5.2 Primitive Recursion

5.2.1 Exponentiation. The binary exponentiation function x^y is a p.r. function by the following primitive recursive definition:

$$x^0 = 1$$
$$x^{y+1} = xx^y.$$

We list here some properties of the exponentiation function:

$$\vdash_{\mathrm{PA}} x^y = 0 \leftrightarrow x = 0 \land y \neq 0 \quad (1)$$
$$\vdash_{\mathrm{PA}} x^y = 1 \leftrightarrow x = 1 \lor y = 0 \quad (2)$$
$$\vdash_{\mathrm{PA}} x^y > 1 \leftrightarrow x > 1 \land y \neq 0 \quad (3)$$
$$\vdash_{\mathrm{PA}} x^{y+z} = x^y x^z \quad (4)$$
$$\vdash_{\mathrm{PA}} x \neq 0 \land y \geq z \rightarrow x^{y \dot{-} z} = x^y \div x^z \quad (5)$$
$$\vdash_{\mathrm{PA}} x > 1 \rightarrow x^y = x^z \leftrightarrow y = z \quad (6)$$
$$\vdash_{\mathrm{PA}} x > 1 \rightarrow x^y \leq x^z \leftrightarrow y \leq z \quad (7)$$
$$\vdash_{\mathrm{PA}} x > 1 \rightarrow x^y < x^z \leftrightarrow y < z. \quad (8)$$

Proof. (1): By induction on y. The base case follows directly from definition because $x^0 = 1 \neq 0$. In the inductive case we have

$$x^{y+1} = 0 \Leftrightarrow xx^y = 0 \Leftrightarrow x = 0 \lor x^y = 0 \overset{\mathrm{IH}}{\Leftrightarrow} x = 0 \lor (x = 0 \land y \neq 0) \overset{(*_1)}{\Leftrightarrow}$$
$$\Leftrightarrow x = 0 \Leftrightarrow x = 0 \land y + 1 \neq 0.$$

The step marked by $(*_1)$ is by case analysis on whether or not $x = 0$.

(2): By induction on y. The base case follows directly from definition. In the inductive case we have

$$x^{y+1} = 1 \Leftrightarrow xx^y = 1 \Leftrightarrow x = 1 \land x^y = 1 \overset{\mathrm{IH}}{\Leftrightarrow} x = 1 \land (x = 1 \lor y = 0) \overset{(*_2)}{\Leftrightarrow}$$
$$\Leftrightarrow x = 1 \Leftrightarrow x = 1 \lor y + 1 = 0.$$

The step marked by $(*_2)$ is by case analysis on whether or not $x = 1$.

(3): Directly from (1) and (3).

(4): By induction on y. The base case follows from

$$x^{0+z} = x^z = 1 \times x^z = x^0 x^z.$$

In the inductive case we have

$$x^{y+1+z} = x^{y+z+1} = xx^{y+z} \overset{\mathrm{IH}}{=} xx^y x^z = x^{y+1} x^z.$$

(5): Assume $x \neq 0$ and prove by induction on y that

$$\forall z(y \geq z \to x^{y \dot{-} z} = x^y \div x^z).$$

In the base case take any z such that $0 \geq z$. Then $z = 0$ and we have

$$x^{0 \dot{-} 0} = x^0 = 1 = 1 \div 1 = x^0 \div x^0.$$

In the inductive case take any z such that $y + 1 \geq z$ and consider two cases. If $z = 0$ then we have

$$x^{y+1 \dot{-} 0} = x^{y+1} = x^{y+1} \div 1 = x^{y+1} \div x^0.$$

If $z = z_1 + 1$ for some z_1 then $y \geq z_1$ and we obtain

$$x^{y+1 \dot{-} (z_1+1)} = x^{y \dot{-} z_1} \overset{\text{IH}}{=} x^y \div x^{z_1} \overset{(*_3)}{=} xx^y \div (xx^{z_1}) = x^{y+1} \div x^{z_1+1}.$$

Note that the induction hypothesis is applied with z_1 in place of z. The step marked by $(*_3)$ follows from the assumption $x \neq 0$.

(6): Assume $x > 1$ and prove by induction on y that

$$\forall z(x^y = x^z \leftrightarrow y = z).$$

In the base case take any z and we obtain

$$x^0 = x^z \Leftrightarrow 1 = x^z \overset{(2)}{\Leftrightarrow} x = 1 \vee z = 0 \Leftrightarrow 0 = z.$$

In the inductive case take any z and consider two cases. If $z = 0$ then we have

$$x^{y+1} = x^0 \Leftrightarrow xx^y = 1 \Leftrightarrow x = 1 \wedge x^y = 1 \Leftrightarrow \bot \Leftrightarrow y + 1 = 0.$$

If $z = z_1 + 1$ for some z_1 then we have

$$x^{y+1} = x^{z_1+1} \Leftrightarrow xx^y = xx^{z_1} \Leftrightarrow x^y = x^{z_1} \overset{\text{IH}}{\Leftrightarrow} y = z_1 \Leftrightarrow y + 1 = z_1 + 1.$$

Note that the induction hypothesis is applied with z_1 in place of z.

The remaining properties are proved similarly. □