1.6 Backward Recursion

1.6.1 Introduction. In this section we will study a simple modification of course of values recursion which is called *backward recursion.* In this new scheme the computation goes from 0 to an arbitrary but fixed upper bound. We show that the class of primitive recursive functions is closed under backward recursion by reducing it to course of values recursion.

1.6.2 Example. Suppose that f is defined by

$$f(x,y) = \begin{cases} g(y) & \text{if } x \geq b(y), \\ h\big(x, f(x+1,y), y\big) & \text{if } x < b(y). \end{cases}$$

We have $x < x+1 \leq b(y)$ for $x < b(y)$ and therefore the definition is legal because the value $b(y) \dot{-} x$ decreases for the arguments of the (only) recursive application $f(x+1,y)$:

$$x < b(y) \to b(y) \dot{-} (x+1) < b(y) < x.$$

We say that f is defined by backward recursion on the difference $b(y) \dot{-} x$.

We want to show that if g, h and b are all p.r. functions then so is f. For that we shall define a new binary function $\hat{f}$ such that

$$v + x = b(y) \to \hat{f}(v,y) = f(x,y). \tag{1}$$

Under the assumption $v + x = b(y)$, if the number x grows from 0 to the upper bound $b(y)$, the number v decreases from $b(y)$ to 0. This suggests the following p.r. derivation of $\hat{f}$. If $0 + x = b(y)$ then $x = b(y)$ and so it must be

$$\hat{f}(0,y) \overset{(1)}{=} f\big(b(y),y\big) = g(y).$$

If $v+1+x = b(y)$ then $v+x+1 = b(y)$ and $x = b(y) \dot{-} (v+1)$, and so it must be

$$\begin{aligned} \hat{f}(v+1,y) \overset{(1)}{=} f(x,y) = h\big(x, f(x+1,y), y\big) &\overset{(1)}{=} \\ = h\big(x, \hat{f}(v,y), y\big) = h\big(b(y) \dot{-} (v+1), \hat{f}(v,y), y\big)&. \end{aligned}$$

It suffices to define $\hat{f}$ as a p.r. function by

$$\begin{aligned} \hat{f}(0,y) &= g(y) \\ \hat{f}(v+1,y) &= h\big(b(y) \dot{-} (v+1), \hat{f}(v,y), y\big). \end{aligned}$$

Now from (1) we get

$$x \leq b(y) \to f(x,y) = \hat{f}\big(b(y) \dot{-} x, y\big).$$

If $x \geq b(y)$ then $b(y) \dot{-} x = 0$ and thus $f(x,y) = g(y) = \hat{f}(0,y) = \hat{f}(b(y) \dot{-} x, y)$. This means that we have also

$$x \geq b(y) \to f(x,y) = \hat{f}\big(b(y) \dot{-} x, y\big).$$

By combining these last two properties together we obtain

$$f(x,y) = \hat{f}\big(b(y) \dot{-} x, y\big).$$

We can take this identity as an explicit definition of f as a p.r. function.

1.6.3 Example. Our second example of backward recursion is the following definition of a binary function f:

$$f(x,y) = \begin{cases} g(y) & \text{if } x \geq b(y), \\ h\Big(x, f\big(\xi[x,y], y\big), y\Big) & \text{if } x < b(y), \end{cases}$$

where $x < \xi[x,y]$ for every x, y. We clearly have

$$x < b(y) \to b(y) \dot{-} \big(\xi[x,y] + 1\big) < b(y) < x$$

and therefore the recursion is legal because the difference $b(y) \dot{-} x$ decreases for the arguments of the recursive application $f(\xi[x,y] + 1, y)$.

