Modified Cantor Pairing Function

$\langle x,y\rangle$	0	1	2	3	4	5	6	$\cdots$
0	1_1	2_2	4_3	7_3	11_4	16_4	22_4	$\cdots$
1	3_2	5_3	8_4	12_4	17_5	23_5	30_5	$\cdots$
2	6_3	9_4	13_5	18_5	24_6	31_6	39_6	$\cdots$
3	10_3	14_4	19_5	25_5	32_6	40_6	49_6	$\cdots$
4	15_4	20_5	26_6	33_6	41_7	50_7	60_7	$\cdots$
5	21_4	27_5	34_6	42_6	51_7	61_7	72_7	$\cdots$
6	28_4	35_5	43_6	52_6	62_7	73_7	85_7	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Definition:

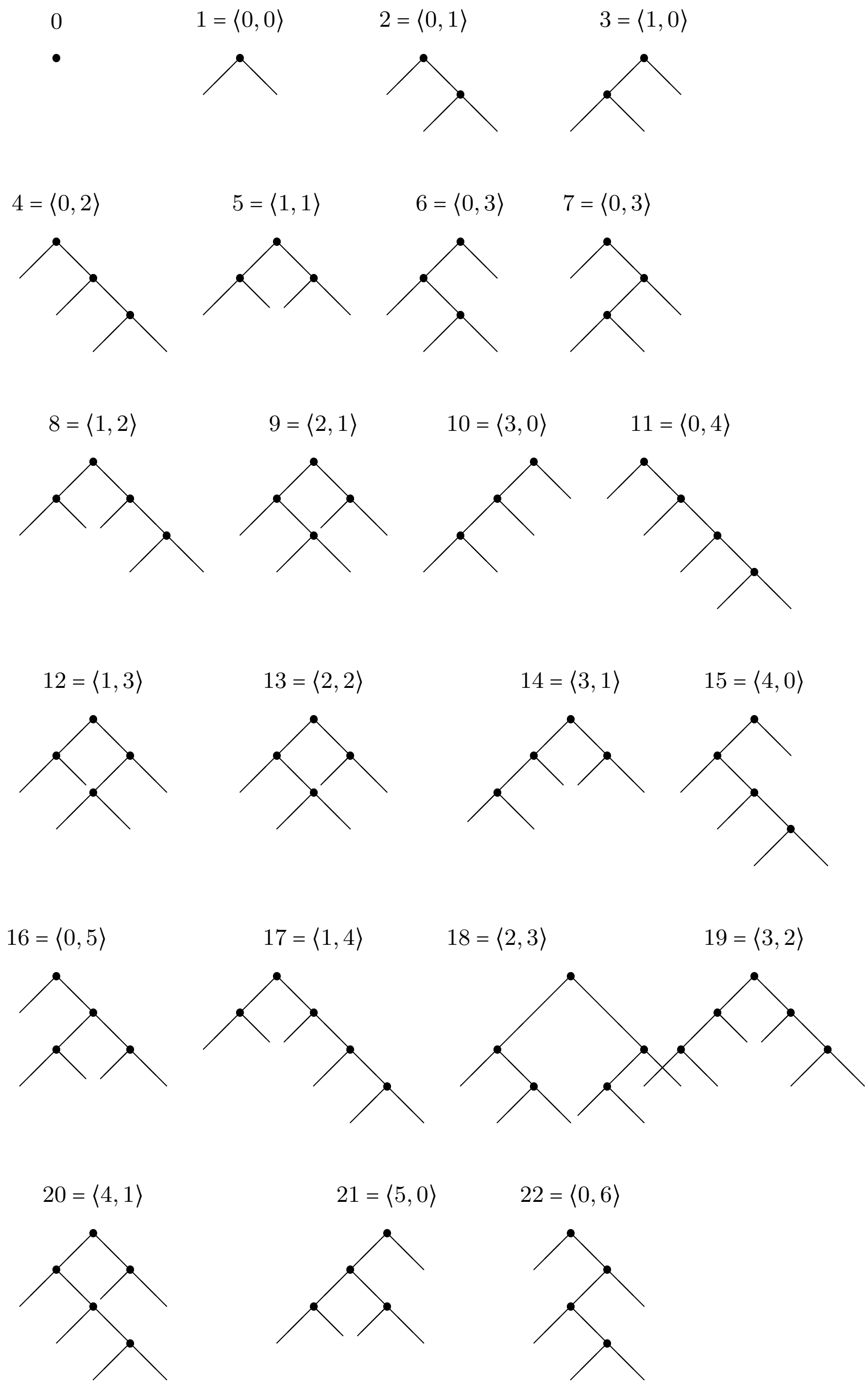
$$\langle x,y\rangle = \sum_{i=0}^{x+y} i + x + 1$$

