Basic bootstrapping of PA

Lecture 6

Recapitulation of extensions

- T' is an **extension** of T if $T' \models A$ for all $A \in T$ (T' **can** prove new facts about formulas of $\mathcal{L}_T$)
- T' is a **conservative extension of** T if for all $A \in \mathcal{L}_T$ from $T' \models A$ we get $T \models A$ (T' **cannot** prove new facts about formulas of $\mathcal{L}_T$ but it can about formulas of $\mathcal{L}_{T'}$),
- Special case of conservative extensions are **extensions by definitions** where **no** new facts about formulas of $\mathcal{L}_{T'}$ can be proved because every theorem of T' can be **translated** to an equivalent theorem of T.

Goals

- For arbitrary **theory** T we have learnt to prove **theorems** A of **extensions** T' as **logical consequences**: $T' \models A$,
- we will now study a particular theory **Peano arithmetic** (PA)
- our **goal** is to show that the **clauses** of legal **CL** definitions are theorems in **definitional extensions** of PA
- Thus all properties of CL programs are provable in PA but the extensions make for **readability** and for the **computability** directly from the clauses

Peano arithmetic

The **language of PA** consists of the **constant** 0 and **function symbols** x', $x + y$, $x \cdot y$.
The **standard structure** $\mathcal{N}$ has the domain $\mathbb{N}$ of natural numbers with the intended interpretaion of symbols in that order as **zero**, **successor**, **addition**, and **multiplication** functions.
The **axioms of PA** are

$$\begin{array}{ll} x' \neq 0 & x' = y' \to x = y \\ 0 + y = y & x' + y = (x + y)' \\ 0 \cdot y = 0 & x' \cdot y = (x \cdot y) + y \end{array}$$

$$A[0, \vec{y}] \wedge \forall x (A[x, \vec{y}] \to A[x', \vec{y}]) \to A[x, \vec{y}]$$

for **all** formulas $A[x, \vec{y}]$ of the language of PA. The last axioms are called the axioms of **mathematical induction** with x called the **induction** variable and $\vec{y}$ (if any) the **parameters**

Incompleteness of PA: Goodstein's sequence

For $x > 0$ write the number $x - 1$ **fully** in **base** $n \geq 2$. For instance, for $x = 528$ and $n = 2$:
$x = 528 = 2^9 + 2^4 + 1 = 2^{2^3+1} + 2^{2^2} = 2^{2^{2^1+1}} + 2^{2^{2^1}}$
$x = 527 = 2^{2^{2^1+1}} + 2^{2^1+1} + 2^{2^1} + 2^1 + 1$ and **change** to base $n + 1 = 3$:
$P_n(x) = 3^{3^{3^1+1}} + 3^{3^1+1} + 3^{3^1} + 3^1 + 1.$
Subtract one and change to base 4, obtain $P_{n+1}(P_n(x))$, and continue. This is called **Goodstein's sequence**
There is a formula $A[n, x]$ of PA which says **Goodstein's sequence for $n \geq 2$ and any x terminates in finitely many steps in** 0
We have $\models^{\mathcal{N}} \forall n \forall x A[n, x]$ **but** PA $\not\vdash \forall n \forall x A[n, x]$.
Hence by **Gödel's completeness** there is a **non-standard** structure $\mathcal{M}$ for natural numbers s.t. $\models^{\mathcal{M}}$ PA $+ \neg \forall n \forall x A[n, x]$.

Incompleteness theorem of Gödel

To every **consistent** extension T of PA in the same language there is a sentence A of PA such that $\models^{\mathcal{N}} T + A$ but neither $T \vdash A$ nor $T \vdash \neg A$.
Thus **arithmetic** is **essentially incomplete**, i.e. to every such T there is a **non-standard model of arithmetic** $\mathcal{M}$ such that $\models^{\mathcal{M}} T + \neg A$.