Symbolic programming with dyadic concatenation

Dyadic concatenation

Recall that for every $x \in \mathbb{N}$ **exactly one** holds:

$$0 \qquad x = y\mathbf{1} = 2{\cdot}y + 1 \qquad x = y\mathbf{2} = 2{\cdot}y + 2$$

Two **strings** of **dyadic digits**, say 1211 and 2122 can be **concatenated** to $12112122 = 1211 \star 2122$ with the function:

$$x \star 0 = y \quad x \star y\mathbf{1} = (x \star y)\mathbf{1} \quad x \star y\mathbf{2} = (x \star y)\mathbf{2}$$

Dyadic concatenation is associative and

$$x \star y = x{\cdot}2^{|y|_d} + y$$

where the **dyadic size** function $|x|_d$ is:

$$|0|_d = 0 \quad |x\mathbf{1}|_d = |x\mathbf{2}|_d = |x|_d + 1$$

We can also define $Drop(n, x)$ and $Take(n, x)$ s.t.:

$$n \leq |x|_d \rightarrow x = Drop(n, x) \star Take(n, x)$$

Dyadic Coding of Turing Machines

Turing has invented his **machines** to investigate **computability**. Such a machine M can be seen as working on a **tape** consisting of symbols 1 and 2. M starts with the tape containing **input** and it terminates with the tape containing **output**. For instance:

$$11112111\underline{2} \overset{M}{\mapsto} 1111111\underline{2}$$

M can be viewed as performing **addition**
$1^4 + 1^3 = 1^7$ in **monadic** where the symbol 2 separates the numbers. And _ designates the **current** position of its **reading-writing head**.

The machine executes a **program** with **instructions**: l (move the head one position **left**), **extending** the tape with the symbol 2 if the head is at the left end,
r (move one position **right**), extending with 2 if needed,
w_1 (**overwrites** the currently read symbol with 1);
w_2 (**overwrites** with 2),
$\mathit{If}_1(p_1, p_2)$ executes the **program** p_1 if the currently **read** symbol is 1 and the **program** p_2 if the symbol is 2
$Wh_1(p)$ executes the program $p\,Wh_1(p)$ if the currently **read** symbol is 1 and does **nothing** if the symbol is 2 .

Program $Rb \equiv r\,Wh_1(r)$ moves **one block right**:

$$\ldots 1^m \underline{2} 1^n 2 \ldots \mapsto \ldots 1^m 2 1^n \underline{2} \ldots$$

Program $Lb \equiv l\,Wh_1(l)$ moves **one block left**:

$$\ldots 1^m 2 1^n \underline{2} \ldots \mapsto \ldots 1^m \underline{2} 1^n 2 \ldots$$