Basic properties of the pairing function:

$$\vdash_{\mathrm{PA}} \langle x_1,x_2\rangle = \langle y_1,y_2\rangle \to x_1 = y_1 \wedge x_2 = y_2$$
$$\vdash_{\mathrm{PA}} x < \langle x,y\rangle \wedge y < \langle x,y\rangle$$
$$\vdash_{\mathrm{PA}} x = 0 \vee \exists y \exists z\, x = \langle y,z\rangle.$$

Ordering properties of the pairing function:

$$\vdash_{\mathrm{PA}} \langle x_1,x_2\rangle \le \langle y_1,y_2\rangle \leftrightarrow x_1 + x_2 < y_1 + y_2 \vee x_1 + x_2 = y_1 + y_2 \wedge x_1 \le y_1$$
$$\vdash_{\mathrm{PA}} \langle x_1,x_2\rangle < \langle y_1,y_2\rangle \leftrightarrow x_1 + x_2 < y_1 + y_2 \vee x_1 + x_2 = y_1 + y_2 \wedge x_1 < y_1.$$



Modified Cantor pairing function and enumeration of binary trees

Catalan Pairing Function

$\langle x,y\rangle$	0	1	2	3	4	5	6	$\cdots$
0	1_1	2_2	4_3	5_3	9_4	10_4	11_4	$\cdots$
1	3_2	6_3	14_4	15_4	37_5	38_5	39_5	$\cdots$
2	7_3	16_4	42_5	43_5	121_6	122_6	123_6	$\cdots$
3	8_3	17_4	44_5	45_5	126_6	127_6	128_6	$\cdots$
4	18_4	46_5	131_6	132_6	399_7	400_7	401_7	$\cdots$
5	19_4	47_5	133_6	134_6	404_7	405_7	406_7	$\cdots$
6	20_4	48_5	135_6	136_6	409_7	410_7	411_7	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Definition:

$$\begin{aligned} C(n) &= \frac{1}{n+1}\binom{2n}{n} \\ \sigma(n) &= \sum_{i<n} C(i) \\ |x|_{\mathrm{p}} &= \mu n \le x\,[x < \sigma(n+1)] \\ Ro(x) &= x \dot{-} \sigma(|x|_{\mathrm{p}}) \end{aligned}$$

