2.5 General Recursive Functions

2.5.1 General recursive functions. The class of *general recursive functions* is generated from the successor function $x+1$ and from the predecessor function $x \dot{-} 1$ by explicit and regular recursive definitions. A predicate is *general recursive* if so is its characteristic function.

2.5.2 Theorem *The class of general recursive functions is primitively recursively closed and thus contains all primitive recursive functions and predicates.*

Proof. The class general recursive functions is closed under explicit definitions of functions and therefore (why?) it contains the identity functions I_i^n and the zero function Z, and it is closed under composition of functions. It remains to show the closure under the operator of primitive recursion. So let f be the function obtained from the general recursive functions g and h by primitive recursion:

$$\begin{aligned} f(0,\vec{y}) &= g(\vec{y}) \\ f(x+1,\vec{y}) &= h(x, f(x,\vec{y}), \vec{y}). \end{aligned}$$

We derive f as general recursive by the following recursive definition

$$f(x,\vec{y}) = \textbf{if } x \neq 0 \textbf{ then } h(x \dot{-} 1, f(x \dot{-} 1, \vec{y}), \vec{y}) \textbf{ else } g(\vec{y}),$$

The definition is regular in the first argument since $x \neq 0 \rightarrow x \dot{-} 1 < x$. □

2.5.3 Theorem *General recursive functions and predicates are closed under explicit definitions of predicates with bounded formulas and under definitions of functions with bounded minimalization.*

Proof. It follows from the fact that the class of general recursive functions is primitively recursively closed and from the proofs of the corresponding theorems for primitive recursive functions and predicates. □

2.5.4 Regular minimalization. For every $n \geq 1$, the operator of *regular minimalization* takes an $(n+1)$-ary function g satisfying the following *condition of regularity*:

$$\forall \vec{x} \exists y \, g(y, \vec{x}) = 1$$

and yields an n-ary function f such that

$$f(\vec{x}) = \text{the least } y \text{ such that } g(y,\vec{x}) = 1 \text{ holds.}$$

This is usually abbreviated to

$$f(\vec{x}) = \mu y[g(y, \vec{x}) = 1].$$

2.5.5 Theorem *General recursive functions are closed under the operator of regular minimalization.*

Proof. Let f be defined by the following regular minimalization

$$f(\vec{x}) = \mu y[g(y, \vec{x}) = 1]$$

of a general recursive function g. Consider the following recursive definition of a function h (written in clausal form):

$$h(y, \vec{x}) = y \leftarrow \exists z \leq y\, g(z, \vec{x}) = 1$$
$$h(y, \vec{x}) = h(y+1, \vec{x}) \leftarrow \forall z \leq y\, g(z, \vec{x}) \neq 1.$$

We claim that the definition is regular in the well-founded relation $\prec$:

$$(y_1, \vec{x}_1) \prec (y_2, \vec{x}_2) \leftrightarrow f(\vec{x}_1) \dot{-} y_1 < f(\vec{x}_2) \dot{-} y_2.$$

Indeed, if $g(z, \vec{x}) \neq 1$ for every $z \leq y$ then clearly $y < y+1 \leq f(\vec{x})$ and thus we have $f(\vec{x}) \dot{-} (y+1) < f(\vec{x}) \dot{-} y$. From this we get

$$\forall z \leq y\, g(z, \vec{x}) \neq 1 \rightarrow (y+1, \vec{x}) \prec (y, \vec{x}).$$

This proves the condition of regularity for the recursive application $h(y+1, \vec{x})$ of h in the second clause of the definition. Consequently, the function h is general recursive.

By a straightforward $\prec$-well-founded induction on $(y, \vec{x})$ we can prove

$$y \leq f(\vec{x}) \rightarrow h(y, \vec{x}) = f(\vec{x}).$$

Now we can take the identity $f(\vec{x}) = h(0, \vec{x})$ as an explicit definition of f as a general recursive function. □

2.5.6 Definitions by regular minimalization. Definitions of functions by *regular minimalization* are of the form

$$f(\vec{x}) = \text{the least } y \text{ such that } \varphi[\vec{x}, y] \text{ holds,}$$

where $\varphi[\vec{x}, y]$ is a bounded formula with at most the indicated variables free and without any application of the symbol f. Moreover we require that the formula φ satisfies the following *condition of regularity*:

$$\forall \vec{x} \exists y \varphi[\vec{x}, y].$$

Every such definition can be viewed as a function operator taking all functions and the characteristic functions of all predicates occurring in the formula φ and yielding the function f.

In the sequel we will abbreviate the definition to

$$f(\vec{x}) = \mu y\big[\varphi[\vec{x}, y]\big].$$

2.5.7 Theorem *General recursive functions are closed under definitions of functions with regular minimalization.*

Proof. Suppose that f is defined by the regular minimalization

$$f(\vec{x}) = \mu y\big[\varphi[\vec{x}, y]\big]$$

from general recursive functions and predicates. We can derive f by

$$\begin{aligned} R(y, \vec{x}) &\leftrightarrow \varphi[\vec{x}, y] \\ f(\vec{x}) &= \mu y[R_*(y, \vec{x}) \simeq 1]. \end{aligned}$$

General recursivness of f follows from Thm. 2.5.3 and Thm. 2.5.5. $\square$