Recursive clausal definitions

Lecture 12

Recursive clausal formulas

For recursive clausal definitions we extend **clausal formulas for** f to **recursive** ones with a new rule:

- $f(\mathbf{s}[\mathbf{x}]) = z \land \mathbf{A}_1[\mathbf{x}, z; v]$ is a recursive clausal formula if
  - $\mathbf{s}$ is a sequence of terms not applying the function symbol f and,
  - $\mathbf{A}_1[\mathbf{x}, z; v]$, is a recursive clausal formula.

Our plan is to extend PA for **suitable** clausal formulas $\mathbf{A}[\mathbf{x}; v]$ by extension by definition of f such that

$$\vdash f(\mathbf{x}) = v \leftarrow \mathbf{A}[\mathbf{x}; v]$$

and then **unfold** this into provably equivalent **recursive clauses** for f

Iterated functions $g_{\mathbf{A}}$

For every **recursive** clausal formula $\mathbf{A}[\mathbf{x}; v]$ we will define an **explicit** clausal formula $\mathbf{B}[n, a, \mathbf{x}; v]$ for an explicit definition of a three-argument function $g_{\mathbf{A}}(x, n, a)$ (below only g) such that

$$\vdash g((x_1; \ldots; x_n), n, a) = v \leftarrow \mathbf{B}[n, a, \mathbf{x}; v]$$

(when x is not an n-tuple then g yields 0) and for an unary measure function μ and a numeral $C \equiv \underline{k}$ we have

$$\vdash g(x, n, a) = v\mathbf{1} \to \mu(v) < \mu(x)$$
$$\vdash 2 \mid g(x, 0, a)$$

Such an **A** is called **regular**.
We then define the **iteration** function g^* and from it explicitly

$$f(\mathbf{x}) = g^*((x_1; \ldots; x_n), C, 0)$$

PA will then prove the **recursive clauses unfolded** from **A**.

Construction of B

By recursion on the structure of **A**. When **A** is:

- $\mathbf{s}[\mathbf{x}] = v$ then $\mathbf{B}[n, a, \mathbf{x}]$ is $(\mathbf{s}[\mathbf{x}])\mathbf{0} = v$.
- $\exists \mathbf{z}_1(\mathbf{D}_1[\mathbf{x}, \mathbf{z}_1] \wedge \mathbf{A}_1[\mathbf{x}, \mathbf{z}_1; v]) \vee \cdots \vee \exists \mathbf{z}_k(\mathbf{D}_k[\mathbf{x}, \mathbf{z}_k] \wedge \mathbf{A}_k[\mathbf{x}, \mathbf{z}_k, v])$ then **B** is $\exists \mathbf{z}_1(\mathbf{D}_1[\mathbf{x}, \mathbf{z}_1] \wedge \mathbf{B}_1[n, a, \mathbf{x}, \mathbf{z}_1; v]) \vee \cdots \vee \exists \mathbf{z}_k(\mathbf{D}_k[\mathbf{x}, \mathbf{z}_k] \wedge \mathbf{B}_k[n, a, \mathbf{x}, \mathbf{z}_k, v])$
- $f(\mathbf{s}[\mathbf{x}]) = z \wedge \mathbf{A}_1[\mathbf{x}, z; v]$ we obtain $\mathbf{B}_1[n, a, \mathbf{x}, z; v]$ by IH and set **B** to

$$Adj(a) = 0 \wedge (n = 0 \wedge (0)\mathbf{0} = v \vee n > 0 \wedge (\mathbf{s}[\mathbf{x}])\mathbf{1} = v) \vee \exists z \exists b(a = z; b \wedge \mathbf{B}_1[m, b, \mathbf{x}, z; v]))$$

Clausal definitions of predicates P

are by **clausal** definitions of their **characteristic** functions f such that $\vdash f(\mathbf{x}) = v \leftarrow \mathbf{A}[\mathbf{x}; v]$ where the (recursive) clausal formula $\mathbf{A}$ has the final **assignments** of the form $1 = v$ (true) or $0 = v$ (false) and **recursions** in it are always followed by **discriminations** on zero:

$$f(\mathbf{s}) = z \wedge (z = 0 \wedge \mathbf{A}_1 \vee z > 0 \wedge \mathbf{A}_1)$$

where neither $\mathbf{A}_1$ nor $\mathbf{A}_2$ contain z free.
We then explicitly **define** $P(\mathbf{x}) \leftrightarrow f(\mathbf{x}) > 0$ and prove in PA the (recursive) clauses for P obtained by unfolding of $\mathbf{A}$ where:

$$f(\mathbf{x}) = v \leftarrow \mathbf{B} \wedge 1 = v \quad \Rightarrow \quad P(\mathbf{x}) \leftarrow \mathbf{B}$$
$$f(\mathbf{x}) = v \leftarrow \mathbf{B} \wedge 0 = v \quad \Rightarrow \quad \neg P(\mathbf{x}) \leftarrow \mathbf{B}$$

We also change all above unfolded recursions as follows:

$$[\neg]P(\mathbf{x}) \leftarrow \mathbf{B} \wedge f(\mathbf{s}){=}z \wedge z{=}0 \wedge \mathbf{A}_1 \Rightarrow [\neg]P(\mathbf{x}) \leftarrow \cdots \neg P(\mathbf{s}) \wedge \mathbf{A}_1$$
$$[\neg]P(\mathbf{x}) \leftarrow \mathbf{B} \wedge f(\mathbf{s}){=}z \wedge z{>}0 \wedge \mathbf{A}_2 \Rightarrow [\neg]P(\mathbf{x}) \leftarrow \cdots P(\mathbf{s}) \wedge \mathbf{A}_2$$