We want to show that if g, h, b and ξ are are all primitive recursive then so is f. For that we shall define a new binary function $\hat{f}$ such that

$$v + x = b(y) \to \hat{f}(v,y) = f(x,y). \tag{1}$$

If $0 + x = b(y)$ then $x = b(y)$ and so it must be

$$\hat{f}(0,y) \overset{(1)}{=} f\big(b(y), y\big) = g(y).$$

Suppose now $v + 1 + x = b(y)$ and for simplicity assume that $\xi[x,y] \leq b(y)$. We have $x = b(y) \dot{-} (v+1)$. Let further $\hat{\xi}[v,y]$ be a term defined by

$$\hat{\xi}[v,y] \equiv b(y) \dot{-} \xi\big[b(y) \dot{-} (v+1), y\big].$$

Then $\hat{\xi}[v,y] + \xi[x,y] = b(y)$ and so it must be

$$\begin{aligned} \hat{f}(v+1,y) &\overset{(1)}{=} f(x,y) = h\Big(x, f\big(\xi[x,y], y\big), y\Big) \overset{(1)}{=} \\ &= h\Big(x, \hat{f}\big(\hat{\xi}[v,y], y\big), y\Big) = h\Big(b(y) \dot{-} (v+1), \hat{f}\big(\hat{\xi}[v,y], y\big), y\Big). \end{aligned}$$

Note also that the inequality $\hat{\xi}[v,y] \leq v$ holds as we have

$$\hat{\xi}[v,y] = b(y) \dot{-} \xi\big[b(y) \dot{-} (v+1), y\big] \le b(y) \dot{-} \Big(\big(b(y) \dot{-} (v+1)\big) + 1\Big) \le$$
$$\le b(y) \dot{-} \big(b(y) + 1 \dot{-} (v+1)\big) = b(y) \dot{-} \big(b(y) \dot{-} v\big) \le v.$$

The following is a course of values recursive definition of $\hat{f}$ as a p.r. function:

$$\hat{f}(0,y) = g(y)$$
$$\hat{f}(v+1,y) = h\Big(b(y) \dot{-} (v+1), \hat{f}\big(\hat{\xi}[v,y], y\big), y\Big)$$

From (1) we get

$$x \le b(y) \to f(x,y) = \hat{f}\big(b(y) \dot{-} x, y\big).$$

If $x \ge b(y)$ then $b(y) \dot{-} x = 0$ and thus $f(x,y) = g(y) = \hat{f}(0,y) = \hat{f}(b(y) \dot{-} x, y)$. Consequently

$$x \ge b(y) \to f(x,y) = \hat{f}\big(b(y) \dot{-} x, y\big).$$

By combining these last two properties together we obtain

$$f(x,y) = \hat{f}\big(b(y) \dot{-} x, y\big).$$

We can take this identity as an explicit definition of f as a p.r. function.

1.6.4 The principle of backward induction. Properties of functions defined by backward recursion are usually verified by backward induction. The principle of backward induction is formalized as follows.

For every formula $\varphi[x, \vec{y}]$ and term $\theta[\vec{y}]$, the formula of *backward induction on the difference* $\theta[\vec{y}] \dot{-} x$ *for* φ is the following one:

$$\forall x \forall \vec{y}\Big(\forall x_1\big(\theta[\vec{y}] \dot{-} x_1 < \theta[\vec{y}] \dot{-} x \to \varphi[x_1, \vec{y}]\big) \to \varphi[x, \vec{y}]\Big) \to \forall x \forall \vec{y} \varphi[x, \vec{y}]. \quad (1)$$

We assume here that the variable x_1 is different from $x, \vec{y}$ and does not occur freely in φ. The formula φ and the term θ may contain additional variables as parameters.

