3.9 Partial μ-Recursive Functions

3.9.1 Introduction. We have defined in Sect. 3.4 the class of partial recursive functions by means of explicit and recursive definitions in the style of Herbrand-Gödel-Kleene. This kind of definitions is traditionally considered by mathematicians to be of *meta-mathematical* character in that it *mentions* terms of some language. While definitions based on languages are completely natural to computer scientists who are used deal with programming languages, mathematicians prefer inductive characterizations with minimal reliance on a language. The characterization of Kleene [1, 2] by partial μ-recursive functions is such.

3.9.2 Composition of partial functions. For every $m \geq 1$ and $n \geq 1$, the operator of *composition of partial functions* takes an m-ary partial function h and m n-ary partial functions $g_1, \ldots, g_m$ and yields an n-ary partial function f satisfying:

$$f(\vec{x}) \simeq h(g_1(\vec{x}), \ldots, g_m(\vec{x})). \tag{1}$$

Note that if $h, g_1, \ldots, g_m$ are total then so is f. Note also that then (1) is equivalent to the ordinary composition:

$$f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_m(\vec{x})).$$

3.9.3 Theorem *Partial recursive functions are closed under the operator of compostion of partial functions.*

Proof. Obvious.

3.9.4 Primitive recursion of partial functions. The operator of *primitive recursion of partial functions* takes an n-ary partial function g and an $(n+2)$-ary partial function h and yields the $(n+1)$-ary partial function f s.t.

$$f(0, \vec{y}) \simeq g(\vec{y}) \tag{1}$$
$$f(x+1, \vec{y}) \simeq h(x, f(x, \vec{y}), \vec{y}). \tag{2}$$

Clearly, there is a unique partial function f satisfying the identities (1),(2).

Note that if g, h are total then so is f. Note also that then the defining equations (1),(2). are equivalent to the ordinary primitive recursion:

$$f(0, \vec{y}) = g(\vec{y})$$
$$f(x+1, \vec{y}) = h(x, f(x, \vec{y}), \vec{y}).$$

3.9.5 Theorem *Partial recursive functions are closed under the operator of primitive recursion of partial functions.*

Proof. Let f be a partial function obtained by primitive recursion 3.9.4(1)(2) from partial recursive functions g and h. We derive f as a partial recursive function by the following recursive definition:

$$f(x, \vec{y}) \simeq \textbf{if } x \neq 0 \textbf{ then } h(x \dot{-} 1, f(x \dot{-} 1, \vec{y}), \vec{y}) \textbf{ else } g(\vec{y}). \qquad \square$$

3.9.6 Partial μ-recursive functions. The class of *partial* μ*-recursive functions* is generated from the successor function $S(x) = x + 1$, the zero function $Z(x) = 0$ and the identity functions $I_i^n(\vec{x}) = x_i$ by the operators of composition, primitive recursion, and unbounded minimalization of partial functions. A μ*-recursive function* is a partial μ-recursive function which is total. A predicate is μ*-recursive* if so is its characteristic function.

A class of partial functions is μ*-recursively closed* if it contains all basic μ-recursive functions and it is closed under composition, primitive recursion and minimalization of partial functions. Clearly, the class of partial μ-recursive functions is the smallest μ-recursively closed class of partial functions.

3.9.7 Theorem *The class of* μ*-recursive functions is primitively recursively closed and thus contains all primitive recursive functions and predicates.*

Proof. Straightforward. □

3.9.8 Theorem *Partial recursive functions are exactly partial* μ*-recursive functions.*

Proof. The class of partial recursive functions is closed under explicit definitions of partial functions and therefore (why?) it contains the zero function and the identity functions. Partial recursive functions are also closed under composition of partial functions by Thm. 3.9.3, primitive recursion of partial functions by Thm. 3.9.5, and the operator of unbounded minimalization by Thm. 3.4.5. The class of partial recursive functions is thus μ-recursively closed and hence every partial μ-recursive function is a partial recursive function.

The converse is proved as follows. By Thm. 3.6.1, every n-ary partial recursive function f is obtained by one minimalization of the Kleene's T-predicate for some number e:

$$f(\vec{x}) \simeq \mathrm{U}\,\mu y[\mathrm{T}_n(e, \vec{x}, y)].$$

Recall that both U and T_n are primive recursive. We claim that the following series of definitions is the derivation of f as a partial μ-recursive function:

$$\begin{aligned} P(y, \vec{x}) &\leftrightarrow \mathrm{T}_n(e, \vec{x}, y) \\ g(\vec{x}) &\simeq \mu y[P_*(y, \vec{x}) \simeq 1] \\ f(\vec{x}) &\simeq U\, g(\vec{x}). \end{aligned}$$

Indeed, the T_n is primitive recursive and so is the auxiliary predicate P. Its characteristic function P_* is thus μ-recursive and so is the auxiliary partial function g defined by the unbounded minimalization. Finally, the f is obtained by the composition of the primitive recursive and hence μ-recursive function U and the partial μ-recursive function g. □