Recursively defined functions

Monadic discrimination

For the **successor** function
$S(x) = x + 1 \equiv x'$ we have:

$$x = 0 \vee \exists! y\, x = S(y)$$

Note that y is the **uniquely** determined **predecessor** of x:

$$\begin{aligned} Pr(x) = \ & \text{if } x = \\ & \quad 0 \to 0 \\ & \quad S(y) \to y \end{aligned}$$

As clauses:

$$Pr(x) = 0 \leftarrow x = 0$$
$$Pr(x) = y \leftarrow x = S(y)$$

or even simpler (writing $x+1$ instead of $S(x)$):

$$Pr(0) = 0$$
$$Pr(x + 1) = x$$

Recursive definition of addition

$$
\begin{aligned}
Plus(x, y) = \text{ if } x = & \\
& 0 \rightarrow y \\
& z + 1 \rightarrow Plus(z, y) + 1
\end{aligned}
$$

As clauses

$$
\begin{aligned}
Plus(x, y) &= 0 & \leftarrow x = 0 \\
Plus(x, y) &= Plus(z, y) + 1 & \leftarrow x = z + 1
\end{aligned}
$$

or even simpler:

$$
\begin{aligned}
Plus(0, y) &= 0 \\
Plus(x + 1, y) &= Plus(x, y) + 1
\end{aligned}
$$

Recursive definition of multiplication

$$
\begin{aligned}
Mul(x,y) = \ & \text{if } x = \\
& \quad 0 \rightarrow 0 \\
& \quad z+1 \rightarrow Plus(Mul(z,y),y)
\end{aligned}
$$

As clauses

$$
\begin{aligned}
Mul(x,y) &= 0 & \leftarrow x = 0 \\
Mul(x,y) &= Plus(Mul(z,y),y) & \leftarrow x = z+1
\end{aligned}
$$

or even simpler:

$$
\begin{aligned}
Mul(0,y) &= 0 \\
Mul(x+1,y) &= Plus(Mul(x,y),y)
\end{aligned}
$$

Recursive definition of modified subtraction

We wish $Sub(x, y) \equiv x \dot{-} y$ such that

$$x \geq y \rightarrow y + (x \dot{-} y) = x$$

and 0 otherwise.

$$\begin{aligned} x \dot{-} y = \text{ if } y = & \\ & 0 \quad \rightarrow x \\ & z+1 \rightarrow \text{if } x = \\ & \qquad 0 \rightarrow 0 \\ & \qquad w+1 \rightarrow w \dot{-} z \end{aligned}$$

As clauses

$$\begin{aligned} x \dot{-} y = x & \leftarrow y = 0 \\ x \dot{-} y = 0 & \leftarrow y = z+1 \wedge x = 0 \\ x \dot{-} y = w \dot{-} z & \leftarrow y = z+1 \wedge x = w+1 \end{aligned}$$

or simpler (**note** the left to right discrimination order)

$$\begin{aligned} & x \dot{-} 0 = x \\ & x \dot{-} (y+1) = 0 \qquad \leftarrow x = 0 \\ & x \dot{-} (y+1) = w \dot{-} y \leftarrow x = w+1 \end{aligned}$$

Recursive definition of division by repeated subtraction

We wish $Div(x, y) \equiv x \div y$ such that

$$y > 0 \rightarrow \exists r(r < y \wedge x = (x \div y) \cdot y + r)$$

$$\begin{aligned} x \div y = \text{ if } & \\ & y = 0 \rightarrow 0 \\ & y > 0 \rightarrow \text{if} \\ & \qquad x < y \rightarrow 0 \\ & \qquad x \geq y \rightarrow (x \dot{-} y) \div y + 1 \end{aligned}$$

where $x < y =$ if $y \dot{-} x = 0 \rightarrow 0; z + 1 \rightarrow 1$

$$x < y \leftarrow y \dot{-} x = z + 1$$

In clauses:

$$\begin{aligned} x \div y &= 0 && \leftarrow y = 0 \\ x \div y &= 0 && \leftarrow y > 0 \wedge x < y \\ x \div y &= (x \dot{-} y) \div y + 1 && \leftarrow y > 0 \wedge x \geq y \end{aligned}$$

Greatest common divisor according to Euclid

$$gcd(x,y) \mid x,y \wedge \forall z(z \mid x,y \rightarrow z \leq gcd(x,y))$$

where $x \mid y \leftrightarrow \exists z\, x{\cdot}z = y$.

$$\begin{aligned} gcd(x,y) = \ & \text{if} \\ & y = 0 \rightarrow x \\ & y > 0 \rightarrow gcd(x, x \text{ mod } y) \end{aligned}$$

In clauses

$$\begin{aligned} gcd(x,y) &= x && \leftarrow y = 0 \\ gcd(x,y) &= gcd(y, x \text{ mod } y) && \leftarrow y > 0 \end{aligned}$$

Measures for recursion

Not every recursive 'definition' defines a function. There is no f satisfying:

$$f(x) = f(x + 1) + 1$$

If for an n-ary recursively defined $f(\vec{x})$ there is an n-ary **measure** function $\mu(\vec{y})$ such that for every recursive call $f(\vec{s})$ in the definition we have $\mu(\vec{s}) < \mu(\vec{x})$, i.e. the recursion **descends** in μ, then there is a **unique** f satisfying the recursive equation.

Measures for the previous recursive definitions

$Plus(x, y)$ has the measure $\mu(x, y) = x$.

$Mul(x, y)$ has the measure $\mu(x, y) = y$

$Sub(x, y)$ has the measure $\mu(x, y) = x$ but also $\mu(x, y) = y$.

$Div(x, y)$ has the measure $\mu(x, y) = x$.

$gcd(x, y)$ has the measure $\mu(x, y) = y$.

With measures $\mu(x, y) = x$ we say that the recursion **descends** in x.

With measures $\mu(x, y) = y$ we say that the recursion **descends** in y.