1.1 Definitions

1.1.1 Basic primitive recursive functions. The *zero* function Z is such that $Z(x) = 0$; the *successor* function S satisfies the equation $S(x) = x + 1$. For every $n \geq 1$ and $1 \leq i \leq n$, the n-ary *identity* function I_i^n yields its i-th argument, i.e. we have

$$I_i^n(x_1, \ldots, x_n) = x_i.$$

We usually write I instead of I_1^1 and we have $I(x) = x$.

1.1.2 Composition. For every $m \geq 1$ and $n \geq 1$, the operator of *composition* takes an m-ary function h and m n-ary functions $g_1, \ldots, g_m$ and yields an n-ary function f satisfying:

$$f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_m(\vec{x})).$$

1.1.3 Primitive recursion. For every $n \geq 1$, the operator of *primitive recursion* takes an n-ary function g and an $(n{+}2)$-ary function h and yields an $(n{+}1)$-ary function f such that

$$\begin{aligned} f(0, \vec{y}) &= g(\vec{y}) \\ f(x + 1, \vec{y}) &= h(x, f(x, \vec{y}), \vec{y}). \end{aligned}$$

The first argument is the *recursive argument* whereas the remaining arguments are *parameters*. Note that the definition has at least one parameter.

1.1.4 Primitive recursive functions. A sequence of functions $f_1, \ldots, f_k$ is called a *primitive recursive derivation* of a function f if

(i) $f = f_k$,
(ii) for every i such that $1 \leq i \leq k$, the function f_i is either one of the basic primitive recursive functions or is obtained from some of the previous functions $f_1, \ldots, f_{i-1}$ by composition or primitive recursion.

A function is primitive recursive if it has a primitive recursive derivation. A predicate is primitive recursive if its characteristic function is.