6.1 Dyadic Words

6.1.1 Arithmetization of word domains with two elements. Consider the two-elements alphabet $\Sigma = \{1, 2\}$. We can code words over Σ with the help of *dyadic successors* functions explicitly defined by:

$$x\mathbf{1} = 2x + 1 \qquad x\mathbf{2} = 2x + 2.$$

It is not difficult to see that every natural number has a unique representation as a *dyadic numeral* which are terms built up from the constant 0 by applications of dyadic successors. This is called *dyadic representation* of natural numbers.

Consider, for instance, the first eight words from the sequence of words over the alphabet Σ which is ordered first on the length and then within the same length lexicographically: $\varnothing, 1, 2, 11, 12, 21, 22, 111$. The corresponding dyadic numerals are shown in Fig. 6.1. Arithmetization is so straightforward that, from now on, we will usually identify dyadic words with their code numbers.

$$\begin{aligned}
0 &= 0 \\
0\mathbf{1} &= 2 \times 0 + 1 = 1 \times 2^0 = 1 \\
0\mathbf{2} &= 2 \times 0 + 2 = 2 \times 2^0 = 2 \\
0\mathbf{11} &= 2 \times (2 \times 0 + 1) + 1 = 1 \times 2^1 + 1 \times 2^0 = 3 \\
0\mathbf{12} &= 2 \times (2 \times 0 + 1) + 2 = 1 \times 2^1 + 2 \times 2^0 = 4 \\
0\mathbf{21} &= 2 \times (2 \times 0 + 2) + 1 = 2 \times 2^1 + 1 \times 2^0 = 5 \\
0\mathbf{22} &= 2 \times (2 \times 0 + 2) + 2 = 2 \times 2^1 + 2 \times 2^0 = 6 \\
0\mathbf{111} &= 2 \times (2 \times (2 \times 0 + 1) + 1) + 1 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 7.
\end{aligned}$$

Fig. 6.1 Dyadic representation of natural numbers

6.1.2 The principle of dyadic case analysis. We have

$$\vdash_{\mathrm{PA}} x = 0 \lor \exists y\, x = y\mathbf{1} \lor \exists y\, x = y\mathbf{2}.$$

6.1.3 The principle of dyadic induction. For every formula $\varphi[x]$, the formula of *dyadic induction on x for φ* is the following one:

$$\phi[0] \land \forall x(\phi[x] \to \phi[x\mathbf{1}]) \land \forall x(\phi[x] \to \phi[x\mathbf{2}]) \to \forall \phi[x].$$

The formula φ may contain additional variables as parameters.

6.1.4 Theorem *The principle of dyadic induction holds for each formula.*

Proof. Dyadic induction is reduced to complete induction as follows. Assume

$$\phi[0] \tag{$\dagger_1$}$$

$$\forall x(\phi[x] \to \phi[x\mathbf{1}]) \tag{$\dagger_2$}$$

$$\forall x(\phi[x] \to \phi[x\mathbf{2}]) \tag{$\dagger_3$}$$

and prove by complete induction on x that $\varphi[x]$ holds for every x. We consider three cases. If $x = 0$ then $\phi[0]$ follows from the assumption ($\dagger_1$). If $x = y\mathbf{1}$ for some y then, since $y < x$, we have $\phi[y]$ from IH and we get $\phi[y\mathbf{1}]$ from ($\dagger_2$). The case when $x = y\mathbf{2}$ for some y is proved similarly. ◻

6.1.5 Dyadic length. The *dyadic length* function $L(x)$ yields the number of dyadic successors in the dyadic numeral denoting the number x. The function is the arithmetization of the word function taking a dyadic word and yielding its length. The function is defined by parameterless dyadic recursion:

$$\begin{aligned} L(0) &= 0 \\ L(x\mathbf{1}) &= L(x) + 1 \\ L(x\mathbf{2}) &= L(x) + 1. \end{aligned}$$

This is a correct definition because recursion decreases the argument since we clearly have $x < x\mathbf{1}$ and $x < x\mathbf{2}$. The function satisfies the following

$$\vdash_{\mathrm{PA}} L(x) = 0 \leftrightarrow x = 0 \tag{1}$$

$$\vdash_{\mathrm{PA}} L(x) = n \leftrightarrow 2^n \le x + 1 < 2^{n+1}. \tag{2}$$

Proof. (1): By a straighforward dyadic case analysis.

