5.4 Recursion with Measure

5.4.1 The principle of measure induction. For every formula $\varphi[\vec{x}]$ and term $\mu[\vec{x}]$, the formula of *induction on* $\vec{x}$ *with measure* $\mu[\vec{x}]$ *for* φ is the following one:

$$\forall\vec{x}\Big(\forall\vec{y}\big(\mu[\vec{y}] < \mu[\vec{x}] \to \varphi[\vec{y}]\big) \to \varphi[\vec{x}]\Big) \to \forall\vec{x}\varphi[\vec{x}]. \tag{1}$$

We assume here that the variables $\vec{y}$ are different from $\vec{x}$ and that they do not occur freely in φ. The formula φ and the term μ may contain additional variables as parameters.

Note that for $\vec{x} \equiv x$ and $\mu[x] \equiv x$, the scheme of measure induction coincides with the scheme of complete induction.

5.4.2 Theorem *The principle of measure induction holds for each formula.*

Proof. The principle of measure induction 5.4.1(1) is reduced to mathematical induction as follows. Under the assumption that φ is μ*-progressive*:

$$\forall\vec{x}\Big(\forall\vec{y}\big(\mu[\vec{y}] < \mu[\vec{x}] \to \varphi[\vec{y}]\big) \to \varphi[\vec{x}]\Big), \tag{$\dagger_1$}$$

we first prove, by induction on n, the auxiliary property:

$$\forall\vec{z}\big(\mu[\vec{z}] < n \to \varphi[\vec{z}]\big). \tag{$\dagger_2$}$$

In the base case there is nothing to prove. In the induction step take any $\vec{z}$ such that $\mu[\vec{z}] < n+1$ and consider two cases. If $\mu[\vec{z}] < n$ then we obtain $\varphi[\vec{z}]$ by IH. If $\mu[\vec{z}] = n$ then by instantiating of $(\dagger_1)$ with $\vec{x} := \vec{z}$ we obtain

$$\forall\vec{y}\big(\mu[\vec{y}] < n \to \varphi[\vec{y}]\big) \to \varphi[\vec{z}].$$

Now we apply IH to get $\varphi[\vec{z}]$.

With the auxiliary property proved we obtain that $\varphi[\vec{x}]$ holds for every $\vec{x}$ by instantiating of $\forall n(\dagger_2)$ with $n := \mu[\vec{x}] + 1$ and $\vec{z} := \vec{x}$. □

5.4.3 Greatest common divisor. Consider the recursive definition of the greatest divisor function of the form

$$
\begin{aligned}
\gcd(x,y) = \ & \textbf{if } x \neq 0 \land y \neq 0 \textbf{ then} \\
& \quad \textbf{case} \\
& \qquad x < y \Rightarrow \gcd(x, y \dot{-} x) \\
& \qquad x = y \Rightarrow x \\
& \qquad x > y \Rightarrow \gcd(x \dot{-} y, y) \\
& \quad \textbf{end} \\
& \textbf{else} \\
& \quad \max(x,y).
\end{aligned}
$$

The definition is an example of regular recursion where recursion goes down in the measure $\max(x,y)$. Its conditions of regularity

$$\vdash_{\mathrm{PA}} \; x \neq 0 \land y \neq 0 \land x < y \to \max(x, y \dot{-} x) < \max(x,y) \qquad (1)$$
$$\vdash_{\mathrm{PA}} \; x \neq 0 \land y \neq 0 \land x > y \to \max(x \dot{-} y, y) < \max(x,y)$$

follow from

$$\vdash_{\mathrm{PA}} \; a > b > 0 \to a \dot{-} b < a.$$

The idea of the algorithm is based on the observation that

$$\vdash_{\mathrm{PA}} \; x < y \land z \mid x \to z \mid y \leftrightarrow z \mid y \dot{-} x. \qquad (2)$$

We claim that

$$\vdash_{\mathrm{PA}} x \neq 0 \lor y \neq 0 \to \gcd(x,y) \mid x \land \gcd(x,y) \mid y \qquad (3)$$
$$\vdash_{\mathrm{PA}} (x \neq 0 \lor y \neq 0) \land z \mid x \land z \mid y \to z \leq \gcd(x,y). \qquad (4)$$

Verification. (3): By measure induction on x, y with the measure $\max(x,y)$. Assume $x \neq 0 \lor y \neq 0$ and consider two cases. If $x = 0 \lor y = 0$ then clearly

$$x \neq 0 \land y = 0 \lor x = 0 \land y = 0.$$

If $x \neq 0 \land y = 0$ then the claim follows from 5.3.4(5)(1) because

$$\gcd(x,y) = \max(x,y) = x \land x \mid x \land x \mid 0.$$

The subcase $x = 0 \land y \neq 0$ is proved similarly. If $x \neq 0 \land y \neq 0$ then we consider three subcases. If $x < y$ then by (1) we have $\max(x, y \dot{-} x) < \max(x,y)$ and thus by IH applied to the pair $(x, y \dot{-} x)$ we obtain

$$\gcd(x, y \dot{-} x) \mid x \land \gcd(x, y \dot{-} x) \mid y \dot{-} x.$$

From definition

$$\gcd(x,y) \mid x \land \gcd(x,y) \mid y \dot{-} x.$$

From this and (2) we finally obtain

$$\gcd(x,y) \mid x \land \gcd(x,y) \mid y.$$

The subcase $x > y$ is proved similarly; the subcase $x = y$ follows from 5.3.4(1) and definition.

(4): By measure induction on x, y with the measure $\max(x,y)$. So assume that $x \neq 0 \lor y \neq 0$ holds and take any number z such that

$$z \mid x \wedge z \mid y.$$

We consider two cases. If $x = 0 \vee y = 0$ then

$$x \neq 0 \wedge y = 0 \vee x = 0 \wedge y = 0.$$

The desired bound $z \leq \gcd(x, y)$ follows from

$$x \neq 0 \wedge y = 0 \wedge z \mid x \overset{5.3.4(13)}{\Rightarrow} z \leq x = \max(x, y) = \gcd(x, y)$$
$$x = 0 \wedge y \neq 0 \wedge z \mid y \overset{5.3.4(13)}{\Rightarrow} z \leq y = \max(x, y) = \gcd(x, y).$$

If $x \neq 0 \wedge y \neq 0$ then we consider three subcases. If $x < y$ then by (2) we have

$$z \mid x \wedge z \mid y \dot{-} x.$$

By (1) we have $\max(x, y \dot{-} x) < \max(x, y)$ and thus by IH applied to the pair $(x, y \dot{-} x)$ we obtain

$$z \leq \gcd(x, y \dot{-} x) = \gcd(x, y).$$

The subcase $x > y$ is proved similarly; the subcase $x = y$ follows from 5.3.4(13) and definition. □