big(\forall\vec{y}(\mu[\vec{y}] < \mu[\vec{x}] \to \varphi[\vec{y}]) \to \varphi[\vec{x}]\big) \to \forall\vec{x}\varphi[\vec{x}]. \tag{1}$$

We assume here that the variables $\vec{y}$ are different from $\vec{x}$ and that they do not occur in φ. The formula φ and the term μ may contain additional variables as parameters.

Note that for $\vec{x} \equiv x$ and $\mu[x] \equiv x$, the schema of measure induction is just the schema of complete induction.

1.8.3 Theorem *The principle of measure induction holds for each formula.*

Proof. The principle of measure induction 1.8.2(1) is reduced to mathematical induction as follows. Under the assumption that φ is *μ-progressive*:

$$\forall\vec{x}\big(\forall\vec{y}(\mu[\vec{y}] < \mu[\vec{x}] \to \varphi[\vec{y}]) \to \varphi[\vec{x}]\big), \tag{$\dagger_1$}$$

we first prove, by induction on n, the following auxiliary property

$$\forall\vec{z}(\mu[\vec{z}] < n \to \varphi[\vec{z}]). \tag{$\dagger_2$}$$

In the base case there is nothing to prove. In the induction step take any $\vec{z}$ such that $\mu[\vec{z}] < n+1$ and consider two cases. If $\mu[\vec{z}] < n$ then we obtain $\varphi[\vec{z}]$ by IH. If $\mu[\vec{z}] = n$ then by instantiating of $(\dagger_1)$ with $\vec{x} := \vec{z}$ we obtain

$$\forall \vec{y}(\mu[\vec{y}] < n \to \varphi[\vec{y}]) \to \varphi[\vec{z}].$$

Now we apply IH to get $\varphi[\vec{z}]$.

With the auxiliary property proved we obtain that $\varphi[\vec{x}]$ holds for every $\vec{x}$ by instantiating of $\forall n(\dagger_2)$ with $n := \mu[\vec{x}] + 1$ and $\vec{z} := \vec{x}$. □

1.8.4 Recursive definitions with measure. Let $\mu[\vec{x}]$ and $\tau[f;\vec{x}]$ be terms in which no other variables than the n indicated ones are free. Consider the following definition of an n-ary function f:

$$f(\vec{x}) = \tau\big[[f]^{\mu}_{\vec{x}};\vec{x}\big], \tag{1}$$

We say that f is defined *by recursion with measure*. Any such definition can be viewed as a function operator taking all functions in the terms τ and μ and yielding the function f as a result.

We keep the notation introduced in this paragraph fixed until Thm. 1.8.7, where we prove that the class of primitive recursive functions is closed under the operator of recursion with measure.

1.8.5 Approximation function. We wish to introduce the function f as a p.r. function with the help of its *approximation* function $f^{+}(z,\vec{x})$. The additional argument z plays the role of the depth of recursion counter. It estimates the depth of recursion needed to compute the value $f(\vec{x})$. If z is sufficiently large then we have $f(\vec{x}) = f^{+}(z,\vec{x})$. As we will see below every number $z > \mu[\vec{x}]$ gives us sufficient estimation of the depth of recursion. This will allow us to defined f explicitly by $f(\vec{x}) = f^{+}(\mu[\vec{x}]+1,\vec{x})$.

The approximation function is introduced with the help of *approximation* terms $\rho^{+}[f^{+};z,\vec{x}]$ which are defined for all subterms ρ of τ to satisfy:

$$x_i^{+} \equiv x_i \qquad (\textit{variable})$$
$$g(\rho_1,\dots,\rho_k)^{+} \equiv g(\rho_1^{+},\dots,\rho_k^{+}) \qquad (\textit{auxiliary function})$$
$$f(\rho_1,\dots,\rho_n)^{+} \equiv f^{+}(z,\rho_1^{+},\dots,\rho_n^{+}). \qquad (\textit{recursive application})$$

We define f^{+} by the following nested simple recursion:

$$f^{+}(0,\vec{x}) = 0 \tag{1}$$
$$f^{+}(z+1,\vec{x}) = \tau^{+}[\dot{\lambda} z_1 \vec{y}.D(\mu[\vec{y}] <_{*} \mu[\vec{x}], f^{+}(z,\vec{y}), 0); z, \vec{x}]. \tag{2}$$

The approximation function is primitive recursive by Thm. 1.7.1.

Below we will use the notation $\tau^{+}[[f^{+}]^{\mu}_{z,\vec{x}};z,\vec{x}]$ (or even $\tau^{+}[[f^{+}];z,\vec{x}]$) as an abbreviation for the term on the right-hand side of the equation (2). We will also use the notation $(\rho_1,\dots,\rho_m)^{+}$ as an abbreviation for $(\rho_1^{+},\dots,\rho_m^{+})$.

1.8.6 Monotonicity of the approximation function. We have

$$\mu[\vec{x}] < z_1 \leq z_2 \rightarrow f^+(z_1, \vec{x}) = f^+(z_2, \vec{x}). \tag{1}$$

The property asserts that the application $f^+(z, \vec{x})$ yields the same result for all numbers $z > \mu[\vec{x}]$.

