Propositional Logic

The language of propositional logic

Propositional formulas are formed from

- **propositional variables** $(P_0, P_1, \ldots)$ by
- **propositional connectives** which are
  - **nullary**: truth ($\top$), falsehood ($\bot$)
  - **unary**: negation ($\neg$)
  - **binary**: disjunction ($\vee$), conjunction ($\wedge$) implication ($\rightarrow$), equivalence ($\leftrightarrow$)

Binary are **infix** ($\rightarrow, \leftrightarrow$ groups to the right, the rest to the left)
Precedence from highest is $\neg$, $\wedge$, $\vee$, $(\rightarrow, \leftrightarrow)$.
Thus
$P_1 \rightarrow P_2 \leftrightarrow P_3 \vee \neg P_4 \wedge P_5$ abbreviates
$P_1 \rightarrow (P_2 \leftrightarrow (P_3 \vee (\neg(P_4) \wedge P_5)))$

Truth functions

We identify the **truth values** *true* and *false* with the nullary symbols $\top$ and $\bot$ respectively. The remaining connectives are **interpreted** as functions over truth values satisfying:

P_1	P_2	$\neg P_1$	$P_1 \wedge P_2$	$P_1 \vee P_2$	$P_1 \rightarrow P_2$	$P_1 \leftrightarrow P_2$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$

We have

$$\begin{aligned} A \leftrightarrow B &\equiv (A \rightarrow B) \wedge (B \rightarrow A) \\ \neg A &\equiv A \rightarrow \bot \\ A \rightarrow B &\equiv \neg A \vee B \\ A \wedge B &\equiv \neg(\neg A \vee \neg B) \end{aligned}$$

Tautologies

Of special interest are those propositional formulas A which are true ($\top$) for all possible truth values of its propositional variables, in writing $\models_p A$.
Every such formula is a **tautology**.
Tautologies are the cornerstones of mathematical logic.
Some examples of (schemas of) tautologies:

$$\models_p (A \to B \to C) \leftrightarrow A \land B \to C$$
$$\models_p (A \to B \to C) \leftrightarrow (A \to B) \to A \to C$$
$$\models_p (A \to B) \leftrightarrow \neg B \to \neg A$$

for any propositional formulas A, B, and C

Propositional satisfaction relation

A **propositional valuation**, or an **propositional assignment** v is a (possibly infinite) set $v \subset \mathbb{N}$ The idea is that the $P_i \equiv \top$ iff $i \in v$.

We say that a formula A **is satisfied in** v, in writing $\models_p^v A$, if A is true for the assignment v. We thus have: $\models_p^v P_i$ iff $i \in v$
$\models_p^v \neg A$ iff not $\models_p^v A$ iff $\not\models_p^v A$
$\models_p^v A \wedge B$ iff $\models_p^v A$ and $\models_p^v B$
$\models_p^v A \vee B$ iff $\models_p^v A$ or $\models_p^v B$
$\models_p^v A \rightarrow B$ iff whenever $\models_p^v A$ also $\models_p^v B$

Thus A **is a tautology** iff $\models^v A$ for all valuations v.

Coincidence property if two valuations v and w are such that $i \in v$ iff $i \in w$ for all P_i occurring in A then $\models_p^v A$ iff $\models_p^w A$

Satisfaction relation for sets of propositional formulas

For T a set of formulas and v a valuation (both possibly infinite), we say that v **satisfies** T, in writing $\models_p^v T$, iff for all $A \in T$ we have $\models_p^v A$.

We say that S **is a propositional (tautological) consequence of** T, in writing $T \models_p S$, iff for all v satisfying T (i.e. $\models_p^v T$) at least one $A \in S$ is satisfied (i.e. $\models_p^v A$)

The special case when $T \models_p \{A\}$ is the most important relation in mathematical logic. We write $T \models_p A$ instead of $T \models_p \{A\}$ and say that A **tautologically follows from** T. Note that $\emptyset \models_p \{A\}$ iff A is tautology.

If $T \models_p S$ holds then we say that the **propositional sequent** $T \models_p S$ is valid

Compactness theorem for propositional consequence

$T \models_p S$ iff there are finite $T' \subset T$ and $S' \subset S$ s.t. $T' \models_p S'$.