1.6.5 Theorem *The principle of backward induction holds for each formula.*

Proof. The principle of backward induction 1.6.4(1) of $\mathcal{L}_T$ is reduced to mathematical induction as follows. Under the assumption

$$\forall x \forall \vec{y}\Big(\forall x_1\big(\theta[\vec{y}] \dot{-} x_1 < \theta[\vec{y}] \dot{-} x \to \varphi[x_1, \vec{y}]\big) \to \varphi[x, \vec{y}]\Big) \quad (\dagger_1)$$

we first prove, by induction on n, the following auxiliary property

$$\forall v\big(\theta[\vec{w}] \dot{-} v < n \to \varphi[v, \vec{w}]\big). \quad (\dagger_2)$$

In the base case there is nothing to prove. In the induction step take any v such that $\theta[\vec{w}] \dot{-} v < n+1$ and consider two cases. If $\theta[\vec{w}] \dot{-} v < n$ then we obtain $\varphi[v,\vec{w}]$ by IH. If $\theta[\vec{w}] \dot{-} v = n$ then by instantiating of $(\dagger_1)$ with $x,\vec{y} \coloneqq v,\vec{w}$ we obtain

$$\forall x_1\big(\theta[\vec{w}] \dot{-} x_1 < n \to \varphi[x_1,\vec{w}]\big) \to \varphi[v,\vec{w}].$$

Now we apply IH to get $\varphi[x,\vec{y}]$.

With the auxiliary property proved we obtain that $\varphi[x,\vec{y}]$ holds for every $x,\vec{y}$ by instantiating of $\forall n \forall \vec{w}(\dagger_2)$ with $n,\vec{w},v \coloneqq \theta[\vec{y}] \dot{-} x+1,\vec{y},x$. ◻

1.6.6 Definitions of functions by backward recursion. Suppose that

$$\rho[x,\vec{y}],\tau[x,\vec{z},\vec{y}],\theta[\vec{y}],\xi_1[x,\vec{y}],\dots,\xi_k[x,\vec{y}]$$

are terms which do not apply f with all their free variables indicated s.t.

$$x < \xi_1[x,\vec{y}] \quad \dots \quad x < \xi_k[x,\vec{y}]. \tag{1}$$

Consider the $(n{+}1)$-ary function f satisfying

$$f(x,\vec{y}) = \begin{cases} \rho[x,\vec{y}] & \text{if } x \geq \theta[\vec{y}], \\ \tau\left[x, f\big(\xi_1[x,\vec{y}],\vec{y}\big),\dots,f\big(\xi_k[x,\vec{y}],\vec{y}\big),\vec{y}\right] & \text{if } x < \theta[\vec{y}]. \end{cases}$$

We say that f is defined by *by backward recursion on the difference* $\theta[\vec{y}] \dot{-} x$. The definition can be viewed as a function operator which takes all functions applied in the terms $\rho,\tau,\theta,\xi_1,\dots,\xi_k$ and yields the function f as a result.

Fixing notation. We keep the notation introduced in this paragraph fixed until the end of this section where we prove in Thm. 1.6.8 that the class of primitive recursive functions is closed under the operator of backward recursion.

1.6.7 Auxiliary function. We will introduce the function f as primitive recursive with the help of an auxiliary $(n{+}1)$-ary function $\hat{f}$ such that

$$v + x = \theta[\vec{y}] \to \hat{f}(v,\vec{y}) = f(x,\vec{y}).$$

Let $\hat{\tau}[v,\vec{z},\vec{y}],\hat{\xi}_1[v,\vec{y}],\dots,\hat{\xi}_k[v,\vec{y}]$ be terms defined by

$$\begin{aligned}
&\hat{\tau}[v, z_1, \ldots, z_k, \vec{y}] \equiv \\
&\qquad \tau\Big[\theta[\vec{y}] \dot{-} (v+1), \\
&\qquad\quad D\Big(\xi_1\big[\theta[\vec{y}] \dot{-} (v+1), \vec{y}\big] <_* \theta[\vec{y}], z_1, \rho\big[\xi_1[\theta[\vec{y}] \dot{-} (v+1), \vec{y}], \vec{y}\big]\Big), \ldots, \\
&\qquad\quad D\Big(\xi_k\big[\theta[\vec{y}] \dot{-} (v+1), \vec{y}\big] <_* \theta[\vec{y}], z_k, \rho\big[\xi_k[\theta[\vec{y}] \dot{-} (v+1), \vec{y}], \vec{y}\big]\Big), \vec{y}\Big] \\
&\hat{\xi}_i[v, \vec{y}] \equiv \theta[\vec{y}] \dot{-} \xi_i\big[\theta[\vec{y}] \dot{-} (v+1), \vec{y}\big].
\end{aligned}$$

