# Equational logic

## Language of equational logic

$\mathcal{L}$ consists of **terms** given by **denumerable** sets of **function symbols** $f_i$ and **predicate symbols** $P_i$ (each with *arity* $n \geq 0$). We always have $=$ among predicate symbols.

**Terms:** are concrete sequences of symbols defined by:

1. (object) variables $x_0$, $x_1$, . . . , $y_0$, $y_1$, . . . are terms,

2. if $\tau_1$, . . . , $\tau_n$ are terms and the function symbol $f_i$ has arity $n \geq 0$ then $f_i(\tau_1, \ldots, \tau_n)$ is a term.

Function symbols of arity 0 are *constants*

**Formulas** are concrete sequences of symbols consisting of:

1. *atomic* formulas $P_i(\tau_1, \ldots, \tau_n)$ with $\tau_1$, ..., $\tau_n$ terms,

2. *propositional* formulas $\bot$, $\top$, $\neg A_1$, $A_1 \vee A_2$, $A_1 \wedge A_2$, $A_1 \rightarrow A_2$, $A_1 \leftrightarrow A_2$ with $A_1$, $A_2$ formulas,

We write $\tau_1 = \tau_2$ for **identities** $=(\tau_1, \tau_2)$.

Predicate symbols of arity 0 are *propositional constants* and they correspond to propositional variables.

## Quasi-tautological consequence

We wish to define **equationally valid** sequents $T \models_i S$ such that if $T \models_p S$ then $T \models_i S$ (i.e. all tautologies are eq. valid). But we also wish to use the properties of $=$, for instance, $\models_i \tau_1 = \tau_2 \rightarrow \tau_2 = \tau_1$ which is not in general a tautology but it is a **quasi-tautology**.

If $T \models_i A$ we say that $A$ is a **quasi-tautological consequence** of $T$.

We wish the **reduction** to propositional logic:

$$T \models_i S \text{ iff } T, \boldsymbol{Eq} \models_p S$$

where $\boldsymbol{Eq}$ are the **axioms of identity**.

## Identity (equational) axioms

for every equational language $\mathcal{L}$ the set of sentences $\tau = \tau$ are **reflexivity** axioms,

$$\tau = \sigma \to \sigma = \tau$$

are **symmetry** axioms,

$$\tau = \sigma \to \sigma = \rho \to \tau = \rho$$

are **transitivity** axioms, and

$$\tau_1 = \sigma_1 \to \cdots \to \tau_n = \sigma_n \to f(\tau_1, \ldots, \tau_n) = f(\sigma_1, \ldots, \sigma_n)$$

$$\tau_1 = \sigma_1 \to \cdots \to \tau_n = \sigma_n \to P(\tau_1, \ldots, \tau_n) \to P(\sigma_1, \ldots, \sigma_n)$$

are **substitution** axioms where $f, P \in \mathcal{L}$

We designate all by $\boldsymbol{Eq}$ and call them **equation** (identity) axioms (for $\mathcal{L}$).

## Interpretation of languages of identity

We wish to introduce **interpretations** $\mathcal{M}$ of $\mathcal{L}$ corresponding to propostional valuations $v$ such that $\models_i^{\mathcal{M}} A$ if $A$ **is satisfied in** $\mathcal{M}$.
This should extend propositional valuations, for instance, we wish:

- $\models_i^{\mathcal{M}} A \wedge B$ iff $\models_i^{\mathcal{M}} A$ and $\models_i^{\mathcal{M}} B$
- $\models_i^{\mathcal{M}} \neg A$ iff not $\models_i^{\mathcal{M}} A$

But we also wish $\models_i^{\mathcal{M}} \boldsymbol{Eq}$, i.e. for instance:

- if $\models_i^{\mathcal{M}} \tau_1 = \tau_2$ then $\models_i^{\mathcal{M}} \tau_2 = \tau_1$

