1.5 Backward Recursion

Backward Recursion and Primitive Recursion

1.5.1 Example. Consider the following definition of the binary function f:

$$f(x,y) = \begin{cases} g(y) & \text{if } x \geq b(y), \\ h\big(x, f(x+1,y), y\big) & \text{if } x < b(y). \end{cases}$$

We want to show that if g, h and b are p.r. functions then so is f.

For that we shall define a new function f_1 by primitive recursion:

$$\begin{aligned} f_1(0,y) &= g(y) \\ f_1(z+1,y) &= h\big(b(y) \dot{-} (z+1), f_1(z,y), y\big). \end{aligned}$$

We have

$$f(x,y) = f_1\big(b(y) \dot{-} x, y\big)$$

and since f_1 is primitive recursive we can take this identity as an alternative, explicit definition of f as a p.r. function.

Proof. The property is proved by induction on the difference $b(y) \dot{-} x$. So take any x, y and consider two cases. If $x \geq b(y)$ then $b(y) \dot{-} x = 0$ and thus

$$f(x,y) = g(y) = f_1(0,y) = f_1(b(y) \dot{-} x, y).$$

If $x < b(y)$ then $b(y) \dot{-} (x+1) < b(y) \dot{-} x$ and thus

$$\begin{aligned} f(x,y) &= h\big(x, f(x+1,y), y\big) \overset{\text{IH}}{=} h\big(x, f_1(b(y) \dot{-} (x+1), y), y\big) = \\ &= h\Big(b(y) \dot{-} \big(b(y) \dot{-} (x+1) + 1\big), f_1\big(b(y) \dot{-} (x+1), y\big), y\Big) = \\ &= f_1\big(b(y) \dot{-} (x+1) + 1, y\big) = f_1\big(b(y) \dot{-} x, y\big). \end{aligned}$$

□

Backward Recursion and Course of Values Recursion

1.5.2 Example. Consider the following definition of the binary function f:

$$f(x,y) = \begin{cases} g(y) & \text{if } x \geq b(y), \\ h\big(x, f(\sigma[x,y], y), y\big) & \text{if } x < b(y), \end{cases}$$

where $x < \sigma[x,y]$. We want to show that if g, h, b and σ are all primitive recursive then so is f.

For that we shall define a new function f_1 by course of values recursion:

$$f_1(0, y) = g(y)$$
$$f_1(z + 1, y) = h\big(b(y) \dot{-} (z + 1), f_1(\xi[z, y], y), y\big),$$

where

$$\xi[z, y] \equiv b(y) \dot{-} \sigma[b(y) \dot{-} (z + 1), y].$$

The inequality $\xi[z, y] \leq z$ follows from

$$\xi[z, y] = b(y) \dot{-} \sigma[b(y) \dot{-} (z + 1), y] \leq b(y) \dot{-} \Big(\big(b(y) \dot{-} (z + 1)\big) + 1\Big) \leq$$
$$\leq b(y) \dot{-} \big(b(y) + 1 \dot{-} (z + 1)\big) = b(y) \dot{-} (b(y) \dot{-} z) \leq z$$

by noting that $\sigma[b(y) \dot{-} (z + 1), y] \geq \big(b(y) \dot{-} (z + 1)\big) + 1$. We have

$$f(x, y) = f_1\big(b(y) \dot{-} x, y\big)$$

and since f_1 is primitive recursive we can take this identity as an alternative, explicit definition of f as a p.r. function.

Proof. The property is proved by induction on the difference $b(y) \dot{-} x$. So take any x, y and consider two cases. If $x \geq b(y)$ then $b(y) \dot{-} x = 0$ and thus

$$f(x, y) = g(y) = f_1(0, y) = f_1\big(b(y) \dot{-} x, y\big).$$

If $x < b(y)$ then $b(y) \dot{-} (x + 1) < b(y) \dot{-} x$ and also

$$\xi[b(y) \dot{-} (x + 1), y] = b(y) \dot{-} \sigma\big[b(y) \dot{-} \big(b(y) \dot{-} (x + 1) + 1\big), y\big] =$$
$$= b(y) \dot{-} \sigma\big[b(y) \dot{-} \big(b(y) + 1 \dot{-} (x + 1)\big), y\big] =$$
$$= b(y) \dot{-} \sigma\big[b(y) \dot{-} \big(b(y) \dot{-} x\big), y\big] = b(y) \dot{-} \sigma[x, y].$$

Therefore

$$f(x, y) = h\big(x, f(\sigma[x, y], y), y\big) \overset{\text{IH}}{=} h\Big(x, f_1\big(b(y) \dot{-} \sigma[x, y], y\big), y\Big) =$$
$$= h\Big(x, f_1\big(\xi[b(y) \dot{-} (x + 1), y], y\big), y\Big) =$$
$$= h\Big(b(y) \dot{-} \big(b(y) \dot{-} (x + 1) + 1\big), f_1\big(\xi[b(y) \dot{-} (x + 1), y], y\big), y\Big) =$$
$$= f_1\big(b(y) \dot{-} (x + 1) + 1, y\big) = f_1(b(y) \dot{-} x, y).$$

□