For every $i = 1, \ldots, k$ we have

$$\hat{\xi}_i[v, \vec{y}] \le v. \tag{1}$$

The function $\hat{f}$ is defined by the following course of values recursion

$$\begin{aligned}
\hat{f}(0, \vec{y}) &= \rho\big[\theta[\vec{y}], \vec{y}\big] \\
\hat{f}(v+1, \vec{y}) &= \hat{\tau}\Big[v, \hat{f}\big(\hat{\xi}_1[v, \vec{y}], \vec{y}\big), \ldots, \hat{f}\big(\hat{\xi}_k[v, \vec{y}], \vec{y}\big), \vec{y}\Big].
\end{aligned}$$

In the sequel we will need the following property of $\hat{f}$:

$$\begin{aligned}
&x < \theta[\vec{y}] \to \hat{f}\big(\theta[\vec{y}] \dot{-} x, \vec{y}\big) = \\
&\quad = \tau\Big[x, D\Big(\xi_1[x, \vec{y}] <_* \theta[\vec{y}], \hat{f}\big(\theta[\vec{y}] \dot{-} \xi_1[x, \vec{y}], \vec{y}\big), \rho\big[\xi_1[x, \vec{y}], \vec{y}\big]\Big), \ldots, \\
&\qquad D\Big(\xi_k[x, \vec{y}] <_* \theta[\vec{y}], \hat{f}\big(\theta[\vec{y}] \dot{-} \xi_k[x, \vec{y}], \vec{y}\big), \rho\big[\xi_k[x, \vec{y}], \vec{y}\big]\Big), \vec{y}\Big].
\end{aligned} \tag{2}$$

Proof. (1): It follows from

$$\begin{aligned}
\hat{\xi}_i[v, y] &= \theta[y] \dot{-} \xi_i\big[\theta[y] \dot{-} (v+1), y\big] \overset{1.6.6(1)}{\le} \theta[y] \dot{-} \Big(\big(\theta[y] \dot{-} (v+1)\big) + 1\Big) \le \\
&\le \theta[y] \dot{-} \big(\theta[y] + 1 \dot{-} (v+1)\big) = \theta[y] \dot{-} \big(\theta[y] \dot{-} v\big) \le v.
\end{aligned}$$

(2): If $x < \theta[\vec{y}]$ then $x + v + 1 = \theta[\vec{y}]$ for some v. We then have

$$\theta[\vec{y}] \dot{-} x = v + 1 \tag{$\dagger_1$}$$

$$\theta[\vec{y}] \dot{-} (v+1) = x \tag{$\dagger_2$}$$

and also

$$\Big(\xi_i\big[\theta[\vec{y}] \dot{-} (v+1), \vec{y}\big] <_* \theta[\vec{y}]\Big) = \big(\xi_i[x, \vec{y}] <_* \theta[\vec{y}]\big) \tag{$\dagger_3$}$$

$$\hat{f}\big(\hat{\xi}_i[v, \vec{y}], \vec{y}\big) = \hat{f}\Big(\theta[\vec{y}] \dot{-} \xi_i\big[\theta[\vec{y}] \dot{-} (v+1), \vec{y}\big], \vec{y}\Big) = \hat{f}\big(\theta[\vec{y}] \dot{-} \xi_i[x, \vec{y}], \vec{y}\big). \tag{$\dagger_4$}$$

