Introduction of dyadic concatenation into PA

Lecture 8

Review: Powers of two

Definition:

$$Pow_2(p) \leftrightarrow \forall d(d \mid p \rightarrow d = 1 \lor 2 \mid d)$$

Provably equivalent properties:

$$\neg Pow_2(0)$$
$$Pow_2(x\mathbf{1}) \leftrightarrow x = 0$$
$$Pow_2(x\mathbf{0}) \leftrightarrow x > 0 \land Pow_2(x)$$

This is equivalent again to **recursive clauses**:

$$Pow_2(x\mathbf{1}) \leftarrow x = 0$$
$$Pow_2(x\mathbf{0}) \leftarrow x > 0 \land Pow_2(x)$$

CL requires a default clause explicit:

$$Pow_2(x\mathbf{0}) \leftarrow x = 0 \land 0 = 1$$

We can now clausally redefine Pow_2 and let CL make **use** commands automatically

Binary case and induction

Clauses for Pow_2 are by **binary discrimination**:

$$x = y\mathbf{0} \lor x = y\mathbf{1}$$

proved from the properties of **division**
This justifies **Binary case rule in CL**: case $Nb; x$:

$$\frac{}{\begin{array}{c} x = y\mathbf{0} \\ y = 0 \end{array} \;\Big|\; \begin{array}{c} x = y\mathbf{0} \\ y > 0 \end{array} \;\Big|\; x = y\mathbf{1}}$$

Complete induction proves the schema of **Binary induction**:

$$\phi[0] \land \forall x(x > 0 \land \phi[x] \to \phi[x\mathbf{0}]) \land \forall x(\phi[x] \to \phi[x\mathbf{1}]) \to \phi[x]$$

This justifies **Binary induction rule in CL**: ind $Nb; x$

$$\frac{}{\begin{array}{c} x = 0 \\ \phi[x\mathbf{0}]* \end{array} \;\Big|\; \begin{array}{c} x > 0 \\ \phi[x] \\ \phi[x\mathbf{0}]* \end{array} \;\Big|\; \begin{array}{c} \phi[x] \\ \phi[x\mathbf{1}]* \end{array}}$$

Towards dyadic concatenation in PA

We wish PA to **prove** the following recurrences as **theorems**

$$x \star 0 = x$$
$$x \star y\mathbf{1} = (x \star y)\mathbf{1}$$
$$x \star y\mathbf{2} = (x \star y)\mathbf{2}$$

For that we need to define $\star$ explicitly:

$$x \star y = x \cdot 2^{|y|} + y$$

For that we need to introduce into PA the **dyadic length power** function: $Dlp(x) \equiv 2^{|x|}$.
Note that we cannot directly define: $|x|$ or 2^x, but we can $2^{|x|}$.

$2^{|x|}$: Illustration

For x such that $7 \leq x \leq 14$ we have

$$2^{|x|} = \begin{cases} \mathbf{0112} & \text{if } x = 7 \;\; = \mathbf{0111} \\ \mathbf{0112} & \text{if } x = 8 \;\; = \mathbf{0112} \\ \mathbf{0112} & \text{if } x = 9 \;\; = \mathbf{0121} \\ \mathbf{0112} & \text{if } x = 10 = \mathbf{0122} \\ \mathbf{0112} & \text{if } x = 11 = \mathbf{0211} \\ \mathbf{0112} & \text{if } x = 12 = \mathbf{0212} \\ \mathbf{0112} & \text{if } x = 13 = \mathbf{0221} \\ \mathbf{0112} & \text{if } x = 14 = \mathbf{0222} \end{cases}$$

Note: $y \star x = y{\cdot}8 + x = y{\cdot}2^3 + x = y{\cdot}2^{|x|} + x$
Also note idempotency: $2^{|2^{|8|}|} = 2^{|8|}$.

