3.6 Normal Form

3.6.1 Normal form theorem (Kleene). *There is a unary primitive recursive function* U *and for each* $n \geq 1$ *there is* $(n{+}2)$*-ary primitive recursive predicate* T_n *such that for every* n*-ary partial recursive function* f *there exists a number* e *such that the following holds for all numbers* $\vec{x}$*:*

$$f(\vec{x}) \simeq \mathrm{U}\,\mu y[\mathrm{T}_n(e,\vec{x},y)].$$

3.6.2 Theorem *The class of recursive functions coincides with the class of general recursive functions.*

Proof. The assertion that every general recursive function is partial recursive follows from the fact proved in Thm. 3.4.2 that the class of recursive functions is generally recursively closed.

The converse is proved as follows. By Thm. 3.6.1, every n-ary recursive function f is obtained by one minimalization of the Kleene's T-predicate for some number e:

$$f(\vec{x}) \simeq \mathrm{U}\,\mu y[\mathrm{T}_n(e,\vec{x},y)].$$

Recall that both U and T_n are primitive recursive and therefore general recursive as well (see Thm. 2.5.2). Since f is total, we have

$$\forall\vec{x}\exists y \mathrm{T}_n(e,\vec{x},y)$$

and thus the following minimalization is regular

$$g(\vec{x}) = \mu y[\mathrm{T}_n(e,\vec{x},y)].$$

By Thm. 2.5.5, the auxiliary function g is general recursive and therefore we can take the following composition

$$f(\vec{x}) = U\,g(\vec{x})$$

as a derivation of f as a general recursive function. □