## Interpretations $\mathcal{M}$ for L

consist of **domains** $D$, of interpretions $f^{\mathcal{M}}$ of functions symbols $f \in \mathcal{L}$ as $n$-ary functions over $D$ and of interpretions $P^{\mathcal{M}}$ of predicate symbols $P \in \mathcal{L}$ as $n$-ary relations over $D$.
$\langle D, \ldots f^{\mathcal{M}}, \ldots P^{\mathcal{M}}, \ldots \rangle$ are **structures for** $\mathcal{L}$.
In **interpretations** $\mathcal{M}$ we also need to **assign objects** $x^{\mathcal{M}}$ from $D$ to the (object) variables $x$.
Terms $\tau$ of $\mathcal{L}$ are *interpreted* in $\mathcal{M}$ by **denotations** $\tau^{\mathcal{M}} \in D$ s.t.
- $f(\tau_1, \ldots, \tau_n)^{\mathcal{M}} = f^{\mathcal{M}}(\tau_1^{\mathcal{M}}, \ldots, \tau_n^{\mathcal{M}})$.

Atomic formulas are interpreted by defining:
- $\models_i^{\mathcal{M}} P(\tau_1, \ldots, \tau_n)$ iff $P^{\mathcal{M}}(\tau_1^{\mathcal{M}}, \ldots, \tau_n^{\mathcal{M}})$,
- $\models_i^{\mathcal{M}} \tau_1 = \tau_2$ iff $\tau_1^{\mathcal{M}} = \tau_2^{\mathcal{M}}$

and we close the satisfaction relation propositionally.

## Saturation of identity sequents

We **define** $T \vDash_i S$ to hold iff for all interpretations $\mathcal{M}$ satisfying $T$, i.e. $\vDash_i^{\mathcal{M}} T$, there is an $A \in S$ s.t. $\vDash_i^{\mathcal{M}} A$.
Thus $\vDash_i A$, i.e. $A$ is a **quasi-tautology**, iff $\vDash_i^{\mathcal{M}} A$ for all $\mathcal{M}$.
We have

- $T \vDash_i S$ iff $\tau = \tau, T \vDash_i S$
- $\tau_1 = \tau_2, T \vDash_i S$ iff $\tau_2 = \tau_1, \tau_1 = \tau_2, T \vDash_i S$
- $\tau_1 = \tau_2, \tau_2 = \tau_3, T \vDash_i S$ iff

$\tau_1 = \tau_3, \tau_1 = \tau_2, \tau_2 = \tau_3, T \vDash_i S$

- $\vec{\tau} = \vec{\rho}, T \vDash_i S$ iff $f(\vec{\tau}) = f(\vec{\rho}), \vec{\tau} = \vec{\rho}, T \vDash_i S$
- $\vec{\tau} = \vec{\rho}, P(\vec{\tau}), T \vDash_i S$ iff $P(\vec{\rho}), \vec{\tau} = \vec{\rho}, P(\vec{\tau}), T \vDash_i S$

plus all saturations corresponding to the propositional ones.

## Tableau rules for identity

**reflexivity** rules:

$$\frac{}{\tau = \tau}$$

**symmetry** rules:

$$\frac{\tau = \sigma}{\sigma = \tau}$$

**transitivity** rules:

$$\frac{\tau = \sigma \quad \sigma = \rho}{\tau = \rho}$$

**substitution** rules:

$$\frac{\tau_1 = \sigma_1 \cdots \tau_n = \sigma_n}{f(\tau_1, \ldots, \tau_n) = f(\sigma_1, \ldots, \sigma_n)} \quad f \in \mathcal{L}$$

$$\frac{\tau_1 = \sigma_1 \cdots \tau_n = \sigma_n \ P(\tau_1, \ldots, \tau_n)}{P(\sigma_1, \ldots, \sigma_n)} \quad P \in \mathcal{L}$$

## Equationally saturated sequents

$T \vDash_i S$ is **equationally saturated** if it is propositionally saturated and

- $\tau = \tau \in T$
- if $\tau_1 = \tau_2 \in T$ then $\tau_2 = \tau_1 \in T$
- if $\tau_1 = \tau_2, \tau_2 = \tau_3 \in T$ then $\tau_1 = \tau_3 \in T$
- if $\vec{\tau} = \vec{\rho} \in T$ then $f(\vec{\tau}) = f(\vec{\rho}) \in T$
- if $\vec{\tau} = \vec{\rho}, P(\vec{\tau}) \in T$ then $P(\vec{\rho}) \in T$

We have $T \vDash_i S$ iff all equationally saturated sons are closed.