Introduction of $2^{|x|}$ into PA

By extension by definition:

$$2^{|x|} = p \leftrightarrow Pow_2(p) \land p \leq x + 1 < 2{\cdot}p$$

because for $x > 0$ we have

$$(\overbrace{1 \cdots 1}^{|x|})_2 = 2^{|x|} - 1 \leq x < 2^{|x|+1} - 1 = (\overbrace{1 \cdots 1}^{|x|+1})_2$$

We extend CL by **minimization**:

$$2^{|x|} = \mu_p[Pow_2(p) \land x + 1 < 2{\cdot}p]$$

We need to prove the **existence condition**:

$$\exists p(Pow_2(p) \land x + 1 < 2{\cdot}p)$$

which says that powers of two are **unbounded**

Comparison of dyadic length

We cannot define in PA the **dyadic length** $|x|$ yet, but we can compare the dyadic length of two numbers:
The numbers x and y have the same dyadic length iff
$2^{|x|} = 2^{|y|}$
or
The number x has a shorter dyadic length than y iff $2^{|x|} < 2^{|y|}$
This is possible because 2^x is **injective**

Recursive clauses for $2^{|x|}$

After defining

$$2^{|x|} = \mu_p[Pow_2(p) \wedge x + 1 < 2{\cdot}p]$$

PA proves:

$$2^{|0|} = 1$$
$$2^{|x\mathbf{1}|} = 2{\cdot}2^{|x|}$$
$$2^{|x\mathbf{2}|} = 2{\cdot}2^{|x|}$$

because **intuitively** $2^{|x\mathbf{1}|} = 2^{|x|+1} = 2{\cdot}2^{|x|}$
The clauses are by **dyadic recursion**.

Dyadic case and induction

Clauses for $2^{|x|}$ are by **dyadic discrimination**:

$$x = 0 \lor x = y\mathbf{1} \lor x = y\mathbf{2}$$

proved by binary case analysis
This justifies **Dyadic case rule in CL**: case $N_2; x$:

$$\frac{}{x = 0 \mid x = y\mathbf{1} \mid x = y\mathbf{2}}$$

Complete induction proves the schema of **Dyadic induction**:

$$\phi[0] \land \forall x(\phi[x] \to \phi[x\mathbf{1}]) \land \forall x(\phi[x] \to \phi[x\mathbf{2}]) \to \phi[x]$$

This justifies **Dyadic induction rule in CL**: ind $N_2; x$

$$\frac{}{\phi[0]* \;\Big|\; \begin{array}{c}\phi[x] \\ \phi[x\mathbf{1}]*\end{array} \;\Big|\; \begin{array}{c}\phi[x] \\ \phi[x\mathbf{2}]*\end{array}}$$

Clauses for $\star$ as theorems of PA

We explicitly define $x \star y = x{\cdot}2^{|y|} + y$ and prove as theorems the clauses for $\star$ by **dyadic recursion**:

$$x \star 0 = x$$
$$x \star y\mathbf{1} = (x \star y)\mathbf{1}$$
$$x \star y\mathbf{2} = (x \star y)\mathbf{2}$$

We can now properties of dyadic concatenation by **dyadic induction** with automatical uses of clauses.

Auxiliary predicate

We explicitly define

$$D\_two(m) \leftrightarrow \exists m_1 \exists m_2\, m = m_1 \mathbf{2} \star m_2$$

Note that

$$m = m_1 \mathbf{2} \star m_2 = (m_1 \star 0\mathbf{2}) \star m_2 = m_1 \star 2 \star m_2$$

And prove as **theorems** its clauses by **dyadic recursion**:

$$D\_two(m\mathbf{1}) \leftarrow D\_two(m)$$
$$D\_two(m\mathbf{2})\ .$$

The relevance of *D_two*

This predicate is *D_two* important because PA proves

$$\exists n(2^{|x+1|} = n \star 2 \land \neg D\_two(n))$$

i.e. $2^{|x+1|} = (1 \cdots 1)_2 \star 2$.
PA then proves the existence of **leading powers**:

$$D\_two(m) \rightarrow \exists x \exists m_1\, m = 2^{|x+1|} \star m_1$$

i.e. if m contains 2 then $m = (\overbrace{1 \cdots 1}^{n})_2 \star 2 \star m_1$ for some m_1, n.
PA also proves the existence of **trailing ones**

$$\exists m_1 \exists n(m = m_1 \mathbf{0} \star n \land \neg D\_two(n))$$

i.e. $m = m_1 \mathbf{0} \star (\overbrace{1 \cdots 1}^{|n|})_2$ for some m_1, n