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Lecture 9

Review: Dyadic concatenation

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The Schema of Nested Iteration in PA

$$x \mid y \leftrightarrow \exists z \, x \cdot z = y$$

$$Pow_2(p) \leftrightarrow \forall d((d+2) \mid p \to 2 \mid d)$$

$$2^{|x|} = \mu_p[Pow_2(p) \land x + 1 < 2 \cdot p]$$

$$x \star y = x \cdot 2^{|y|} + y$$

We have also defined

$$D_{-}two(m) \leftrightarrow \exists n_1 \exists n_2 \ m = n_1 \star 2 \star n_2$$
,

i.e. $\neg D_{-}two(m)$ iff $m = 1 \star \cdots \star 1$.

However, we have for
$$x > 0$$
 $2^{|x|} = \underbrace{1 \star \cdots \star 1 \star 2}_{|x|}$

Also $2^{|0|} = 1$. Thus, for any $x \quad 2^{|x|} \cdot 1 = \underbrace{1 \cdot \cdots \cdot 1}$ i.e.

$$\neg D_two(m) \leftrightarrow \exists x \ m+1 = 2^{|x|} \leftrightarrow \exists x \ m = 2^{|x|} \stackrel{\bullet}{\longrightarrow} 1$$

Marker sequences

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The Schema of Nested Iteration in PA To every number m there are unique numbers n and b as well as numbers a_1, a_2, \ldots, a_n of unique dyadic length such that

$$m = \overbrace{1 \star \cdots \star 1 \star 2 \star \overbrace{1 \star \cdots \star 1 \star 2}^{|a_1+1|}}^{n} \underbrace{|a_2+1|}_{|a_2+1|} \underbrace{|a_n+1|}_{|a_n+1|} \underbrace{|b|}_{|a_1+1|}$$

This can be also written as:

$$m = 2^{|a_1+1|} \star 2^{|a_2+1|} \star \cdots \star 2^{|a_{n+1}|} \star (2^{|b|}-1)$$

Note that that x+1 and $2^{|x+1|}$ have the same **length** because of **idempotency**: $2^{|x+1|} = 2^{|2^{|x+1|}|}$.

To every m and w of the same **length** there are **unique** n, b, a_1, a_2, \ldots, a_n s.t.:

$$w = (a_1 + 1) \star (a_2 + 1) \star \cdots \star (a_n + 1) \star b$$

$$m = 2^{|a_1 + 1|} \star 2^{|a_1 + 1|} \star \cdots \star 2^{|a_n + 1|} \star (2^{|b|} - 1)$$

Properties of squares

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The Schema of Nested Iteration in

Square is an **injection**:

$$x^2 = y^2 \to x = y$$

and $\sqrt{2}$ is irrational: $y>0\to (\frac{x}{y})^2\neq 2$ for $x,y\in\mathbb{Z}.$ This is expressed in PA as

$$x^2 = 2 \cdot y^2 \to x = 0 \land y = 0$$

Hence

$$i \cdot x^2 = j \cdot y^2 \land x > 0 \land 0 < i, j \le 2 \rightarrow i = j \land x = y$$

This property is used in the **uniqueness** property on the following slide.

Two lemmas for Splits

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Existence of splits

$$\exists w \exists i \exists m (t = w \star i \star m \wedge 2^{|w|} = 2^{|m|} \wedge 2^{|i|} \leq 2)$$

We can view this as every number t can be decomposed into

two numbers of length almost
$$\sqrt{|t|}$$
: $t = \begin{bmatrix} w & i \\ & \star \end{bmatrix}$ and $i = 0, 1, 2$

Uniqueness of splits:

$$w_1 \star i_1 \star m_1 = w_2 \star i_2 \star m_2 \wedge 2^{|i_1|} \le 2 \wedge 2^{|w_1|} = 2^{|m_1|} \wedge 2^{|i_2|} \le 2 \wedge 2^{|w_2|} = 2^{|m_2|} \to w_1 = w_2 \wedge i_1 = i_2 \wedge m_1 = m_2$$

Definition of splits

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$$t \doteq \begin{bmatrix} v \\ m \end{bmatrix} \leftrightarrow t = v \star m \land \exists w \exists i (v = w \star i \land 2^{|m|} = 2^{|w|} \land 2^{|i|} \leq 2)$$

We can vizualize the definition as: $t = \frac{v}{\star}$

Splits **exist** $\exists v \exists m \ t \doteq \begin{bmatrix} v \\ m \end{bmatrix}$ and they are **unique**:

$$t \doteq \begin{bmatrix} v_1 \\ m_1 \end{bmatrix} \wedge t \doteq \begin{bmatrix} v_2 \\ m_2 \end{bmatrix} \rightarrow v_1 = v_2 \wedge m_1 = m_2$$

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pairing into PA The Schema of Nested Iteration in If $t \doteq \begin{bmatrix} v \\ m \end{bmatrix}$ then $t = w \star i$ where i = 0, 1, 2 and $w = (a_1 + 1) \star (a_2 + 1) \star \cdots \star (a_n + 1) \star b$ $m = 2^{|a_1+1|} \star 2^{|a_2+1|} \star \cdots \star 2^{|a_n+1|} \star (2^{|b|} - 1)$

The **tail** part such that: $b\star i\star (2^{|b|}-1)\doteq \begin{bmatrix} b\star i\\ 2^{|b|}-1 \end{bmatrix}$ is an

atom. We now define $Adj(t) \doteq \begin{bmatrix} w' \\ m' \end{bmatrix}$ removing the atom:

$$w' = (a_1 + 1) \star (a_2 + 1) \star \cdots \star (a_n + 1)$$

$$m' - 2^{|a_1+1|} + 2^{|a_2+1|} + \cdots + 2^{|a_n+1|}$$

 $m' = 2^{|a_1+1|} \star 2^{|a_2+1|} \star \cdots \star 2^{|a_n+1|}$ $Adj(t) = s \leftrightarrow \exists v_1 \exists v_2 \exists m \exists b (t \doteq \begin{bmatrix} v_1 \star v_2 \\ m\mathbf{0} \star (2^{|b|} \div 1) \end{bmatrix} \land 2^{|v_1|} = 2^{|m\mathbf{0}|} \land$

$$s \doteq egin{bmatrix} v_1 \\ m 0 \end{bmatrix}$$

Note that: $v_1 = w'$, $m\mathbf{0} = m'$, and $v_2 = b \star \dot{b} + c \star \dot{c} + c \star \dot{$

Dyadic list concatenation and pairing

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The Schema of Nested Iteration in We define the **concatenation**:

$$s \boxplus t = u \leftrightarrow \exists v_1 \exists v_2 \exists m \exists x (Adj(s) \doteq \begin{bmatrix} v_1 \\ m_1 \end{bmatrix} \land t \doteq \begin{bmatrix} v_2 \\ m_2 \end{bmatrix} \land u \doteq \begin{bmatrix} v_1 \star v_2 \\ m_1 \star m_2 \end{bmatrix})$$

and the pairing:

$$x$$
; $t = ((x+1) \star 2^{|x+1|}) \boxplus t$

Thus

$$t \doteq \begin{bmatrix} v \\ m \end{bmatrix} \rightarrow s; t \doteq \begin{bmatrix} (x+1) \star v \\ 2^{|x+1|} \star m \end{bmatrix}$$