Universal Function for Primitive Recursive Functions

Primitive recursive function symbols. For every $n \geq 1$, the class PR^n of n-ary *primitive recursive function symbols* is defined inductively as follows:

- $Z \in \mathrm{PR}^1$, $S \in \mathrm{PR}^1$ and $I_i^n \in \mathrm{PR}^n$ for $1 \leq i \leq n$,
- if $h \in \mathrm{PR}^m$ and $g_1, \ldots, g_m \in \mathrm{PR}^n$ then $\mathit{Comp}_m^n(h, g_1, \ldots, g_m) \in \mathrm{PR}^n$,
- if $g \in \mathrm{PR}^n$ and $h \in \mathrm{PR}^{n+2}$ then $\mathit{Rec}_{n+1}(g, h) \in \mathrm{PR}^{n+1}$.

We set $\mathrm{PR} = \bigcup_{n \geq 1} \mathrm{PR}^n$.

Arithmetization of primitive recursive function symbols. The PR-symbols are arithmetized with the help of the following pair constructors:

$$
\begin{aligned}
Zx \equiv \boldsymbol{Z} &= \langle 0, 0 \rangle && \text{(zero)} \\
Sx \equiv \boldsymbol{S} &= \langle 1, 0 \rangle && \text{(successor)} \\
Ix(n, i) \equiv \boldsymbol{I}_i^n &= \langle 2, n, i \rangle && \text{(identities)} \\
Px(g, gs) \equiv \boldsymbol{\langle} g\boldsymbol{,} gs \boldsymbol{\rangle} &= \langle 3, g, gs \rangle && \text{(contraction)} \\
Cx(n, m, h, gs) \equiv \boldsymbol{Comp}_m^n(h, gs) &= \langle 4, n, m, h, gs \rangle && \text{(composition)} \\
Rx(n, g, h) \equiv \boldsymbol{Rec}_n(g, h) &= \langle 5, n, g, h \rangle. && \text{(primitive recursion)}
\end{aligned}
$$

The arities of the constructors are as shown in their definitions. We postulate that the binary constructor $\boldsymbol{\langle} g\boldsymbol{,} gs \boldsymbol{\rangle}$ groups to the right and has the same precedence as the pairing function $\langle x, y \rangle$.

The assignment of the code $\ulcorner f \urcorner$ to the p.r. function symbol f is defined inductively on the structure of p.r. function symbols:

$$
\begin{aligned}
\ulcorner Z \urcorner &= \boldsymbol{Z} \\
\ulcorner S \urcorner &= \boldsymbol{S} \\
\ulcorner I_i^n \urcorner &= \boldsymbol{I}_i^n \\
\ulcorner \mathit{Comp}_m^n(h, g_1, \ldots, g_m) \urcorner &= \boldsymbol{Comp}_m^n\big(\ulcorner h \urcorner, \boldsymbol{\langle} \ulcorner g_1 \urcorner\boldsymbol{,} \ldots\boldsymbol{,} \ulcorner g_m \urcorner \boldsymbol{\rangle}\big) \\
\ulcorner \mathit{Rec}_n(g, h) \urcorner &= \boldsymbol{Rec}_n(\ulcorner g \urcorner, \ulcorner h \urcorner).
\end{aligned}
$$

Interpreter of primitive recursive function symbols. The interpreter $e \bullet x$ of p.r. function symbols is defined by

$\boldsymbol{Z} \bullet x = 0$
$\boldsymbol{S} \bullet x = x + 1$
$\boldsymbol{I}_i^n \bullet x = [x]_i^n$
$\boldsymbol{\langle} g\boldsymbol{,} gs \boldsymbol{\rangle} \bullet x = \langle g \bullet x, gs \bullet x \rangle$
$\boldsymbol{Comp}_m^n(h, gs) \bullet x = h \bullet (gs \bullet x)$
$\boldsymbol{Rec}_n(g, h) \bullet \langle 0, y \rangle = g \bullet y$
$\boldsymbol{Rec}_n(g, h) \bullet \big\langle x + 1, y \big\rangle = h \bullet \big\langle x, \boldsymbol{Rec}_n(g, h) \bullet \langle x, y \rangle, y \big\rangle.$