Proof. The property is proved by induction on z_2 as $\forall\vec{x}\forall z_1(1)$. In the base case there is nothing to prove. In the induction step, take any numbers $\vec{x}, z_1$ such that $\mu[\vec{x}] < z_1 \leq z_2 + 1$ and prove by inner induction of subterms $\rho[f; \vec{x}]$ of the term τ the following identity

$$\rho^+\big[[f^+]; z_1 \dot{-} 1, \vec{x}\big] = \rho^+\big[[f^+]; z_2, \vec{x}\big]. \tag{$\dagger_1$}$$

We continue by the case analysis of ρ. If $\rho \equiv f(\vec{\theta})$ then by inner IH there are numbers $\vec{y} \equiv y_1, \dots, y_n$ such that

$$\theta_i^+\big[[f^+]; z_1 \dot{-} 1, \vec{x}\big] = y_i = \theta_i^+\big[[f^+]; z_2, \vec{x}\big]$$

for every $i = 1, \dots, n$. We consider two subcases. The subcase $\mu[\vec{y}] \geq \mu[\vec{x}]$ is obvious. In the subcase $\mu[\vec{y}] < \mu[\vec{x}]$ we have $\mu[\vec{y}] < z_1 \dot{-} 1 \leq z_2$ and thus

$$\begin{aligned} &D\big(\mu[\vec{y}] <_* \mu[\vec{x}], f^+(z_1 \dot{-} 1, \vec{y}), 0\big) = f^+(z_1 \dot{-} 1, \vec{y}) \overset{\text{outer IH}}{=} f^+(z_2, \vec{y}) = \\ &\quad = D\big(\mu[\vec{y}] <_* \mu[\vec{x}], f^+(z_2, \vec{y}), 0\big). \end{aligned}$$

The remaining cases when $\rho \equiv x_i$ or $\rho \equiv g(\vec{\theta})$ are straightforward.

With the auxiliary property proved the induction step of the outer induction follows from

$$f^+(z_1, \vec{x}) = \tau^+\big[[f^+]; z_1 \dot{-} 1, \vec{x}\big] \overset{(\dagger_1)}{=} \tau^+\big[[f^+]; z_2, \vec{x}\big] = f^+(z_2 + 1, \vec{x}). \qquad \square$$

1.8.7 Theorem *Primitive recursive functions are closed under recursion with measure.*

Proof. Let f be defined by recursion with measure as in Par. 1.8.4 from p.r. functions. Let further f^+ be its approximation function as in Par. 1.8.5. We claim that we have

$$f(\vec{x}) = f^+(\mu[\vec{x}] + 1, \vec{x}). \tag{$\dagger_1$}$$

The function f^+ is primitive recursive and so is f.

The property is proved by measure induction on $\vec{x}$ with measure $\mu[\vec{x}]$. So take any $\vec{x}$ and prove by (the inner) induction on the structure of subterms $\rho[f; \vec{x}]$ of τ the property

$$\rho\big[[f]; \vec{x}\big] = \rho^+\big[[f^+]; \mu[\vec{x}], \vec{x}\big]. \tag{$\dagger_2$}$$

We continue by case analysis of ρ. The case when $\rho \equiv f(\vec{\theta})$ follows from

$$\begin{aligned}
&D\Big(\mu\big[\vec{\theta}[[f];\vec{x}]\big] <_* \mu[\vec{x}], f\big(\vec{\theta}[[f];\vec{x}]\big), 0\Big) \overset{\text{outer IH}}{=} \\
&\quad = D\Big(\mu\big[\vec{\theta}[[f];\vec{x}]\big] <_* \mu[\vec{x}], f^+\big(\mu\big[\vec{\theta}[[f];\vec{x}]\big] + 1, \vec{\theta}[[f];\vec{x}]\big), 0\Big) \overset{1.8.6(1)}{=} \\
&\quad = D\Big(\mu\big[\vec{\theta}[[f];\vec{x}]\big] <_* \mu[\vec{x}], f^+\big(\mu[\vec{x}], \vec{\theta}[[f];\vec{x}]\big), 0\Big) \overset{\text{inner IH's}}{=} \\
&\quad = D\Big(\mu\big[\vec{\theta}^+[[f^+];\mu[\vec{x}],\vec{x}]\big] <_* \mu[\vec{x}], f^+\big(\mu[\vec{x}], \vec{\theta}^+[[f^+];\mu[\vec{x}],\vec{x}]\big), 0\Big).
\end{aligned}$$

The remaining cases when $\rho \equiv x_i$ or $\rho \equiv g(\vec{\theta})$ are straightforward. With the auxiliary property proved the equality $(\dagger_1)$ is obtained from

$$f(\vec{x}) \overset{1.8.4(1)}{=} \tau\big[[f];\vec{x}\big] \overset{(\dagger_2)}{=} \tau^+\big[[f^+];\mu[\vec{x}],\vec{x}\big] \overset{1.8.5(2)}{=} f^+(\mu[\vec{x}] + 1, \vec{x}). \qquad \square$$