12 Arithmetical Hierarchy

12.1 Introduction. We will study properties of the following index sets:

$$\begin{aligned} Tot &= \{e \mid \varphi_e \text{ is total}\} = \{e \mid W_e = \mathrm{N}\} \\ Inf &= \{e \mid W_e \text{ is infinite}\} \\ Con &= \{e \mid \varphi_e \text{ is total and constant}\} \\ Fin &= \{e \mid W_e \text{ is finite}\} \\ Comp &= \{e \mid W_e \text{ is computable}\} \end{aligned}$$

12.2 Exercise. Prove that *Tot*, *Inf* and *Con* are Π_2 sets.

12.3 Exercise. Prove that *Fin* is Σ_2.

12.4 Exercise. Prove that *Comp* is Σ_3. *Hint.* Use Post's theorem.

12.5 Definition. The set A is *many-one reducible* (*m-reducible* for short) to the set B, written $A \leq_{\mathrm{m}} B$, if there is a computable function f such that $x \in A \leftrightarrow f(x) \in B$ for every number x.

The sets A and B are *many-one equivalent* (*m-equivalent* for short), written $A \equiv_{\mathrm{m}} B$, if $A \leq_{\mathrm{m}} B$ and $B \leq_{\mathrm{m}} A$.

12.6 Exercise. Prove that if A is Σ_n (Π_n) and $B \leq_{\mathrm{m}} A$ then B is Σ_n (Π_n).

12.7 Exercise. Prove that *Tot* is Π_2-complete, i.e. prove $A \leq_{\mathrm{m}}$ *Tot* for every Π_2 set A.

12.8 Exercise. Prove that *Tot* $\equiv_{\mathrm{m}}$ *Inf* $\equiv_{\mathrm{m}}$ *Con*. *Hint.* Use Kleene's s-m-n theorem.

12.9 Exercise. Prove that *Tot*, *Inf* and *Con* are not Σ_2 sets.

12.10 Exercise. Prove that *Fin* is not Π_2.