(2): By induction on n with induction formula $\forall x\,(2)$. In the base take any x and we have

$$L(x) = 0 \overset{(1)}{\Leftrightarrow} x = 0 \Leftrightarrow 1 \le x + 1 < 2 \Leftrightarrow 2^0 \le x + 1 < 2^{0+1}.$$

In the inductive case take any number x and consider three cases. If $x = 0$ then the claim follows from the following two simple facts: $L(0) = 0 \ne n + 1$ and $2^{n+1} \nleq 1 = 0 + 1$. If $x = y\mathbf{1}$ for some y then we have

$$\begin{aligned} L(y\mathbf{1}) = n + 1 \Leftrightarrow L(y) + 1 = n + 1 \Leftrightarrow L(y) = n \overset{\mathrm{IH}}{\Leftrightarrow} 2^n \le y + 1 < 2^{n+1} \Leftrightarrow \\ \Leftrightarrow 22^n \le 2(y + 1) < 22^{n+1} \Leftrightarrow 2^{n+1} \le y\mathbf{1} + 1 < 2^{n+2}. \end{aligned}$$

If $x = y\mathbf{2}$ for some y then we have

$$\begin{aligned} L(y\mathbf{2}) = n + 1 \Leftrightarrow L(y) + 1 = n + 1 \Leftrightarrow L(y) = n \overset{\mathrm{IH}}{\Leftrightarrow} 2^n \le y + 1 < 2^{n+1} \Leftrightarrow \\ \Leftrightarrow 22^n \le 2(y + 1) + 1 < 22^{n+1} \Leftrightarrow 2^{n+1} \le y\mathbf{2} + 1 < 2^{n+2}. \end{aligned}$$

◻

6.1.6 Dyadic concatenation. The binary *dyadic concatenation* function $x \star y$ is the arithmetization of the word function concatenating two dyadic words. The function is defined by dyadic recursion on y:

$$\begin{aligned} x \star 0 &= x \\ x \star y\mathbf{1} &= (x \star y)\mathbf{1} \\ x \star y\mathbf{2} &= (x \star y)\mathbf{2}. \end{aligned}$$

The function satisfies the following properties:

$$\vdash_{\mathrm{PA}} x \star y = x2^{L(y)} + y \tag{1}$$

$$\vdash_{\mathrm{PA}} x \star y = 0 \leftrightarrow x = 0 \wedge y = 0 \tag{2}$$

$$\vdash_{\mathrm{PA}} 0 \star y = y \tag{3}$$

$$\vdash_{\mathrm{PA}} (x \star y) \star z = x \star (y \star z) \tag{4}$$

$$\vdash_{\mathrm{PA}} L(x \star y) = L(x) + L(y)\,. \tag{5}$$

Proof. (1): By dyadic induction on y. In the base case when $y = 0$ we have

$$x \star 0 = x = x2^0 + 0 = x2^{L(0)} + 0.$$

The induction step for $y\mathbf{1}$ follows from

$$x \star y\mathbf{1} = (x \star y)\mathbf{1} \overset{\mathrm{IH}}{=} (x2^{L(y)} + y)\mathbf{1} = x2^{L(y)+1} + y\mathbf{1} = x2^{L(y\mathbf{1})} + y\mathbf{1}.$$

The other induction step is proved similarly.

(2): By a straightforward dyadic case analysis on y.

(3): By a straightforward dyadic induction.

