Introduction of dyadic pairing into PA

Lecture 9

Review: Dyadic concatenation

$$x \mid y \leftrightarrow \exists z\, x{\cdot}z = y$$
$$Pow_2(p) \leftrightarrow \forall d((d+2) \mid p \rightarrow 2 \mid d)$$
$$2^{|x|} = \mu_p[Pow_2(p) \wedge x+1 < 2{\cdot}p]$$
$$x \star y = x{\cdot}2^{|y|} + y$$

We have also defined

$$D\_two(m) \leftrightarrow \exists n_1 \exists n_2\, m = n_1 \star 2 \star n_2 \ ,$$

i.e. $\neg D\_two(m)$ iff $m = 1 \star \cdots \star 1$.

However, we have for $x > 0$ $\quad 2^{|x|} = \overbrace{1 \star \cdots \star 1 \star 2}^{|x|}$

Also $2^{|0|} = 1$. Thus, for any x $\quad 2^{|x|} \dot{-} 1 = \overbrace{1 \star \cdots \star 1}^{|x|}$ i.e.

$$\neg D\_two(m) \leftrightarrow \exists x\, m+1 = 2^{|x|} \leftrightarrow \exists x\, m = 2^{|x|} \dot{-} 1$$

Marker sequences

To every number m there are unique numbers n and b as well as numbers $a_1, a_2, \ldots, a_n$ of unique dyadic length such that

$$m = \overbrace{\underbrace{1 \star \cdots \star 1 \star 2}^{|a_1+1|} \star \underbrace{1 \star \cdots \star 1 \star 2}^{|a_2+1|} \star \cdots \star \underbrace{1 \star \cdots \star 1 \star 2}^{|a_n+1|}}^{n} \star \overbrace{1 \star \cdots \star 1}^{|b|}$$

This can be also written as:

$$m \quad = 2^{|a_1+1|} \quad \star 2^{|a_2+1|} \quad \star \cdots \quad \star 2^{|a_{n+1}|} \quad \star (2^{|b|} - 1)$$

Note that that $x + 1$ and $2^{|x+1|}$ have the same **length** because of **idempotency**: $2^{|x+1|} = 2^{|2^{|x+1|}|}$.

To every m and w of the same **length** there are **unique** n, b, $a_1, a_2, \ldots, a_n$ s.t.:

$$w = (a_1 + 1) \star (a_2 + 1) \star \cdots \star (a_n + 1) \star b$$

$$m = 2^{|a_1+1|} \quad \star 2^{|a_1+1|} \quad \star \cdots \star 2^{|a_n+1|} \quad \star (2^{|b|} - 1)$$

Properties of squares

Square is an **injection**:

$$x^2 = y^2 \rightarrow x = y$$

and $\sqrt{2}$ is irrational: $y > 0 \rightarrow (\frac{x}{y})^2 \neq 2$ for $x, y \in \mathbb{Z}$. This is expressed in PA as

$$x^2 = 2{\cdot}y^2 \rightarrow x = 0 \land y = 0$$

Hence

$$i{\cdot}x^2 = j{\cdot}y^2 \land x > 0 \land 0 < i,j \leq 2 \rightarrow i = j \land x = y$$

This property is used in the **uniqueness** property on the following slide.

Two lemmas for Splits

Existence of splits

$$\exists w \exists i \exists m (t = w \star i \star m \wedge 2^{|w|} = 2^{|m|} \wedge 2^{|i|} \leq 2)$$

We can view this as every number t can be decomposed into two numbers of length almost $\sqrt{|t|}$: $t = \begin{array}{|c|c|} \hline w & i \\ \hline \end{array} \star \begin{array}{|c|c|} \hline m & \\ \hline \end{array}$ and $i = 0, 1, 2$

Uniqueness of splits:

$$w_1 \star i_1 \star m_1 = w_2 \star i_2 \star m_2 \wedge 2^{|i_1|} \leq 2 \wedge 2^{|w_1|} = 2^{|m_1|} \wedge 2^{|i_2|} \leq 2 \wedge 2^{|w_2|} = 2^{|m_2|} \rightarrow w_1 = w_2 \wedge i_1 = i_2 \wedge m_1 = m_2$$

Definition of splits

$$t \doteq \begin{bmatrix} v \\ m \end{bmatrix} \leftrightarrow t = v \star m \land \exists w \exists i (v = w \star i \land 2^{|m|} = 2^{|w|} \land 2^{|i|} \leq 2)$$

We can vizualize the definition as: $t = \begin{array}{c} \boxed{w \mid i} \\ \boxed{\quad v \quad} \\ \star \\ \boxed{m \mid } \end{array}$.

Splits **exist** $\exists v \exists m \, t \doteq \begin{bmatrix} v \\ m \end{bmatrix}$ and they are **unique**:

$$t \doteq \begin{bmatrix} v_1 \\ m_1 \end{bmatrix} \land t \doteq \begin{bmatrix} v_2 \\ m_2 \end{bmatrix} \rightarrow v_1 = v_2 \land m_1 = m_2$$

Adjustment of splits

If $t \doteq \begin{bmatrix} v \\ m \end{bmatrix}$ then $t = w \star i$ where $i = 0, 1, 2$ and

$$w = (a_1 + 1) \star (a_2 + 1) \star \cdots \star (a_n + 1) \star b$$

$$m = 2^{|a_1+1|} \star 2^{|a_2+1|} \star \cdots \star 2^{|a_n+1|} \star (2^{|b|} - 1)$$

The **tail** part such that: $b \star i \star (2^{|b|} - 1) \doteq \begin{bmatrix} b \star i \\ 2^{|b|} - 1 \end{bmatrix}$ is an **atom**.

We now define $Adj(t) \doteq \begin{bmatrix} w' \\ m' \end{bmatrix}$ removing the atom:

$$w' = (a_1 + 1) \star (a_2 + 1) \star \cdots \star (a_n + 1)$$

$$m' = 2^{|a_1+1|} \star 2^{|a_2+1|} \star \cdots \star 2^{|a_n+1|}$$

$$Adj(t) = s \leftrightarrow \exists v_1 \exists v_2 \exists m \exists b (t \doteq \begin{bmatrix} v_1 \star v_2 \\ m\mathbf{0} \star (2^{|b|} \dot{-} 1) \end{bmatrix} \land 2^{|v_1|} = 2^{|m\mathbf{0}|} \land s \doteq \begin{bmatrix} v_1 \\ m\mathbf{0} \end{bmatrix})$$

Note that: $v_1 = w'$, $m\mathbf{0} = m'$, and $v_2 = b \star i$.

Dyadic list concatenation and pairing

We define the **concatenation**:

$$s \boxplus t = u \leftrightarrow \exists v_1 \exists v_2 \exists m \exists x (Adj(s) \doteq \begin{bmatrix} v_1 \\ m_1 \end{bmatrix} \wedge t \doteq \begin{bmatrix} v_2 \\ m_2 \end{bmatrix} \wedge u \doteq \begin{bmatrix} v_1 \star v_2 \\ m_1 \star m_2 \end{bmatrix})$$

and the **pairing**:

$$x; t = ((x+1) \star 2^{|x+1|}) \boxplus t$$

Thus

$$t \doteq \begin{bmatrix} v \\ m \end{bmatrix} \rightarrow s; t \doteq \begin{bmatrix} (x+1) \star v \\ 2^{|x+1|} \star m \end{bmatrix}$$