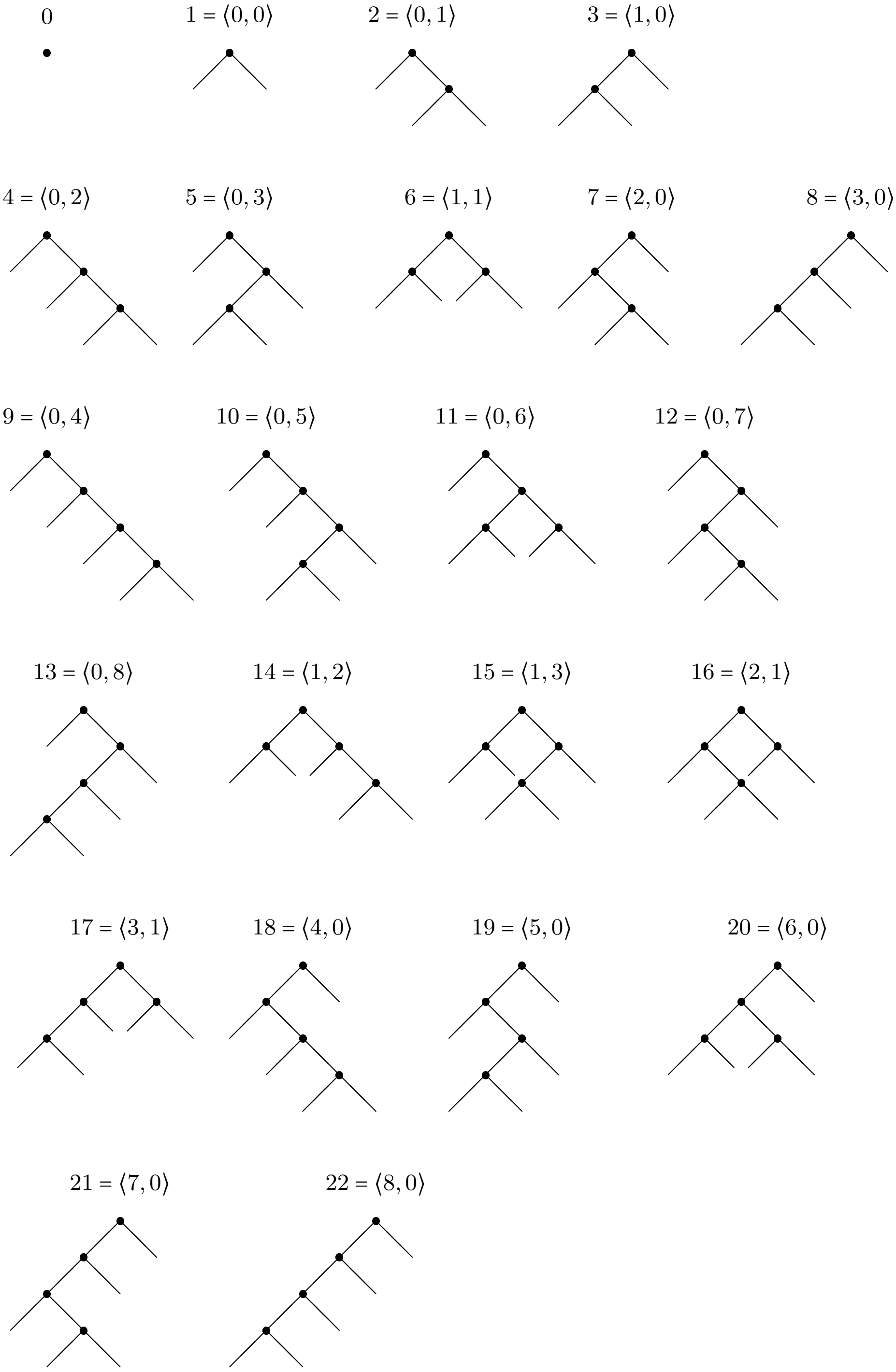
$$\langle x,y\rangle = \sigma(|x|_{\mathrm{p}} + |y|_{\mathrm{p}} + 1) + \sum_{i<|x|_{\mathrm{p}}} C(i)C(|x|_{\mathrm{p}} + |y|_{\mathrm{p}} \dot{-} i) + Ro(x)C(|y|_{\mathrm{p}}) + Ro(y)$$

Basic properties of the pairing function:

$$\begin{aligned} &\vdash_{\mathrm{PA}} \langle x_1,x_2\rangle = \langle y_1,y_2\rangle \rightarrow x_1 = y_1 \wedge x_2 = y_2 \\ &\vdash_{\mathrm{PA}} x < \langle x,y\rangle \wedge y < \langle x,y\rangle \\ &\vdash_{\mathrm{PA}} x = 0 \vee \exists y \exists z\, x = \langle y,z\rangle. \end{aligned}$$

Ordering properties of the pairing function:

$$\begin{aligned} \vdash_{\mathrm{PA}} \langle x_1,x_2\rangle \le \langle y_1,y_2\rangle \leftrightarrow{} & |\langle x_1,x_2\rangle|_{\mathrm{p}} < |\langle y_1,y_2\rangle|_{\mathrm{p}} \vee |\langle x_1,x_2\rangle|_{\mathrm{p}} = |\langle y_1,y_2\rangle|_{\mathrm{p}} \wedge \\ & \wedge (x_1 < y_1 \vee x_1 = y_1 \wedge x_2 \le y_2) \\ \vdash_{\mathrm{PA}} \langle x_1,x_2\rangle < \langle y_1,y_2\rangle \leftrightarrow{} & |\langle x_1,x_2\rangle|_{\mathrm{p}} < |\langle y_1,y_2\rangle|_{\mathrm{p}} \vee |\langle x_1,x_2\rangle|_{\mathrm{p}} = |\langle y_1,y_2\rangle|_{\mathrm{p}} \wedge \\ & \wedge (x_1 < y_1 \vee x_1 = y_1 \wedge x_2 < y_2). \end{aligned}$$



Catalan pairing function and enumeration of binary trees