(4): By dyadic induction on z. In the base case we have

$$(x \star y) \star 0 = x \star y = x \star (y \star 0).$$

In the inductive step for $z\mathbf{1}$ we have

$$(x \star y) \star z\mathbf{1} = \big((x \star y) \star z\big)\mathbf{1} \overset{\mathrm{IH}}{=} \big(x \star (y \star z)\big)\mathbf{1} = x \star (y \star z)\mathbf{1} = x \star (y \star z\mathbf{1}).$$

The other induction step is proved similarly.

(5): By a straightforward dyadic induction on y. □

6.1.7 Dyadic reversal. The function *Rev* is the arithmetization of the word function reverting the order of the elements of dyadic words. The function is defined by dyadic recursion as follows:

$$\begin{aligned} &Rev(0) = 0 \\ &Rev(x\mathbf{1}) = 0\mathbf{1} \star Rev(x) \\ &Rev(x\mathbf{2}) = 0\mathbf{2} \star Rev(x). \end{aligned}$$

The function has the following properties:

$$\vdash_{\mathrm{PA}} Rev(x) = 0 \leftrightarrow x = 0 \tag{1}$$
$$\vdash_{\mathrm{PA}} Rev(x \star y) = Rev(y) \star Rev(x) \tag{2}$$
$$\vdash_{\mathrm{PA}} Rev\, Rev(x) = x \tag{3}$$
$$\vdash_{\mathrm{PA}} \exists y\, x = Rev(y) \tag{4}$$
$$\vdash_{\mathrm{PA}} Rev(x) = Rev(y) \rightarrow x = y \tag{5}$$
$$\vdash_{\mathrm{PA}} L\, Rev(x) = L(x)\,. \tag{6}$$

Proof. (1): By dyadic case analysis with the help 6.1.6(2).

(2): By dyadic induction on y. In the base case we have

$$Rev(x \star 0) = Rev(x) \overset{6.1.6(3)}{=} 0 \star Rev(x).$$

In the inductive case for $y\mathbf{1}$ we have

$$Rev(x \star y\mathbf{1}) = Rev\big((x \star y)\mathbf{1}\big) = 0\mathbf{1} \star Rev(x \star y) \overset{\mathrm{IH}}{=}$$
$$0\mathbf{1} \star \big(Rev(y) \star Rev(x)\big) \overset{6.1.6(4)}{=} \big(0\mathbf{1} \star Rev(y)\big) \star Rev(x) = Rev(y\mathbf{1}) \star Rev(x).$$

The other induction case is proved similarly.

(3): By dyadic induction on x. The base case follows directly from definition. In the inductive case for $x\mathbf{1}$ we have

$$Rev\, Rev(x\mathbf{1}) = Rev\big(0\mathbf{1} \star Rev(x)\big) \overset{(2)}{=} Rev\, Rev(x) \star Rev(0\mathbf{1}) \overset{\mathrm{IH}}{=} x \star 0\mathbf{1} = x\mathbf{1}.$$

The other induction case is proved similarly.

(4),(5): Directly from (3). (6): By a straightforward dyadic induction with the help 6.1.6(5). □

6.1.8 Cancellation laws for dyadic concatenation. We have

$$\vdash_{\mathrm{PA}} x \star z = y \star z \rightarrow x = y \tag{1}$$
$$\vdash_{\mathrm{PA}} z \star x = z \star y \rightarrow x = y. \tag{2}$$

Proof. (1): By dyadic induction on z. The base case follows directly from definition. In the inductive case for $z\mathbf{1}$ we have

$$x \star z\mathbf{1} = y \star z\mathbf{1} \Rightarrow (x \star z)\mathbf{1} = (y \star z)\mathbf{1} \Rightarrow x \star z = y \star z \overset{\mathrm{IH}}{\Rightarrow} x = y.$$

(2): It follows from

$$z \star x = z \star y \Rightarrow Rev(z \star x) = Rev(z \star y) \overset{6.1.7(2)}{\Rightarrow}$$
$$Rev(x) \star Rev(z) = Rev(y) \star Rev(z) \overset{(1)}{\Rightarrow} Rev(x) = Rev(y) \overset{6.1.7(5)}{\Rightarrow} x = y.$$ □