If $T' = \{A_1, \ldots, A_n\}$ and $S' = \{B_1, \ldots, B_m\}$ we have $T' \models_p S'$ iff

$$\models_p A_1 \wedge \cdots \wedge A_n \to B_1 \vee \cdots \vee B_m$$

Saturation of propositional sequents

Closure:
$A, T \models_p A, S$; $\bot, T \models_p S$; $T \models_p \top, S$ are valid
Flattenings:
- $T \models_p A \to B, S$ iff $A, T \models_p B, A \to B, S$
- $T \models_p A \vee B, S$ iff $T \models_p A, B, A \vee B, S$
- $A \wedge B, T \models_p S$ iff $A, B, A \wedge B, T \models_p S$

Splits:
- $A \to B, T \models_p S$ iff
$B, A \to B, T \models_p S$ and $A \to B, T \models_p A, S$
- $A \vee B, T \models_p S$ iff
$A, A \vee B, T \models_p S$ and $B, A \vee B, T \models_p S$
- $T \models_p A \wedge B, S$ iff
$T \models_p A, A \wedge B, S$ and $T \models_p B, A \wedge B, S$

Inversions:
- $T \models_p \neg A, S$ iff $A, T \models_p \neg A, S$
- $\neg A, T \models_p S$ iff $\neg A, T \models_p A, S$

Cuts:
$T \models_p S$ iff $A, T \models_p S$ and $T \models_p A, S$

Here $A_1, \ldots, A_k$ stands for $A_1, \ldots, A_k, \emptyset$ and $A_1, \ldots, A_k, S$ stands for $S \cup \{A_1, \ldots, A_k\}$.

Propositional tableaux as trees of sequents

A branch with formulas $A_1, \ldots, A_n, B_1*, \ldots, B_m*$ can be viewed as a finite *sequent*:
$\{A_1, \ldots, A_n\} \vDash_p \{B_1, \ldots B_m\}$ A tableau can be viewed as a conjunction of sequents corresponding to its branches.
Tableau rules: correspond to saturation of sequents:
A branch **closes** when it contains $\bot$, $\top*$, $A, A*$
Flattens:

$$\frac{A \to B*}{A,\ B*} \quad \frac{A \vee B*}{A*,\ B*} \quad \frac{A \wedge B}{A,\ B}$$

Splits:

$$\frac{A \to B}{B \mid A*} \quad \frac{A \vee B}{A \mid B} \quad \frac{A \wedge B*}{A* \mid B*}$$

Inversions, Cuts, and Axioms:

$$\frac{\neg A*}{A} \quad \frac{\neg A}{A*} \quad \frac{}{A \mid A*} \quad \frac{}{A} \text{ when } A \in T$$

Saturated sequents

A sequent $T \vDash_p S$ (a branch of a tableau) is **saturated** if no rule can be applied to it, i.e.

- if $A \rightarrow B \in S$ then $A \in T$ and $B \in S$
- if $A \vee B \in S$ then $A \in S$ and $B \in S$
- it $A \wedge B \in T$ then $A \in T$ and $B \in T$

- if $A \rightarrow B \in T$ then $B \in T$ or $A \in S$
- if $A \vee B \in T$ then $A \in T$ or $B \in T$
- if $A \wedge B \in S$ then $A \in S$ or $B \in S$

- if $\neg A \in S$ then $A \in T$
- if $\neg A \in T$ then $A \in S$

If a saturated sequent is closed then it is valid because it cannot be falsified in any v.

If a saturated sequent is valid then it is closed, because if not closed then $v = \{i \mid P_i \in T\}$ falsifies the sequent.

Soundness and completeness of propositional tableaux

We write $T \vDash_p S \rhd_p T' \vDash_p S'$ when the first sequent is a **father** of the second one.

We have $T \vDash_p S$ iff $T' \vDash_p S'$ for all saturated sons.

When we write $T \vdash_p A$ for **there is a closed tableau for the goal** A then we have

Soundness: if $T \vdash_p A$ then $T \vDash_p A$

Completeness: if $T \vDash_p A$ then $T \vdash_p A$