Extension of theories

Lecture 5

Recapitulation of Predicate logic

- **Gödel's completeness and soundness**:

$$T \models A \text{ iff } T \vdash A$$

- **Reduction of predicate logic to propositional:**

$$T \models A \text{ iff } T, \mathbf{Eq}, \mathbf{Q} \models_p A$$

Extensions of theories

- **Extension of languages**: $\mathcal{L}'$ is an **extension** of $\mathcal{L}$ if every symbol of $\mathcal{L}$ is a symbol of $\mathcal{L}'$,
- **Extension of theories**: T' is an **extension** of T if the language of T' extends the language of T and $T' \vdash A$ for all $A \in T$,
- **Conservative extensions:** An extension T' of T is **conservative** iff from $T' \vdash A$ where A is in the language of T we have $T \vdash A$
- **Consistent theories:** A theory T is **consistent** if $T \nvdash \bot$. Clearly, if T' is conservative over a consistent T then also T' is consistent

Extension by definitions with predicate symbols

- Let T be a theory in $\mathcal{L}$ which does not contain n-ary predicate symbol P, and $A[\vec{x}]$ a formula of $\mathcal{L}$ with just the n-variables $\vec{x}$ free,
- then $T' = T + \forall\vec{x}(P(\vec{x}) \leftrightarrow A[\vec{x}])$ is an **extension** of T in the language $\mathcal{L} + P$,
- **Elimination of** P: Let B^* be like B but with every $P(\vec{\tau})$ replaced by $A[\vec{\tau}]$,
- $T' \vdash B \leftrightarrow B^*$, proof is straightforward,
- T' is **conservative over** T: If $T' \models B \in \mathcal{L}$ take any $\models^{\mathcal{M}} T$, expand it to $\models^{\mathcal{M}'} T'$, conclude $\models^{\mathcal{M}'} B$, and $\models^{\mathcal{M}} B$. Hence $T \models B$
- **Translation:** $T' \vdash B$ iff $T \vdash B^*$ for any $B \in \mathcal{L} + P$

Extension by definitions with function symbols

- Let T be a theory in $\mathcal{L}$ which does not contain n-ary function symbol f, and $A[\vec{x}, y]$ a formula of $\mathcal{L}$ with just the $n+1$-variables $\vec{x}, y$ free,
- if the **existence condition**: $T \vdash \exists y A[\vec{x}, y]$ holds then
- $T' = T + A[\vec{x}, f(\vec{x})]$ is **conservative** over T: If $T' \models B \in \mathcal{L}$ take any $\models^{\mathcal{M}} T$, expand it to $\models^{\mathcal{M}'} T'$, conclude $\models^{\mathcal{M}'} B$, and $\models^{\mathcal{M}} B$. Hence $T \models B$
- if also the **uniqueness condition**: $T \vdash A[\vec{x}, y_1] = A[\vec{x}, y_2] \rightarrow y_1 = y_2$ holds
- then $T'' = T + \forall \vec{x}(f(\vec{x}) = y \leftrightarrow A[\vec{x}, y])$ is **conservative over** T because T' extends T''.
- **Elimination of** f: Let B^* be like B but with every atomic subformula $C_y[f(\vec{\tau})]$ replaced by $\exists y(A[\vec{\tau}, y] \wedge C[y])$,
- $T'' \vdash B \leftrightarrow B^*$, proof is straightforward,
- **Translation:** $T'' \vdash B$ iff $T \vdash B^*$ for any $B \in \mathcal{L} + P$