Propositional Logic I (semantics)

The language of propositional logic

Propositional formulas are formed from

- **propositional variables** $(P_0, P_1, \ldots)$ by
- **propositional connectives** which are
  - **nullary**: truth $(\top)$, falsehood $(\bot)$
  - **unary**: negation $(\neg)$
  - **binary**: disjunction $(\vee)$, conjunction $(\wedge)$ implication $(\rightarrow)$, equivalence $(\leftrightarrow)$

Binary are **infix** $(\rightarrow, \leftrightarrow$ groups to the right, the rest to the left)
Precedence from highest is $\neg$, $\wedge$, $\vee$, $(\rightarrow, \leftrightarrow)$.
Thus
$P_1 \rightarrow P_2 \leftrightarrow P_3 \vee \neg P_4 \wedge P_5$ abbreviates
$P_1 \rightarrow (P_2 \leftrightarrow (P_3 \vee (\neg(P_4) \wedge P_5)))$

Truth functions

We identify the **truth values** *true* and *false* with the nullary symbols $\top$ and $\bot$ respectively. The remaining connectives are **interpreted** as functions over truth values satisfying:

P_1	P_2	$\neg P_1$	$P_1 \wedge P_2$	$P_1 \vee P_2$	$P_1 \rightarrow P_2$	$P_1 \leftrightarrow P_2$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$

We have

$$A \leftrightarrow B \equiv A \rightarrow B \wedge B \rightarrow A$$

$$\neg A \equiv A \rightarrow \bot$$

$$A \rightarrow B \equiv \neg A \vee B$$

$$A \wedge B \equiv \neg(\neg A \vee \neg B)$$

Complete sets of connectives

We can define all propositional connectives either from $\neg$ and $\rightarrow$, or from $\neg$ and $\vee$, or from $\neg$ and $\wedge$, or from $\bot$ and $\rightarrow$.

Actually, we can define all connectives from the single connective **Sheffer's stroke**: *not both A and B*

$$A \mid B \equiv \neg(A \wedge B)$$

because

$$\neg A \equiv A \mid A$$
$$A \wedge B \equiv (A \mid B) \mid (A \mid B)$$

Tautologies

Of special interest are those propositional formulas A which are true ($\top$) for all possible truth values of its propositional variables, in writing $\models_p A$.
Every such formula is a **tautology**.
Tautologies are the cornerstones of mathematical logic.
Some examples of (schemas of) tautologies:

$$\models_p (A \to B \to C) \leftrightarrow A \land B \to C$$
$$\models_p (A \to B \to C) \leftrightarrow (A \to B) \to A \to C$$
$$\models_p (A \to B) \leftrightarrow \neg B \to \neg A$$

for any propositional formulas A, B, and C

Propositional satisfaction relation

A **propositional valuation**, or an **propositional assignment** v is a (possibly infinite) set $v \subset \mathbb{N}$ The idea is that the $P_i \equiv \top$ iff $i \in v$.

We say that a formula A **is satisfied in** v, in writing $v \models_p A$, if A is true for the assignment v.

We thus have: $v \models P_i$ iff $i \in v$

$v \models_p \neg A$ iff not $v \models_p A$ iff $v \not\models_p A$

$v \models_p A \wedge B$ iff $v \models_p A$ and $v \models_p B$

$v \models_p A \vee B$ iff $v \models_p A$ or $v \models_p B$

$v \models_p A \rightarrow B$ iff whenever $v \models_p A$ also $v \models_p B$

Thus A **is a tautology** iff $v \models A$ for all valuations v.

Coincidence property if two valuations v and w are such that $i \in v$ iff $i \in w$ for all P_i occurring in A then $v \models A$ iff $w \models A$

Arithmetization of propositional logic

We wish to show that the property of A **being a tautology** is **decidable**, i.e. that the predicate $\vDash_p A$ is **computable**. For that we have to **encode** (arithmetize) propositional logic into natural numbers.