We now obtain

$$\begin{aligned}
&\hat{f}\big(\theta[\vec{y}] \dot{-} x, \vec{y}\big) \overset{(\dagger_1)}{=} \hat{f}\big(\theta[v+1, \vec{y}\big) = \\
&\quad = \hat{\tau}\Big[v, \hat{f}\big(\hat{\xi}_1[v,\vec{y}],\vec{y}\big), \dots, \hat{f}\big(\hat{\xi}_k[v,\vec{y}],\vec{y}\big), \vec{y}\Big] \overset{(\dagger_2),(\dagger_3),(\dagger_4)}{=} \\
&\quad = \tau\Bigg[x, D\Big(\xi_1[x,\vec{y}] <_* \theta[\vec{y}], \hat{f}\big(\theta[\vec{y}] \dot{-} \xi_1[x,\vec{y}],\vec{y}\big), \rho\big[\xi_1[x,\vec{y}],\vec{y}\big]\Big), \dots, \\
&\qquad\qquad D\Big(\xi_k[x,\vec{y}] <_* \theta[\vec{y}], \hat{f}\big(\theta[\vec{y}] \dot{-} \xi_k[x,\vec{y}],\vec{y}\big), \rho\big[\xi_k[x,\vec{y}],\vec{y}\big]\Big), \vec{y}\Bigg].
\end{aligned}$$

This proves (2). □

1.6.8 Theorem *Primitive recursive functions are closed under backward recursion.*

Proof. Let f be defined by backward recursion as in Par. 1.6.6 from p.r. functions. Let further $\hat{f}$ be the function from Par. 1.6.7. We claim that

$$f(x,\vec{y}) = D\Big(x <_* \theta[\vec{y}], \hat{f}\big(\theta[\vec{y}] \dot{-} x, \vec{y}\big), \rho[x,\vec{y}]\Big).$$

The auxiliary function $\hat{f}$ is primitive recursive and so is f.

The property is proved by backward induction on the difference $\theta[\vec{y}] \dot{-} x$. So take any $x, \vec{y}$ and consider two cases by dichotomy. If $x \geq \theta[\vec{y}]$ then

$$\begin{aligned}
f(x,\vec{y}) = \rho[x,\vec{y}] &= D\Big(0, \hat{f}\big(\theta[\vec{y}] \dot{-} x, \vec{y}\big), \rho[x,\vec{y}]\Big) = \\
&= D\Big(x <_* \theta[\vec{y}], \hat{f}\big(\theta[\vec{y}] \dot{-} x, \vec{y}\big), \rho[x,\vec{y}]\Big).
\end{aligned}$$

If $x < \theta[\vec{y}]$ then $\theta[\vec{y}] \dot{-} \xi_i[x,\vec{y}] < \theta[\vec{y}] \dot{-} x$ by 1.6.6(1) for all $i = 1, \dots, k$. Thus

$$\begin{aligned}
f(x,\vec{y}) &= \tau\Big[x, f\big(\xi_1[x,\vec{y}],\vec{y}\big), \dots, f\big(\xi_k[x,\vec{y}],\vec{y}\big), \vec{y}\Big] \overset{k\times\mathrm{IH}}{=} \\
&= \tau\Bigg[x, D\Big(\xi_1[x,\vec{y}] <_* \theta[\vec{y}], \hat{f}\big(\theta[\vec{y}] \dot{-} \xi_1[x,\vec{y}],\vec{y}\big), \rho\big[\xi_1[x,\vec{y}],\vec{y}\big]\Big), \dots, \\
&\qquad D\Big(\xi_k[x,\vec{y}] <_* \theta[\vec{y}], \hat{f}\big(\theta[\vec{y}] \dot{-} \xi_k[x,\vec{y}],\vec{y}\big), \rho\big[\xi_k[x,\vec{y}],\vec{y}\big]\Big), \vec{y}\Bigg] \overset{1.6.7(2)}{=} \\
&= \hat{f}\big(\theta[\vec{y}] \dot{-} x, \vec{y}\big) = D\Big(1, \hat{f}\big(\theta[\vec{y}] \dot{-} x, \vec{y}\big), \rho[x,\vec{y}]\Big) = \\
&= D\Big(x <_* \theta[\vec{y}], \hat{f}\big(\theta[\vec{y}] \dot{-} x, \vec{y}\big), \rho[x,\vec{y}]\Big).
\end{aligned}$$

□