## 3.4 Partial Recursive Functions

**3.4.1 Partial recursive functions.** The class of *partial recursive functions* is generated from the successor function $x + 1$ and from the predecessor function $x \dot{-} 1$ by explicit and recursive definitions of partial functions. A *recursive function* is a partial recursive function which is total. A predicate is *recursive* if its characteristic function is.

**3.4.2 Theorem** *The class of recursive functions is primitively and generally recursively closed. Consequently, every primitive (or general) recursive function and predicate is* μ*-recursive as well.*

*Proof.* The class of recursive functions is obviously generally recursively closed and thus, by the inspection of the proof of Thm. 2.5.2, primitively recursively closed as well. □

**3.4.3 Theorem** *Recursive functions and predicates are closed under explicit definitions of predicates with bounded formulas and under definitions of functions with bounded and regular minimalization.*

*Proof.* The claim follows from the fact that the class of recursive functions is generally recursively closed and from the proofs of the corresponding theorems for general recursive functions and predicates (see Thm. 2.5.3 and Thm. 2.5.7 for details). □

**3.4.4 Unbounded minimalization.** For every $n \geq 1$, the operator of *unbounded minimalization* takes an $(n{+}1)$-ary partial function $g$ and yields an $n$-ary partial function $f$ satisfying:

$$f(\vec{x}) \simeq y \leftrightarrow g(y, \vec{x}) \simeq 1 \wedge \forall z < y \exists v \big(g(z, \vec{x}) \simeq v \wedge v \neq 1\big). \qquad (1)$$

The partial function $f$ is such that $f(\vec{x})$ is the smallest number $y$ such that $g(y, \vec{x}) \simeq 1$ and for all $z < y$ the applications $g(z, \vec{x})$ are defined (and hence different from 1). The application $f(\vec{x})$ is undefined if there is no such number. This is usually abbreviated to

$$f(\vec{x}) \simeq \mu y[g(y, \vec{x}) \simeq 1].$$

Note that if $g$ is total such that $\forall \vec{x} \exists y\, g(y, \vec{x}) = 1$ then $f$ is total as well. Note also that then the above unbounded minimalization is equivalent to the ordinary regular minimalization:

$$f(\vec{x}) = \mu y[g(y, \vec{x}) = 1].$$

**3.4.5 Theorem** *Partial recursive functions are closed under the operator of unbounded minimalization.*

*Proof.* Let $f$ be defined by the unbounded minimalization

$$f(\vec{x}) \simeq \mu y[g(y,\vec{x}) \simeq 1]$$

from a partial recursive function $g$. First, we define an auxiliary partial recursive function $h$ by the following recursive definition:

$$h(y,\vec{x}) \simeq \textbf{if } (g(y,\vec{x}) =_* 1) \neq 0 \textbf{ then } y \textbf{ else } h(y+1,\vec{x}).$$

The partial function $h$ satisfies:

$$h(y,\vec{x}) \simeq z \leftrightarrow y \leq z \land g(z,\vec{x}) \simeq 1 \land \forall z_1\big(y \leq z_1 < z \rightarrow \exists v(g(z_1,\vec{x}) \simeq v \land v \neq 1)\big).$$

Consequently $f(\vec{x}) \simeq h(0,\vec{x})$ and thus we can take the last identity as the explicit definition of $f$ as a partial recursive function. □

**3.4.6 Definitions by unbounded minimalization.** Definitions of partial functions by *unbounded minimalization* are of a form

$$f(\vec{x}) \simeq y \leftrightarrow \varphi[\vec{x},y] \land \forall z < y\, \neg\varphi[\vec{x},z], \tag{1}$$

where $\varphi[\vec{x},y]$ is a bounded formula with at most the indicated variables free and without any application of $f$. The function $f$ defined by (1) is such that $f(\vec{x})$ is the smallest number $y$ such that $\varphi[\vec{x},y]$ holds. The application $f(\vec{x})$ is undefined if there is no such number. In the sequel we abbreviate (1) to

$$f(\vec{x}) \simeq \mu y\big[\varphi[\vec{x},y]\big].$$

Every such definition can be viewed as a function operator taking all functions and the characteristic functions of all predicates occurring in the formula $\varphi$ and yielding the partial function $f$.

Note that if $\varphi$ is regular, i.e. $\forall\vec{x}\exists y\, \varphi[\vec{x},y]$, then $f$ is total. Note also that then the above unbounded minimalization is equivalent to the ordinary regular minimalization:

$$f(\vec{x}) = \mu y[\varphi[\vec{x},y]].$$

**3.4.7 Theorem** *Partial recursive functions are closed under definitions of partial functions with unbounded minimalization.*

*Proof.* Suppose that $f$ is defined by the unbounded minimalization

$$f(\vec{x}) \simeq \mu y\big[\varphi[\vec{x},y]\big]$$

from recursive functions and predicates. We can explicitly define $f$ by

$$P(y, \vec{x}) \leftrightarrow \varphi[\vec{x}, y]$$
$$f(\vec{x}) \simeq \mu y[P_*(y, \vec{x}) \simeq 1].$$

Recursivity of $f$ follows from Thm. 3.4.3 and Thm. 3.4.5. □